

# Accumulation

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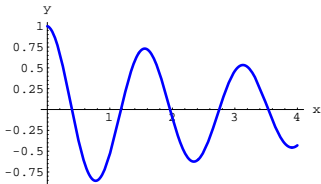
## 2.05 2D Integrals and the Gauss-Green Formula Basics

### B.1) 2D integrals $\iint_R f[x, y] dx dy$ for volume measurements

Here's a plot of a function  $y = f[x]$ :

```
Clear[f, x];
f[x_] = E^-0.2*x Cos[4 x];
{a, b} = {0, 4};

fplot = Plot[f[x], {x, a, b}, PlotStyle -> {{Blue, Thickness[0.01]}},
  AxesLabel -> {"x", "y"}];
```



To get the points on the plot, you go to a point

$$\{x_0, 0\}$$

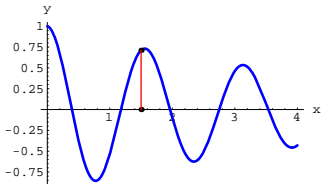
on the x-axis underneath or above the plot and then you run a line to the point

$$\{x_0, f[x_0]\}$$

on the curve like this:

```
x0 = 1.5;

Show[fplot,
  Graphics[{PointSize[0.02], Point[{x0, 0}]}],
  Graphics[{PointSize[0.02], Point[{x0, f[x0]}]}],
  Graphics[{Red, Line[{x0, 0}, {x0, f[x0]}]}]]];
```

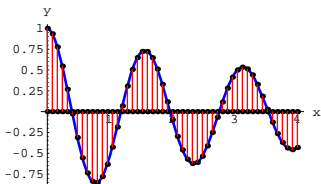


Play with this by resetting  $x_0$  and rerunning.

You can get a pretty good idea of what the curve looks like by just looking at the tips of the little line segments:

```
xjump = (b - a) / 50;

Show[fplot,
  Table[{Graphics[{PointSize[0.02], Point[{x, 0}]}],
    Graphics[{PointSize[0.02], Point[{x, f[x]}]}],
    Graphics[{Red, Line[{x, 0}, {x, f[x]}]}]}],
  {x, a, b, xjump}]]];
```



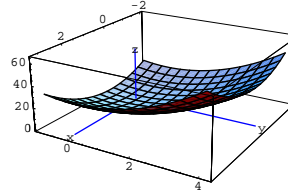
You can plot a surface  $z = f[x, y]$  like this:

```
Clear[f, x, y];
f[x_, y_] = 3.1 x^2 + 2.3 y^2;
{{a, b}, {c, d}} = {{-2, 3}, {-1, 4}};

surfaceplot = Plot3D[f[x, y], {x, a, b}, {y, c, d},
  DisplayFunction -> Identity];
```

```
spacer = 0.2;
threedims =
  Graphics3D[{{
    {Blue, Line[{{a, 0, 0}, {b, 0, 0}}]},
    Text["x", {b + spacer, 0, 0}],
    {Blue, Line[{{0, c, 0}, {0, d, 0}}]},
    Text["y", {0, d + spacer, 0}],
    {Blue, Line[{{0, 0, 0}, {0, 0, 50}}]},
    Text["z", {0, 0, 50 + spacer}]}];

CMView = {2.7, 1.6, 1.2};
fplot = Show[surfaceplot, threedims,
  ViewPoint -> CMView, PlotRange -> All,
  DisplayFunction -> $DisplayFunction];
```



#### □B.1.a)

Explain the meaning of the plotted points that make up the surface.

□ Answer:

You do it just as you did it above.

