Can your computer do complex analysis?

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1 Introduction

The purpose of this paper is to elaborate on the results in my earlier paper on multiple-valued complex functions ([Aslak96]) using the unwinding number notation introduced by Corless and Jeffrey ([Corle96]).

2 Basic problems of multiple-valued complex functions

For $z = x + iy$, the complex exponential function is defined by

$$e^z = e^x(\cos y + i \sin y).$$

We define the principal argument by $z = |z|e^{i \operatorname{Arg} z}$ where $\operatorname{Arg} z \in (-\pi, \pi]$. We do not define the principal argument of 0, and we will from now on assume that z is different from 0. Notice that we have defined the principal argument on the negative axis, too, but it is not continuous there. We have instead what is called "Counter-Clockwise Continuity". We then define the principal logarithm $\operatorname{Log} z$ by

$$\operatorname{Log} z = \ln |z| + i \operatorname{Arg} z,$$

where $\ln|z|$ denotes the usual real logarithm of $|z|$. We finally define the general power/exponential function by

$$z^w = e^{w \operatorname{Log} z}. \tag{1}$$

In complex analysis, the logarithm and power/exponential functions are considered to be multiple-valued functions that are made single-valued by choosing a branch of the logarithm. When doing computer algebra, we choose the principal logarithm for all our computations. One disadvantage of this is that formulas that in complex analysis are interpreted as identities between multiple-valued functions are now not always true. We clearly have $e^{z+w} = e^z e^w$ and $e^{\operatorname{Log} z} = z$, but consider the following five formulas:

Question 1

$$\operatorname{Log} e^z = z, \tag{2}$$
$$\operatorname{Log} zw = \operatorname{Log} z + \operatorname{Log} w, \tag{3}$$
$$\operatorname{Log} z^w = w \operatorname{Log} z, \tag{4}$$
$$(zw)^a = z^a w^a, \tag{5}$$
$$(z^a)^b = z^{ab}. \tag{6}$$

If all the numbers involved are real, and we take $\operatorname{Log} z$ to be the real logarithm, then all the formulas are true whenever they are defined. But for complex numbers using the principal logarithm, all five formulas are in general wrong! Let us look at some quick counter-examples and paradoxes.

For (2), we have $\operatorname{Log} e^{i2\pi} = \operatorname{Log} 1 = 0$ rather than $i2\pi$. For (3), we have $\operatorname{Log}(-i) = -i\pi/2$ while $\operatorname{Log}(-1) + \operatorname{Log} i = i\pi + i\pi/2 = i3\pi/2$. For (4), start with $(-z)^2 = z^2$. Taking the logarithm and assuming that (4) holds, we get $2\operatorname{Log}(-z) = 2\operatorname{Log} z$, or $\operatorname{Log}(-z) = \operatorname{Log} z$. If (5) holds, we would have

$$1 = \sqrt{1} = \sqrt{(-1)(-1)} = ii = -1. \tag{7}$$

For (6), we consider a paradox due to the Danish mathematician Thomas Clausen ([Claus27], [Remme91]). It was published as an exercise in Crelle's journal in 1827.

Let n be an integer. Then

$$e^{1+2n\pi i} = e,$$

and

$$(e^{1+2n\pi i})^{1+2n\pi i} = e^{1+2n\pi i} = e.$$

If we assume (6), we get

$$(e^{1+2n\pi i})^{1+2n\pi i} = e^{1+4n\pi i - 4n^2\pi^2} = ee^{-4n^2\pi^2},$$

and it follows that

$$e^{-4n^2\pi^2} = 1.$$

Another way to see that (6) is problematic, is to set $z = e$ and observe that e^{ab} is just the usual (single-valued) exponential function, while in order to evaluate $(e^a)^b$, we need to consider an exponential function with base e^a. In order to get a single-valued function we use (1), which involves making a choice of branch for the logarithm. It is therefore not reasonable to expect (6) to always be true.

