Vincent's Theorem from a modern point of view

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1 Introduction

In this paper, after having summarized the main results we obtained in [2], we suggest some ideas which may lead to future developments.

The reader may wonder at the very particular nature of our subject, and whether it is inconsistent with the generality of category theory; but as emphasized in [6] the peculiar nature of mathematics resides exactly in the force it gains by contrasting general ideas to 'facts' in a never ending dialectics.

André Weil loved to quote Euler's maxim: "nihil est in numerico quod non est in algebraico".

In fact, even the most trivial numerical identity may be the starting point for a deep understanding of the mathematical structure upon which it may depend in a subtle and unforeseeable way. On the other hand, no abstract mathematical structure is meaningful if it isn't able to generate concrete and particular results.

Vincent's theorem originally appeared as a sort of complement to Lagrange's method to approximate the roots of algebraic equations via continued fractions. We described in great detail this aspect of the theorem in [2]. In this paper we underline its geometrical features which, in principle, make it applicable also in other situations to obtain different kinds of algorithms.

We conclude our work by giving an example in terms of Farey sequences (which are very similar to continued fractions...) to emphasize the independence of the theorem from the particular kind of approximation we devise for the roots.

The polynomials considered throughout the paper have real coefficients and, for the sake of simplicity, they are assumed to have simple roots, even if (as we have shown in [2]) this is not a real limitation.

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2 Positive real roots and variations

The number of sign variations of a polynomial[1] gives precise information about its roots only in the cases it has the value 0 or 1.

The value 0, the absence of variations, points out that the polynomial has no positive roots, while the value 1 indicates the presence of a single positive root.

But how can it happen that a polynomial has a number of variations greater than the number of its positive roots ? In this case, how are its complex roots located in the complex plane ?

The examination of the number of variations of a third degree polynomial gives a precise suggestion about a general situation.

Consider the polynomial $p(x)$ which has the real positive root a and the roots $\alpha \pm i\beta$.

Then

$$p(x) = x^3 - (2\alpha + a)\,x^2 + \left(\alpha^2 + \beta^2 + 2a\alpha\right) x - a\left(\alpha^2 + \beta^2\right). \tag{1}$$

The possibility that $p(x)$ has 3 variations corresponds to

$$2\alpha + a > 0 \quad \wedge \quad \alpha^2 + \beta^2 + 2a\alpha > 0. \tag{2}$$

Let us look at (2) from a geometrical point of view, with reference to the following figure:

[1]Given the sequence of the real coefficients of a polynomial

$$\alpha_0, \alpha_1, \alpha_2, \ldots,$$

we say that a sign variation exists between two coefficients α_p and α_q if one of the following conditions holds:
1) $q = p + 1$ and α_p and α_q have opposite signs;
2) $q > p+1$ and the numbers $\alpha_{p+1}, \alpha_{p+2}, \ldots, \alpha_{q-1}$ are all zero while α_p and α_q have opposite signs. We will say 'a variation of the polynomial' to mean a variation in the sequence of its coefficients.

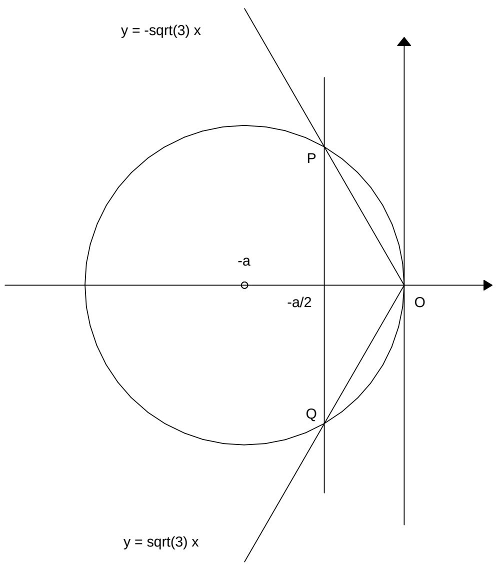

Fig. 1

The possibility that the polynomial (1) has 3 variations corresponds to the fact that the points $\alpha \pm i\beta$ are 'on the right ' of the line parallel to the imaginary axis through the point $-\frac{a}{2}$ and at the same time to the fact that they are exterior to the circle having equation