To get the points on the plot, you go to a point

$$\{x_0, y_0, 0\}$$

on the xy-plane underneath or above the surface. Then you run a line to the 3D point

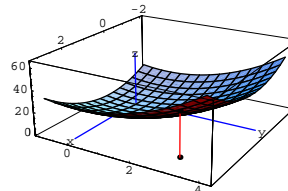
$$\{x_0, y_0, f[x_0, y_0]\}$$

like this:

```
{x0, y0} = {2.5, 3};

Show[fplot,
```

```
Graphics3D[{PointSize[0.02], Point[{x0, y0, 0}]}],
Graphics3D[{PointSize[0.02], Point[{x0, y0, f[x0, y0]}]}],
Graphics3D[{Red, Line[{x0, y0, 0}, {x0, y0, f[x0, y0]}]}]]];
```

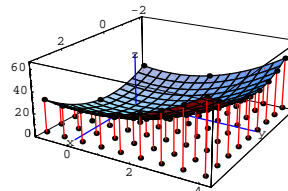


Play with this by resetting  $x_0$  and  $y_0$  and rerunning.

You can get a pretty good idea of what the surface looks like by just looking at the tips of the little line segments:

```
xjump = (b - a) / 8;
yjump = (d - c) / 8;

Show[fplot,
  Table[{Graphics3D[{PointSize[0.02], Point[{x, y, 0}]}],
    Graphics3D[{PointSize[0.02], Point[{x, y, f[x, y]}]}],
    Graphics3D[{Red, Line[{x, y, 0}, {x, y, f[x, y]}]}]}],
  {x, a, b, xjump}, {y, c, d, yjump}]]];
```



Think of the line segments as poles propping up the surface.

If the base of the pole is at  $\{x, y\}$ , then the height of the pole is  $f[x, y]$ .

#### □B.1.b.i) How to use a double integral to measure volume

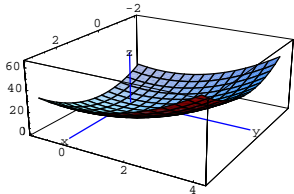
Take another look at the plot from part a) immediately above:

```
Clear[f, x, y]
f[x_, y_] = 3.1 x^2 + 2.3 y^2;
{{a, b}, {c, d}} = {{-2, 3}, {-1, 4}};

surfaceplot =
  Plot3D[f[x, y], {x, a, b}, {y, c, d},
  DisplayFunction -> Identity];

spacer = 0.2;
threedims =
  Graphics3D[{{
    {Blue, Line[{{a, 0, 0}, {b, 0, 0}]},
    Text["x", {b + spacer, 0, 0}],
    {Blue, Line[{{0, c, 0}, {0, d, 0}]},
    Text["y", {0, d + spacer, 0}],
    {Blue, Line[{{0, 0, 0}, {0, 0, 50}]},
    Text["z", {0, 0, 50 + spacer}]}],
  CMView = {2.7, 1.6, 1.2};

fplot =
  Show[surfaceplot, threedims,
  ViewPoint -> CMView, PlotRange -> All,
  DisplayFunction -> $DisplayFunction];
```



How do you use the double integral

$$\iint_R f(x, y) \, dx \, dy$$

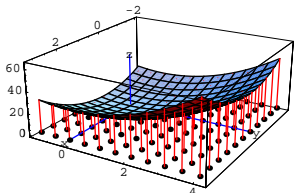
to measure the volume of the solid consisting of everything between the surface plotted above and the xy-plane?

□ Answer:

Look again at the poles like those plotted above:

```
xjump = (b - a) / 10;
yjump = (d - c) / 10;
```

```
Show[fplot,
  Table[{Graphics3D[PointSize[0.02], Point[{x, y, 0}]],
  Graphics3D[Red, Line[{{x, y, 0}, {x, y, f[x, y]}]}],
  {x, a, b, xjump}, {y, c, d, yjump}]]];
```



The solid region you're talking about consists of all the poles you get when you plot all the possible poles (not just those plotted above).

When you plotted this surface on the top, you ran

x from a to b

and you ran

y from c to d.

The volume of the solid is given by the double integral

$$\iint_R f(x, y) \, dx \, dy$$

where R is the rectangle in the xy-plane consisting of all points

{x, y, 0} with  $a \leq x \leq b$  and  $c \leq y \leq d$ .