In my earlier paper ([Aslak96]), I gave correct versions of the formulas in Question 1 using several auxiliary functions. Corless and Jeffrey derived similar formula independently in [Corle96], but they used only one new function, namely the "unwinding number" $\mathcal{K}(z)$ defined by

$$\operatorname{Log} e^z = z + i2\pi\mathcal{K}(z). \tag{8}$$

We have

$$\operatorname{Log} e^{x+iy} = \ln|e^x e^{iy}| + i \operatorname{Arg}(e^x e^{iy}) = x + i \operatorname{Arg} e^{iy},$$

so

$$\mathcal{K}(z) = (\operatorname{Arg} e^{iy} - y)/2\pi.$$

It follows that

$$\mathcal{K}(z) = -n \quad \text{if } (2n-1)\pi < \operatorname{Im} z \leq (2n+1)\pi.$$

Formula (8) gives the correct version of formula (2). It is now also easy to derive the correct version of formulas (3) – (6). The following theorem was stated in [Corle96].

Theorem 1 *We have*

$$\operatorname{Log} e^z = z + i2\pi\mathcal{K}(z), \tag{9}$$
$$\operatorname{Log} zw = \operatorname{Log} z + \operatorname{Log} w + i2\pi\mathcal{K}(\operatorname{Log} z + \operatorname{Log} w), \tag{10}$$
$$\operatorname{Log} z^w = w \operatorname{Log} z + i2\pi\mathcal{K}(w \operatorname{Log} z), \tag{11}$$
$$(zw)^a = z^a w^a \exp[ai2\pi\mathcal{K}(\operatorname{Log} z + \operatorname{Log} w)], \tag{12}$$
$$(z^a)^b = z^{ab} \exp[bi2\pi\mathcal{K}(a \operatorname{Log} z)]. \tag{13}$$

Proof: Formula (9) is now just a definition. To prove (10) we write

$$\operatorname{Log} zw = \operatorname{Log}(e^{\operatorname{Log} z + \operatorname{Log} w}) = \operatorname{Log} z + \operatorname{Log} w + i2\pi\mathcal{K}(\operatorname{Log} z + \operatorname{Log} w).$$

To prove (11) we write

$$\operatorname{Log} z^w = \operatorname{Log} e^{w \operatorname{Log} z} = w \operatorname{Log} z + i2\pi\mathcal{K}(w \operatorname{Log} z).$$

To prove (12) we write

$$(zw)^a = \exp[\operatorname{Log}(zw)^a] = \exp[a \operatorname{Log} zw + i2\pi\mathcal{K}(a \operatorname{Log} zw)] =$$
$$\exp[a(\operatorname{Log} z + \operatorname{Log} w + i2\pi\mathcal{K}(\operatorname{Log} z + \operatorname{Log} w))] =$$
$$z^a w^a e^{ia2\pi\mathcal{K}(\operatorname{Log} z + \operatorname{Log} w)}.$$

To prove (13) we write

$$(z^a)^b = \exp[\operatorname{Log}(z^a)^b] = \exp[b \operatorname{Log} z^a + i2\pi\mathcal{K}(b \operatorname{Log} z^a)] =$$
$$\exp[b(a \operatorname{Log} z + i2\pi\mathcal{K}(a \operatorname{Log} z))] = z^{ab} e^{ib2\pi\mathcal{K}(a \operatorname{Log} z)}. \quad \square$$

Let us next address the question of when exactly the formulas from Question 1 do hold. (This is related to the concept of clearcut region introduced by Corless and Jeffrey in [Corle96].)

Theorem 2 *We have*

$$\operatorname{Log} e^z = z \iff -\pi < \operatorname{Im} z \le \pi, \tag{14}$$

$$\operatorname{Log} zw = \operatorname{Log} z + \operatorname{Log} w \iff -\pi < \operatorname{Arg} z + \operatorname{Arg} w \le \pi, \tag{15}$$

$$\operatorname{Log} z^w = w \operatorname{Log} z \iff -\pi < \ln|z| \operatorname{Im} w + \operatorname{Arg} z \operatorname{Re} w \le \pi. \tag{16}$$

If a is an integer, then $(zw)^a = z^a w^a$ for all z and w, (17)

$$\text{if } a \text{ is not an integer, then} \quad (zw)^a = z^a w^a \iff -\pi < \operatorname{Arg} z + \operatorname{Arg} w \le \pi. \tag{18}$$

$$\text{If } b \text{ is an integer, then } (z^a)^b = z^{ab} \text{ for all } z, \tag{19}$$

$$\text{if } b = p/q \text{ with } p,\ q \in \mathbb{Z} \text{ and } \gcd(p,q) = 1, \text{ then } (z^a)^b = z^{ab} \iff (2kq-1)\pi < \ln|z| \operatorname{Im} a + \operatorname{Arg} z \operatorname{Re} a \le (2kq+1)\pi \text{ for some } k \in \mathbb{Z}, \tag{20}$$

$$\text{if } b \text{ is irrational, then} \quad (z^a)^b = z^{ab} \iff -\pi < \ln|z| \operatorname{Im} a + \operatorname{Arg} z \operatorname{Re} a \le \pi. \tag{21}$$