$$|z + a| = a.$$

The points P and Q, in which the line and the circle intersect, have coordinates $\left(-\frac{a}{2}, \pm\frac{\sqrt{3}}{2}a\right)$. Hence *independently of the value* a they are on the lines

$$\operatorname{Im} z = \pm\sqrt{3}\operatorname{Re} z. \tag{3}$$

Suppose that the points $\alpha \pm i\beta$ are in the interior of the sector $S_{\sqrt{3}}$ defined by

$$S_{\sqrt{3}} = \left\{ z \mid \operatorname{Re} z < 0 \quad \wedge \quad |\operatorname{Im} z| < \sqrt{3}\,|\operatorname{Re} z| \right\}. \tag{4}$$

Then the polynomial (1) has exactly one variation.

This sector is the particular case for $r = 1$ of Obreschkoff's definition:

$$S = \left\{ z \,|z = -\rho\left(\cos\varphi + i\sin\varphi\right),\ \rho > 0,\ |\varphi| < \frac{\pi}{r+2} \right\} \tag{5}$$

The result for the third degree polynomials suggests a general behavior.

A Lemma contained in [7, p.81], implies that a real polynomial which has only one positive root while all the other roots are in the sector $S_{\sqrt{3}}$ defined by (4) has *exactly one variation.* We gave a simple and constructive proof of this result in [2].

More generally we proved the

Lemma 1 *A polynomial which has r positive real roots and all the other roots within the sector (5) has exactly r variations.*

Proof. *See [2, section 8.2].* ■

3 The properties of a simple geometrical transformation

We expose some geometrical properties of the transformation $T:\ \mathbb{C}\to\mathbb{C}$, defined by

$$T(z) = \frac{z-a}{b-z}. \tag{6}$$

where a and b are positive real numbers.

1) The circle

$$\left|z - \frac{a+b}{2}\right| = \frac{|b-a|}{2}$$

whose diameter lies on the real axis, with endpoints a and b, is mapped by the transformation (6) onto the imaginary axis. The exterior points are mapped into the half-plane $\mathrm{Re}(z) < 0$.

2) The lines

$$\mathrm{Im}\,(z) = \pm s\,\mathrm{Re}\,(z) \quad (s\in\mathbb{R})$$

are the images of the circles having center

$$c^{\pm} = \frac{a+b}{2} \pm i\frac{|b-a|}{2s},$$

and radius

$$r = \frac{|b-a|}{2}\sqrt{1+\frac{1}{s^2}}.$$

It easily follows (see Fig. 2) that the sector S_s of the complex plane defined by

$$S_s = \{z \mid \mathrm{Re}\,(z) < 0 \text{ and } \ |\mathrm{Im}\,(z)| \le s\cdot|\mathrm{Re}\,(z)|\} \tag{7}$$

is the image of the exterior of the eight-shaped figure R given by the union of the two disks

$$\left|z - c^{\pm}\right| \le r.$$

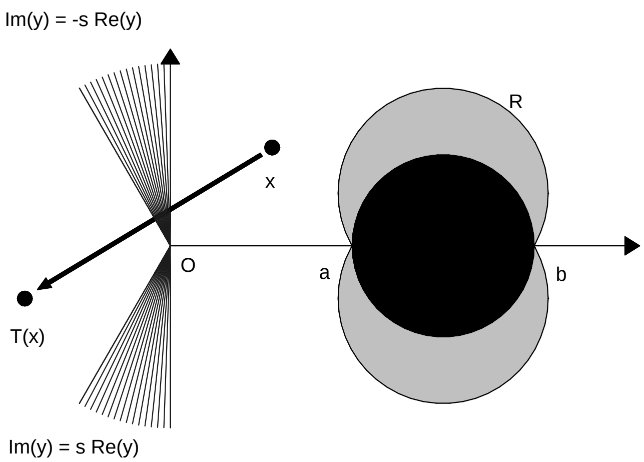

Fig. 2

4 How the geometrical transformation influences the number of variations

Given the real polynomial $f(z)$, without multiple roots, let Δ be the minimum distance of its roots $z_1, z_2, \ldots, z_n$ i.e.:

$$\Delta = \min_{i<k} |z_i - z_k| .$$

Let the numbers a and b of the previous section be such that:

$$|b-a| < \frac{\sqrt{3}}{2}\Delta. \tag{8}$$

Then, in particular, $|b-a| < \Delta$, and the circle whose diameter is (a, b) contains no complex roots and at most one positive root of the equation $f(z) = 0$.