You can calculate the volume measurement

$$\iint_R f(x, y) \, dx \, dy$$

by writing it as

$$\int_c^d \int_a^b f(x, y) \, dx \, dy$$

which you get from *Mathematica* with:

```
∫ab ∫cd f[x, y] dy dx
430.
```

The volume of the solid measures out to 430 cubic units.

□ B.1.b.ii) Why double integrals measure volume

What is the physical process that confirms that the double integral

$$\iint_R f(x, y) \, dx \, dy = \int_c^d \int_a^b f(x, y) \, dx \, dy$$

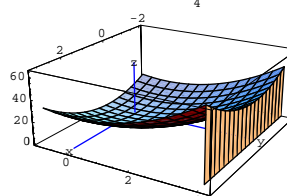
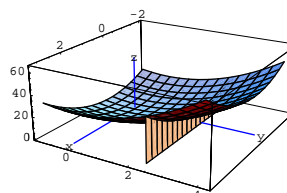
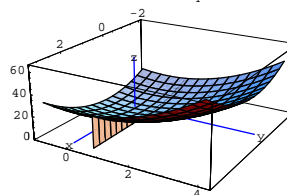
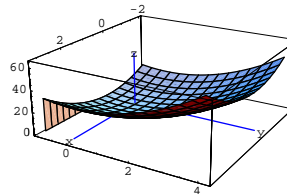
actually measures the volume of the solid described in part i)?

□ Answer:

Look at some slices of the solid consisting of everything between the surface plotted above and the xy-plane:

```
Clear[slice, x, y, t];
slice[y_] := ParametricPlot3D[{x, y, 0} + t {0, 0, f[x, y]}, {x, a, b},
  {t, 0, 1}, PlotPoints -> {Automatic, 2}, DisplayFunction -> Identity];
jump = (d - c) / 3;
```

```
Table[Show[fplot, slice[y], DisplayFunction -> $DisplayFunction],
  {y, c, d, jump}];
```



Animate these.

The physical process of calculating

$$\int_c^d \int_a^b f(x, y) \, dx \, dy$$

involves two steps.

First step:

For each y with  $c \leq y \leq d$ , you calculate

$$\int_a^b f(x, y) \, dx:$$

```
Clear[first, x, y];
first[y_] = ∫ab f[x, y] dx
36.1667 + 11.5 y^2
```

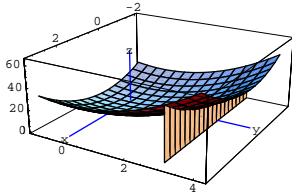
As y advances from c to d, first[y] measures the area of the slices that sweep out the solid.

For instance,

```
first[3]
139.667
```

measures the area of this slice of the solid:

```
Show[fplot, slice[3], DisplayFunction -> $DisplayFunction];
```



Second step:

Calculate  $\int_c^d \text{first}[y] dy = \int_c^d \int_a^b f[x, y] dx dy$ .

This second integral sums up the area measurements of the individual slices and measures the volume.

```
second = Integrate[first[y], {y, c, d}];
```

Compare to *Mathematica's* calculation of

```
Integrate[Integrate[f[x, y], {x, a, b}], {y, c, d}];
```

Bingo.

## B.2) What $\int_R \int f[x, y] dx dy$ means when $f[x, y]$ isn't always positive

### □B.2.a)

Here's a calculation of

$\int_0^4 \int_0^2 f[x, y] dx dy$   
for  $f[x, y] = (x - 3)E^{0.43y}$ :

```
Clear[f, x, y];
f[x_, y_] = (x - 3) E^{0.43 y};
Integrate[Integrate[f[x, y], {x, 0, 2}], {y, 0, 4}];
```

Negative.

When the dweeby Calculus Cal looked at this, he said, "This has got to be wrong because double integrals measure volume and volume can't be negative."

What do you say to Cal in response?

□Answer:

First, you tell Cal to stop picking his nose.