Proof: Recall that $\mathcal{K}(z) = 0$ if and only if $-\pi < \operatorname{Im} z \le \pi$. This gives us formula (14), and applying the same method to $\mathcal{K}(\operatorname{Log} z + \operatorname{Log} w)$ and $\mathcal{K}(w \operatorname{Log} z)$ we get (15) and (16). For (17) and (18) we need to determine when $a\mathcal{K}(\operatorname{Log} z + \operatorname{Log} w) \in \mathbb{Z}$. If a is an integer, then $a\mathcal{K}(\operatorname{Log} z + \operatorname{Log} w)$ will also be an integer. Since $-2\pi < \operatorname{Arg} z + \operatorname{Arg} w \le 2\pi$ for all z and w, we have $\mathcal{K}(\operatorname{Log} z + \operatorname{Log} w) \in \{0, \pm 1\}$. It follows that the unwinding number cannot cancel out the denominator if a is rational. So if a is not an integer, then $a\mathcal{K}(\operatorname{Log} z + \operatorname{Log} w)$ can only be an integer if the unwinding number is 0, i.e., if $-\pi < \operatorname{Arg} z + \operatorname{Arg} w \le \pi$.

For (19)–(21) we need to determine when $b\mathcal{K}(a \operatorname{Log} z) \in \mathbb{Z}$. The argument is straightforward. □

I would like to finish this section with some general comments. First of all, I think it would be very interesting to include the formulas from Question 1 in an exam in a complex analysis course. I believe that a lot of students would be confused. I believe that it is important for students to be aware that these formulas do not always hold.

I have shown the formulas from Theorem 1 to several people in complex analysis, and they are usually not very interested. I think the reason is simply that this an example of how computers are changing mathematics. In traditional complex analysis the emphasis was never on computations, but thanks to computers, new problems become interesting.

3 Computer tests

Many people have attempted to test the capabilities of different computer algebra systems. The most well known is probably the tests developed by Michael Wester ([Weste94]). While such tests definitely serve a purpose, I am sometimes troubled by their use in comparing different systems. It is not clear to me that the way a program performs on such problems truly reflects its capabilities. There is nothing canonical about the choice of problems, and a different choice of problems could give different results.

The purpose of this short article is rather to look at the theoretical issues behind some of the problems that computer algebra systems (and humans) face when they try to do computational complex analysis. My goal is simply to make the readers aware of some of the problems and their solutions, and to encourage the readers to sit down and experiment with their favorite programs. I hope the following eight tests (adapted from [Stout91]) will serve as a starting point for interesting explorations.

Computer algebra systems are in general much better at reducing the difference between two equivalent expressions to 0, rather than simplifying an expression to a specific form. Notice that some programs simplify expressions automatically, while others only do so when you use an explicit `simplify` command. Sometimes you can control the behavior by using a special option to the `simplify` command, or a different command such as `PowerExpand`. In some programs you can explicitly restrict the domain of a variable, use statements like `on expandlogs` or program your own transformation rules to change the behavior.

For each of these tests, I will first state the correct formula then state when the "wrong" formula is right, and then state what I consider to be reasonable behavior from the computer.

Test 1 *We have* $\sqrt{zw} = \sqrt{z}\sqrt{w}e^{i\pi\mathcal{K}(\operatorname{Log} z+\operatorname{Log} w)}$*, so* $\sqrt{zw} - \sqrt{z}\sqrt{w} = 0$ *if and only if* $-\pi < \operatorname{Arg} z + \operatorname{Arg} w \leq \pi$*. (Recall that* $\mathcal{K}(\operatorname{Log} z + \operatorname{Log} w) \in \{0, \pm 1\}$*.)*

(a) $\sqrt{zw} - \sqrt{z}\sqrt{w}$ *should not simplify when* z *and* w *are complex.*

(b) $\sqrt{zw} - \sqrt{z}\sqrt{w}$ *should simplify to* 0 *when* z *and* w *are both positive.*

Test 2 *We have* $\sqrt{z^2} = ze^{i\pi\mathcal{K}(2\operatorname{Log} z)}$*, so* $\sqrt{z^2} - z = 0$ *if and only if* $-\pi/2 < \operatorname{Arg} z \leq \pi/2$.