Let now R be the union of the two disks centered at

$$c^{\pm} = \frac{a+b}{2} \pm i\frac{|b-a|}{2\sqrt{3}} \tag{9}$$

with radius

$$r = \frac{|b-a|}{\sqrt{3}}, \tag{10}$$

which correspond to the half-planes bounded by the lines

$$\mathrm{Im}\,(z) = \pm\sqrt{3}\,\mathrm{Re}\,(z)\,.$$

It follows from (8) that R contains at most one real root: indeed, the maximum distance between points of R and points of the interval (a, b) is

$$2r = 2\frac{|b-a|}{\sqrt{3}} < \Delta.$$

The following alternative is then possible:

- all the roots of $f\,(z) = 0$ lie outside the circle whose diameter is (a, b);
- if a (necessarily unique and real) root lies inside this circle, then all the other roots lie in the complement of R.

In the first case T maps all the roots of $f\,(z) = 0$ into the left complex half-plane, while in the second case the image of the positive root is still positive and all the other roots are mapped into the sector:

$$S_{\sqrt{3}} = \left\{ z \mid \mathrm{Re}\, z < 0 \quad \wedge \quad |\mathrm{Im}\, z| < \sqrt{3}\,|\mathrm{Re}\, z| \right\}.$$

The inverse transformation of T is:

$$S\,(z) = \frac{a+bz}{1+z},$$

hence the polynomial

$$\phi\,(z) = (1+z)^n f\left(\frac{a+bz}{1+z}\right)$$

has no variations in the former case, while *it has exactly one variation* in the latter.

Example 2 *The polynomial*

$$f\,(z) = z^3 - z^2 - 2z + 2$$

has the roots $\pm\sqrt{2}, 1$. *The least distance is*

$$\Delta = \sqrt{2} - 1 \approx 0.414\,21,$$

and

$$\Delta\frac{\sqrt{3}}{2} \approx 0.3587 > \frac{1}{3}.$$

The length of the interval $\left[1 - \frac{1}{8}, 1 + \frac{1}{8}\right] = \left[\frac{7}{8}, \frac{9}{8}\right]$ *is less than* $\frac{1}{3}$ *and it contains only one root. We see that*

$$(1+z)^3 f\left(\frac{\frac{7}{8} + \frac{9}{8}z}{1+z}\right) = -\frac{47}{512}z^3 - \frac{83}{512}z^2 + \frac{51}{512}z + \frac{79}{512}$$

has only one variation. The interval $\left[\frac{2}{8}, \frac{3}{8}\right]$ *does not contain any root and we see that*

$$(1+z)^3 f\left(\frac{\frac{2}{8}+\frac{3}{8}z}{1+z}\right) = \frac{595}{512}z^3 + \frac{967}{256}z^2 + \frac{521}{128}z + \frac{93}{64}$$

has no variations.

5 Vincent's Theorem

We observed at the beginning that Vincent's theorem was originally formulated in terms of continued fractions. But, as we ourselves learned by experience, its original formulation looks rather enigmatic.

We prefer to formulate Vincent's result in the following form:

Theorem 3 *(Vincent) Let* $f(z)$ *be a real polynomial of degree* n *which has only simple roots. It is possible to determine a positive quantity* δ *so that for every pair of positive real numbers* a, b *with* $|b-a| < \delta$, *every transformed polynomial of the form*

$$\phi(z) = (1+z)^n f\left(\frac{a+bz}{1+z}\right) \tag{11}$$

has exactly 0 *or* 1 *variations. The second case is possible if and only if* $f(z)$ *has a single root within* (a, b).

Proof. Let Δ denote the least distance of the roots of $f(z)$, and set $\delta = \frac{\sqrt{3}}{2}\Delta$. ■

Remark 4 *Usually, every algorithm to isolate the real roots of a polynomial equation depends on a scan of an interval, which contains all the roots, by subintervals of decreasing amplitude.*
Lagrange's famous 'équation au carré des différences' may be used (in principle, but it is a highly impractical tool) to find an upper bound for the least distance of the roots, and hence to divide the original interval into subintervals of amplitude less than the least distance. The presence of a real root is marked by the fact that the polynomial must have opposite signs at the endpoints of every sub interval containing a root.
Vincent's theorem, without any need of knowing a priori the least distance of the roots, gives a test to determine when an arbitrary method based on a subdivision into subintervals reaches its goal.