But then you realize that Cal's question does have scientific merit and you answer it by telling Cal to look at the integral

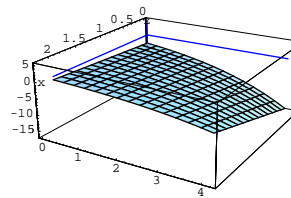
$$\int_0^4 \int_0^2 f[x, y] dx dy$$

and the corresponding plot of  $f[x, y]$ :

```
{a, b} = {0, 2};
{c, d} = {0, 4};
fplot = Plot3D[f[x, y], {x, a, b}, {y, c, d},
  DisplayFunction -> Identity];

spacer = 0.2;
threedims =
Graphics3D[
  {Blue, Line[{{a, 0, 0}, {b, 0, 0}}],
  Text["x", {b + spacer, 0, 0}],
  {Blue, Line[{{0, c, 0}, {0, d, 0}}],
  Text["y", {0, d + spacer, 0}],
  {Blue, Line[{{0, 0, -10}, {0, 0, 5}}],
  Text["z", {0, 0, 5 + spacer}]}];
CMView = {2.7, 1.6, 1.2};

Show[fplot, threedims, ViewPoint -> CMView,
  DisplayFunction -> $DisplayFunction];
```



You say that the plot reveals that  $f[x, y]$  is negative for the  $x$ 's and  $y$ 's involved in the calculation of

$$\int_0^4 \int_0^2 f[x, y] dx dy.$$

This is why  $\int_0^4 \int_0^2 f[x, y] dx dy$  is negative.

You go on to tell Cal that if he wants a measurement of the volume of the solid whose BOTTOM skin is the plotted part of the surface and whose TOP skin is the rectangle in the  $xy$ -plane consisting of the points  $\{x, y, 0\}$  with  $0 \leq x \leq 2$  and  $0 \leq y \leq 4$ , then he can get the measurement by calculating

$$-\int_0^4 \int_0^2 f[x, y] dx dy;$$

Positive.

Now Cal is happy and so are you.

### □B.2.b.i)

Here's a calculation of

$\int_0^4 \int_0^5 f[x, y] dx dy$   
for  $f[x, y] = (x - 3)E^{0.43y}$ :

```
Clear[f, x, y];
f[x_, y_] = (x - 3) E^{0.43 y};
Integrate[Integrate[f[x, y], {x, 0, 5}], {y, 0, 4}];
```

-26.6542

Try to interpret this calculation.

□Answer:

Look at the integral

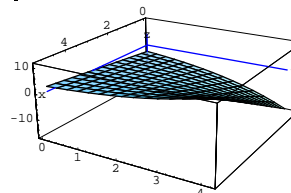
$$\int_0^4 \int_0^5 f[x, y] dx dy$$

and the corresponding plot of  $f[x, y]$ :

```
{a, b} = {0, 5};
{c, d} = {0, 4};
fplot = Plot3D[f[x, y], {x, a, b}, {y, c, d},
  DisplayFunction -> Identity];

spacer = 0.2;
threedims =
Graphics3D[
  {Blue, Line[{{a, 0, 0}, {b, 0, 0}}],
  Text["x", {b + spacer, 0, 0}],
  {Blue, Line[{{0, c, 0}, {0, d, 0}}],
  Text["y", {0, d + spacer, 0}],
  {Blue, Line[{{0, 0, -10}, {0, 0, 5}}],
  Text["z", {0, 0, 5 + spacer}]}];
CMView = {2.7, 1.6, 1.2};

Show[fplot, threedims, ViewPoint -> CMView,
  DisplayFunction -> $DisplayFunction];
```



Look at the plot to see that  $f[x, y]$  is sometimes positive and sometimes negative as  $x$  varies between 0 and 5 and  $y$  varies between 0 and 4.

The upshot:

$$\int_0^4 \int_0^5 f[x, y] dx dy$$

does not measure any specific volume.