(a) $\sqrt{z^2}$ *should not simplify, or simplify to* $\operatorname{csgn}(z)z$ *when* z *is complex.*

(b) $\sqrt{z^2}$ *should not simplify, or simplify to* $\operatorname{sgn}(z)z = |z|$ *when* z *is real.*

(c) $\sqrt{z^2}$ *should simplify to* z *when* z *is positive.*

Test 3 *We have* $\sqrt{1/z} = 1/\sqrt{z}e^{i\pi\mathcal{K}(-\operatorname{Log} z)}$, *so* $\sqrt{1/z} - 1/\sqrt{z} = 0$ *if and only if* z *is not on the negative real axis.*

(a) $\sqrt{1/z} - 1/\sqrt{z}$ *should not simplify when* z *is complex.*

(b) $\sqrt{1/z} - 1/\sqrt{z}$ *should simplify to* 0 *when* z *is not real* ($\operatorname{Im} z \neq 0$).

(c) $\sqrt{1/z} - 1/\sqrt{z}$ *should not simplify, or simplify to* $(\operatorname{sgn}(z) - 1)/\sqrt{z}$ *when* z *is real.*

(d) $\sqrt{1/z} - 1/\sqrt{z}$ *should simplify to* 0 *when* z *is positive.*

Test 4 *We have* $\sqrt{e^z} = e^{z/2}e^{i\pi\mathcal{K}(z)}$, *so* $\sqrt{e^z} - e^{z/2} = 0$ *if and only if* $(4k - 1)\pi < \operatorname{Im} z \leq (4k + 1)\pi$ *for some* $k \in \mathbb{Z}$.

(a) $\sqrt{e^z} - e^{z/2}$ *should not simplify when* z *is complex.*

(b) $\sqrt{e^z} - e^{z/2}$ *should simplify to* 0 *when* z *is real.*

Test 5 *We have* $\operatorname{Log} zw = \operatorname{Log} z + \operatorname{Log} w + i2\pi\mathcal{K}(\operatorname{Log} z + \operatorname{Log} w)$, *so* $\operatorname{Log} zw - \operatorname{Log} z - \operatorname{Log} w = 0$ *if and only if* $-\pi < \operatorname{Arg} z + \operatorname{Arg} w \leq \pi$.

(a) $\operatorname{Log} zw - \operatorname{Log} z - \operatorname{Log} w$ *should not simplify when* z *and* w *are complex.*

(b) $\operatorname{Log} zw - \operatorname{Log} z - \operatorname{Log} w$ *should simplify to* 0 *when* z *and* w *are both positive.*

Test 6 *We have* $\operatorname{Log} z^2 = 2\operatorname{Log} z + i2\pi\mathcal{K}(2\operatorname{Log} z)$, *so* $\operatorname{Log} z^2 - 2\operatorname{Log} z = 0$ *if and only if* $-\pi/2 < \operatorname{Arg} z \leq \pi/2$.

(a) $\operatorname{Log} z^2 - 2\operatorname{Log} z$ *should not simplify when* z *is complex.*

(b) $\operatorname{Log} z^2 - 2\operatorname{Log} z$ *should simplify to* 0 *when* z *is positive.*

Test 7 *We have* $\operatorname{Log}(1/z) = -\operatorname{Log} z + i2\pi\mathcal{K}(-\operatorname{Log} z)$, *so* $\operatorname{Log}(1/z) + \operatorname{Log} z = 0$ *if and only if* z *is not on the negative real axis.*

(a) $\operatorname{Log}(1/z) + \operatorname{Log} z$ *should not simplify when* z *is complex.*

(b) $\operatorname{Log}(1/z) + \operatorname{Log} z$ *should simplify to* 0 *when* z *is positive.*

Test 8 *We have* $\operatorname{Log} e^z = z + i2\pi\mathcal{K}(z)$, *so* $\operatorname{Log} e^z - z = 0$ *if and only if* $-\pi < \operatorname{Im} z \leq \pi$.

(a) $\operatorname{Log} e^z - z$ *should not simplify when* z *is complex.*

(b) $\operatorname{Log} e^z - z$ *should simplify to* 0 *when* z *is real.*

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