Remark 5 *Let us look at the form of the polynomial* $\phi(x)$ *in (11). Since*

$$\phi(z) = (1+z)^n f\left(b + \frac{a-b}{1+z}\right),$$

by the help of the Taylor formula we get

$$\phi(z) = (1+z)^n \left\{ f(b) + \frac{f'(b)}{1!}\frac{a-b}{1+z} + \frac{f''(b)}{2!}\frac{(a-b)^2}{(1+z)^2} + \dots \right\} \tag{12}$$

From (12) it is quite easy to obtain a vector representation of the coefficients of ϕ. Let us write

$$\phi(z) = \alpha_0 + \alpha_1 z + \ldots + \alpha_n z^n.$$

Then

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \ldots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \binom{n}{0} & \binom{n-1}{0} & \binom{n-2}{0} & \ldots & \binom{0}{0} \\ \binom{n}{1} & \binom{n-1}{1} & \binom{n-2}{1} & \ldots & 0 \\ \binom{n}{2} & \binom{n-1}{2} & \binom{n-2}{2} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \binom{n}{n} & 0 & 0 & \ldots & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & a-b & \ldots & 0 \\ 0 & & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & (a-b)^n \end{bmatrix} \begin{bmatrix} f(b) \\ f'(b) \\ \frac{f''(b)}{2!} \\ \ldots \\ \frac{f^{(n)}(b)}{n!} \end{bmatrix}$$

A look at the previous formula shows that the vector of the coefficients is obtained by the product of the matrix

$$\begin{bmatrix} \binom{n}{0} & \binom{n-1}{0} & \binom{n-2}{0} & \ldots & \binom{0}{0} \\ \binom{n}{1} & \binom{n-1}{1} & \binom{n-2}{1} & \ldots & 0 \\ \binom{n}{2} & \binom{n-1}{2} & \binom{n-2}{2} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \binom{n}{n} & 0 & 0 & \ldots & 0 \end{bmatrix}, \tag{13}$$

which depends ***only on the degree*** n*, by the matrix*

$$\begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & a-b & \ldots & 0 \\ 0 & & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & (a-b)^n \end{bmatrix} \tag{14}$$

which ***depends only on*** a ***and*** b*, and at last by the vector*

$$\left[f(b), f'(b), \ldots \frac{f^{(n)}(b)}{n!}\right]$$

which collects all the information about the polynomial $f(z)$ at b.
This representation of the transformed polynomial, which was the one originally used by Vincent to prove his theorem (see [2, section 4]), might be exploited to obtain the same kind of results which are usually obtained by the help of Sturm's theorem. Just to give the most obvious example: Newton rule to find an upper bound of the positive roots appears as an obvious corollary.[2]
Suppose

$$b < a$$

[2]In fact, we use a more general result. If the polynomial $\phi(z)$ has no variations, the polynomial $f(z)$ cannot have roots in (a, b).

and that

$$f(b) > 0, \ldots, f^{(n)}(b) > 0.$$

Then the matrices (13) and (14) have only positive coefficients as well as their product. The polynomial $\phi(z)$ *has no variations and hence* $f(z)$ *has no roots between* b *and* a*. Since* a *can be chosen arbitrarily,* $f(z)$ *has no roots greater than* b*.*

An idea of the future developments we devise may be given by the following

Proposition 6 *Suppose* $f(z)$ *has only real roots and let* a*,* b *be positive real numbers. Then the number of variations of the polynomial (12) is exactly the number of real roots of* $f(z)$ *contained in the interval* (a, b)*.*

Proof. Since all the roots are real, the eight-shaped figure R of Fig. 2 does not contain any root. Hence all the roots exterior to the interval (a, b) are mapped by $T(z)$ onto the negative real axis. The application of Lemma 1 concludes the proof. ■

Corollary 7 *The number of variations of the polynomial (12) is greater than the number of real roots in* (a, b)*, and the difference is an even number.*

Proof. It is enough to decompose $f(z)$ as $f_1(z) \cdot f_2(z)$ where $f_1(z)$ contains all the real roots of $f(z)$. The result follows from Lemma 1(observe that the zero degree coefficient of $f_2(z)$ is positive. The parity of the number of roots of $f(z)$ is the same as the one of $f_1(z)$). ■

We give an example to show how the proposition and the corollary may be used.