### □B.2.b.ii)

Now you know that when you go with

$$f[x, y] = (x - 3)E^{0.43y},$$

then you can calculate

$$\int_0^4 \int_0^5 f[x, y] dx dy.$$

```
Clear[f, x, y];
f[x_, y_] = (x - 3) E^{0.43 y};
Integrate[f[x, y], {x, 0, 4}, {y, 0, 5}]
-26.6542
```

But the integral doesn't measure any specific volume because  $f[x, y]$  takes on both positive and negative values as  $x$  varies between 0 and 5 and  $y$  varies between 0 and 4.

Just how can you interpret this integral?

□Answer:

Look at:

$$f[x, y] = E^{0.43y}(-3 + x)$$

You can see that

$$f[x, y] \geq 0 \text{ for } x \geq 3$$

and

$$f[x, y] \leq 0 \text{ for } x \leq 3.$$

To get the interpretation of  $\int_0^4 \int_0^5 f[x, y] dx dy$ , write

$$\int_0^4 \int_0^5 f[x, y] dx dy = \int_0^4 \int_0^3 f[x, y] dx dy + \int_0^4 \int_3^5 f[x, y] dx dy.$$

→ In the first integral,  $f[x, y] \geq 0$ ; so

$$\int_0^4 \int_0^3 f[x, y] dx dy$$

measures the volume of the solid whose BOTTOM skin is the rectangle in the  $xy$ -plane consisting of all points  $\{x, y, 0\}$  with  $0 \leq x \leq 3$  and  $0 \leq y \leq 4$  and whose TOP skin is the surface  $z = f[x, y]$  plotted with  $0 \leq x \leq 3$  and  $0 \leq y \leq 4$ .

→ In the second integral,  $f[x, y] \leq 0$ ; so

$$\int_0^4 \int_3^5 f[x, y] dx dy$$

measures the NEGATIVE of the volume of the solid whose TOP skin is the rectangle in the  $xy$ -plane consisting of all points  $\{x, y, 0\}$  with  $3 \leq x \leq 5$  and  $0 \leq y \leq 4$  and whose BOTTOM skin is the surface  $z = f[x, y]$  plotted with  $3 \leq x \leq 5$  and  $0 \leq y \leq 4$ .

Consequently,

$$\int_0^4 \int_0^5 f[x, y] dx dy = \int_0^4 \int_0^3 f[x, y] dx dy + \int_0^4 \int_3^5 f[x, y] dx dy$$

measures the difference of the two volumes of the two solids calculated above.

Just a little bit esoteric.

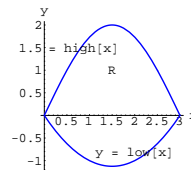
## B.3) Trying to calculate $\int_R f[x, y] dx dy$ when R isn't a rectangle

### □B.3.a) Low to high

Here is a little region R on the  $xy$ -plane.

```
Clear[high, low, x];
high[x_] = 2 Sin[Pi x / 3];
low[x_] = 1/2 x (x - 3);
regionplot = Plot[{high[x], low[x]}, {x, 0, 3},
```

```
PlotStyle -> {{Blue, Thickness[0.01]}, AspectRatio -> Automatic,
AxesLabel -> {"x", "y"}, Epilog -> {Text["y = high[x]", {0.7, 1.5}],
Text["y = low[x]", {2.0, -0.8}], Text["R", {1.5, 1}]}];
```



The region R consists of all points between and on the plotted curves.

Calculate

$$\iint_R f[x, y] dx dy$$

for

$$f[x, y] = E^{-0.4x - 0.3y}.$$

□Answer:

Enter the function  $f[x, y]$ :

```
Clear[f, x, y];
f[x_, y_] = E^{-0.4 x - 0.3 y};
```

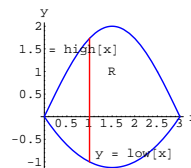
To set up

$$\iint_R f[x, y] dx dy$$

for calculation, you need a convenient way of slicing R.