Example 8 *Consider the symmetric matrix*

$$\begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 2 & 1 & 5 \\ -2 & 1 & 0 & 7 \\ 3 & 5 & 7 & 1 \end{bmatrix}.$$

The roots of the characteristic polynomial

$$f(z) = z^4 - 4z^3 - 83z^2 + 216z - 36,$$

are all real. We want to find the number of the roots in $(0, 1)$*.*
Considering what we explained in Remark 5, we set $b = 0$*,* $a = 1$ *and consequently we simply consider the matrix product*

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & 3 & 2 & 1 & 0 \\ 6 & 3 & 1 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -36 \\ 216 \\ -83 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 94 \\ 334 \\ 349 \\ 72 \\ -36 \end{bmatrix}.$$

Since the resulting vector has one variation, we have one root in $(0, 1)$*.*

Example 9 *Consider the polynomial*

$$f(z) = z^6 - 5z^5 + 7z^4 - 5z^3 + 9z^2 - 15z + 17. \tag{15}$$

We want to determine the maximum number of its roots in $(1,4)$. *We have*

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 6 & 5 & 4 & 3 & 2 & 1 & 0 \\ 15 & 10 & 6 & 3 & 1 & 0 & 0 \\ 20 & 10 & 4 & 1 & 0 & 0 & 0 \\ 15 & 5 & 1 & 0 & 0 & 0 & 0 \\ 6 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -27 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 81 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -243 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 729 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & -3 & 9 & -27 & 81 & -243 & 729 \\ 6 & -15 & 36 & -81 & 162 & -243 & 0 \\ 15 & -30 & 54 & -81 & 81 & 0 & 0 \\ 20 & -30 & 36 & -27 & 0 & 0 & 0 \\ 15 & -15 & 9 & 0 & 0 & 0 & 0 \\ 6 & -3 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the sequence of values

$$f(1), \frac{f'(1)}{1!}, \ldots, \frac{f^{(6)}(1)}{6!}$$

is

$$9, -3, 1, -7, -3, 1, 1, \tag{16}$$

and the product

$$\begin{bmatrix} 1 & -3 & 9 & -27 & 81 & -243 & 729 \\ 6 & -15 & 36 & -81 & 162 & -243 & 0 \\ 15 & -30 & 54 & -81 & 81 & 0 & 0 \\ 20 & -30 & 36 & -27 & 0 & 0 & 0 \\ 15 & -15 & 9 & 0 & 0 & 0 & 0 \\ 6 & -3 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 9 \\ -3 \\ 1 \\ -7 \\ -3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 459 \\ -27 \\ 603 \\ 495 \\ 189 \\ 63 \\ 9 \end{bmatrix}$$

gives a vector which has two variations, we may have at most 2 *roots in* $(1,4)$. *On the other hand, the sequence of values*

$$f(4), \frac{f'(4)}{1!}, \ldots, \frac{f^{(6)}(4)}{6!}$$

is

$$549, 1353, 1261, 587, 147, 19, 1. \tag{17}$$

Since the difference of the number of variations of the two sequences (16) and (17) is 4, *the theorem of Budan and Fourier allows to conclude that there are* *at most* 4 *roots in* $(1,4)$. *The polynomial (15) actually has two roots in* $(1,4)$, *hence the estimate given by Vincent's algorithm is, at least in this case, more precise. But the comparison of the two algorithms will be a matter for future developments.*

6 Algorithms

A Farey series of order N, which we denote by $\mathcal{F}_N$ (we take the definition from [5, p. 118]), is the set of all reduced fractions between 0 and 1 whose denominators are N or less, arranged in increasing order.

For example, for $N=3$ we have

$$\mathcal{F}_3 = \left\{ \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1} \right\}.$$

Recalling that a *mediant* of two fractions $\frac{m}{n} < \frac{p}{q}$ is given by $\frac{m+p}{n+q}$ and that $\frac{m}{n} < \frac{m+p}{n+q} < \frac{p}{q}$, it is evident that we can obtain $\mathcal{F}_N$ from $\mathcal{F}_{N-1}$ by inserting mediants whenever it is possible to do so without getting a denominator greater than N (see [5, ibidem]).

Hence

$$\mathcal{F}_4 = \left\{ \frac{0}{1}, \frac{0+1}{1+3}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{2+1}{3+1}, \frac{1}{1} \right\} = \left\{ 0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, 1 \right\}.$$

The Farey series, or even better the Stern-Brocot tree, gives an interesting way to obtain all the reduced fractions within $(0,1)$.

It is possible that Vincent's Theorem might be connected with the Farey series in the same fruitful way it was connected with continued fractions. This may be a direction of future research, but for the moment we content ourselves to present an example of its use to separate the roots.[3]

Example 10 *Consider the polynomial*

$$f(z) = 12z^3 + 54z^2 - 34z + 5. \tag{18}$$

We want to find its roots in the interval $(0,1)$
We begin with $\mathcal{F}_2 = \left\{0, \frac{1}{2}, 1\right\}$ *Then*

$$(z+1)^3 f\left(\frac{0+\frac{1}{2}z}{z+1}\right) = 3z^3 - \frac{11}{2}z^2 - 2z + 5$$

$$(z+1)^3 f\left(\frac{\frac{1}{2}+z}{1+z}\right) = 37z^3 + 56z^2 + \frac{47}{2}z + 3.$$

[3] We are indebted to Donato Saeli, who, at the end of a conference where we described the contents and applications of Vincent's theorem, suggested to look at its connections with the Farey series.

The second polynomial has no variations, hence there are no roots in $\left[\frac{1}{2}, 1\right]$ *while the presence of two variations in the first polynomial shows the possibility of two roots in* $\left[0, \frac{1}{2}\right]$.

Now $\mathcal{F}_3 = \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right\}$ *and we have to consider the intervals* $\left[0, \frac{1}{3}\right]$ *and* $\left[\frac{1}{3}, \frac{1}{2}\right]$. *We have*

$$(z+1)^3 f\left(\frac{0 + \frac{1}{3}z}{1+z}\right) = \frac{1}{9}z^3 - \frac{5}{3}z^2 + \frac{11}{3}z + 5,$$

$$(z+1)^3 f\left(\frac{\frac{1}{3} + \frac{1}{2}z}{1+z}\right) = 3z^3 + \frac{25}{6}z^2 + \frac{4}{3}z + \frac{1}{9}.$$

We are reduced to the consideration of the interval $\left[\frac{0}{1}, \frac{1}{3}\right]$. *The further intervals we need of* $\mathcal{F}_4$ *are simply obtained by the insertion of the mediant:* $\left[0, \frac{1}{4}\right], \left[\frac{1}{4}, \frac{1}{3}\right]$.

$$(z+1)^3 f\left(\frac{0 + \frac{1}{4}z}{1+z}\right) = \frac{1}{16}z^3 + \frac{11}{8}z^2 + \frac{13}{2}z + 5,$$

$$(z+1)^3 f\left(\frac{\frac{1}{4} + \frac{1}{3}z}{1+z}\right) = \frac{1}{9}z^3 - \frac{1}{6}z^2 - \frac{5}{24}z + \frac{1}{16}.$$

Now we have to try with $\left[\frac{1}{4}, \frac{2}{7}\right]$, $\left[\frac{2}{7}, \frac{1}{3}\right]$, *since there are no fractions of* $\mathcal{F}_5$ *and* $\mathcal{F}_6$ *in the interval* $\left(\frac{1}{4}, \frac{1}{3}\right)$.

$$(z+1)^3 f\left(\frac{\frac{1}{4} + \frac{2}{7}z}{1+z}\right) = -\frac{9}{343}z^3 - \frac{1}{14}z^2 + \frac{1}{56}z + \frac{1}{16},$$

$$(z+1)^3 f\left(\frac{\frac{2}{7} + \frac{1}{3}z}{1+z}\right) = \frac{1}{9}z^3 + \frac{1}{21}z^2 - \frac{13}{147}z - \frac{9}{343},$$

Each polynomial has a single variation, hence every interval $\left[\frac{1}{4}, \frac{2}{7}\right]$, $\left[\frac{2}{7}, \frac{1}{3}\right]$ *contains exactly one root.*

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