Here's one:

```
Clear[slice, sliceplot, x];
slice[x_] :=
Graphics[{Red, Thickness[0.01], Line[{x, low[x]}, {x, high[x]}]}];
sliceplot[x_] :=
Show[regionplot, slice[x], DisplayFunction -> Identity];
Show[sliceplot[1], DisplayFunction -> $DisplayFunction];
```



You slice from low to high, and this means that you integrate with respect to  $y$  first. This tells you how to set up

$$\iint_R f[x, y] dx dy = \int_0^3 \left( \int_{\text{low}[x]}^{\text{high}[x]} f[x, y] dy \right) dx$$

for calculation:

```
Clear[first];
first[x_] = Integrate[f[x, y], {y, low[x], high[x]}];
3.33333 E^{(0.05-0.15 x) x} - 3.33333 E^{-0.4 x - 0.6 Sin[Pi x / 3]}
```

You can't use `NIntegrate` on the first integral because the formula for the first integral includes an `X`. You can use `NIntegrate` on the second integral.

```
second = NIntegrate[first[x], {x, 0, 3}]
3.20682
```

Here's how to get *Mathematica* to calculate the double integral

$$\iint_R f[x, y] dx dy = \int_0^3 \int_{\text{low}[x]}^{\text{high}[x]} f[x, y] dy dx$$

in one sweet instruction with `NIntegrate`:

```
NIntegrate[f[x, y], {x, 0, 3}, {y, low[x], high[x]}]
3.20682
```

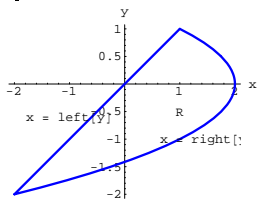
Done.

### □B.3.b) Left to right

Here is another little region on the  $xy$ -plane.

```
Clear[left, right, y];
left[y_] = y;
right[y_] = 2 - y^2;
```

```
regionplot = ParametricPlot[{{left[y], y}, {right[y], y}}, {y, -2, 1},
PlotStyle -> {{Blue, Thickness[0.01]}}, AspectRatio -> Automatic,
AxesLabel -> {"x", "y"}, Epilog -> {Text["x = left[y]", {-1, -0.6}],
Text["x = right[y]", {1.5, -1}], Text["R", {1, -0.5}]}];
```



The region R consists of all points between and on the plotted curves. Calculate

$$\iint_R f[x, y] dx dy$$

for

$$f[x, y] = x y.$$

□ Answer:

Enter the function f[x, y]:

```
Clear[f, x, y];
f[x_, y_] := x y
```

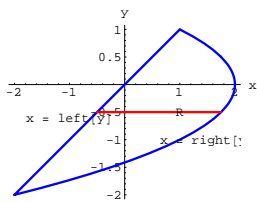
To set up

$$\iint_R f[x, y] dx dy$$

for calculation, you need a convenient way of slicing R.

Here's one:

```
Clear[slice, sliceplot, y];
slice[y_] :=
Graphics[{{Red, Thickness[0.01], Line[{{left[y], y}, {right[y], y}}]}}];
sliceplot[y_] :=
Show[regionplot, slice[y], DisplayFunction -> Identity];
Show[sliceplot[-0.5], DisplayFunction -> $DisplayFunction];
```



You slice from left to right, and this means that you integrate with respect to x first. This tells you how to set up

$$\iint_R f[x, y] dx dy = \int_{-2}^1 \int_{\text{left}[y]}^{\text{right}[y]} f[x, y] dx dy$$

for calculation:

```
Clear[first];
first[y_] = \int_{\text{left}[y]}^{\text{right}[y]} f[x, y] dx
2 y - \frac{5 y^3}{2} + \frac{y^5}{2}
```

You can't use `NIntegrate` on the first integral because the formula for the first integral includes an `X`. You can use `NIntegrate` on the second integral.

```
second = \int_{-2}^1 first[x] dx
\frac{9}{8}
```

Here's how to get *Mathematica* to calculate the double integral

$$\iint_R f[x, y] dx dy = \int_{-2}^1 \int_{\text{left}[y]}^{\text{right}[y]} f[x, y] dx dy$$

in one sweet instruction:

```
\int_{-2}^1 \int_{\text{left}[y]}^{\text{right}[y]} f[x, y] dx dy
\frac{9}{8}
```

Done.

## B.4) The Gauss-Green formula helps you calculate

$$\iint_R f[x, y] dx dy$$

One part of the art of mathematics is knowing how to avoid miserable calculations. The Gauss-Green formula is a tool for avoiding miserable calculations.

The Gauss-Green formula says this:

When you have a parameterization  $\{x[t], y[t]\}$  of the boundary of a region R in the xy-plane with the extra feature that as t advances from a to b, the points  $\{x[t], y[t]\}$  advance around the boundary of R exactly once in the counterclockwise way, then you are guaranteed that

$$\begin{aligned} \iint_R \partial_x n[x, y] - \partial_y m[x, y] dx dy \\ = \int_a^b m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t] dt. \end{aligned}$$

It's a funny-looking formula, but don't let it intimidate you. Its weirdness gives it the utility of a crescent wrench because you can adjust it to many situations.

Get to feel comfortable with this formula because you can use it to side-step a lot of miserable calculations.

### □ B.4.a) Gauss-Green for calculating double integrals

Use the Gauss-Green formula to help calculate the double integral

$$\iint_R (x^2 + y^2) E^{-y} dx dy$$

where R is the region inside the ellipse

$$\left(\frac{x}{2}\right)^2 + y^2 = 1.$$

□ Answer:

Here's how you use the Gauss-Green formula

$$\begin{aligned} \iint_R \partial_x n[x, y] - \partial_y m[x, y] dx dy \\ = \int_a^b m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t] dt \end{aligned}$$

to calculate

$$\iint_R (x^2 + y^2) E^{-y} dx dy.$$

You just say

$$f[x, y] = (x^2 + y^2) E^{-y}$$

$$m[x, y] = 0 \text{ and}$$

$$n[x, y] = \int_0^x f[s, y] ds:$$

```
Clear[x, y, n, m, t, s];
f[x_, y_] = (x^2 + y^2) E^-y;
m[x_, y_] = 0;
n[x_, y_] = \int_0^x f[s, y] ds
\frac{1}{3} E^{-y} x^3 + E^{-y} x y^2
```

This gives you

$$f[x, y] = D[n[x, y], x] - D[m[x, y], y]:$$

```
{D[n[x, y], x] - D[m[x, y], y], f[x, y]}
{E^{-y} x^2 + E^{-y} y^2, E^{-y} (x^2 + y^2)}
```

Now you know that

$$\begin{aligned} \iint_R (x^2 + y^2) E^{-y} dx dy &= \iint_R f[x, y] dx dy \\ &= \iint_R (D[n[x, y], x] - D[m[x, y], y]) dx dy \\ &= \int_a^b m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t] dt \end{aligned}$$

where  $\{x[t], y[t]\}$  is a parameterization of the ellipse

$$\left(\frac{x}{2}\right)^2 + y^2 = 1$$

chosen so that the points  $\{x[t], y[t]\}$  advance around the ellipse exactly once in the counterclockwise fashion as t advances from a to b.

One parameterization of

$$\left(\frac{x}{2}\right)^2 + y^2 = 1$$

that does this nicely is

$$\{x[t], y[t]\} = \{2 \text{Cos}[t], \text{Sin}[t]\}$$

with  $a = 0$  and  $b = 2\pi$ .

The upshot:

$$\iint_{\mathbb{R}} (x^2 + y^2) E^{-y} dx dy$$

is given by:

```

a = 0;
b = 2 Pi;
Clear[x, y, t];
{x[t_], y[t_]} = {2 Cos[t], Sin[t]};
NIntegrate[
m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t], {t, a, b}]
8.80785
    
```

That's all there is to it.

You're out of here without worrying about slicing or anything like that.

**□B.4.b)**

To use the Gauss-Green formula to calculate

$$\iint_{\mathbb{R}} f[x, y] dx dy,$$

you need a parameterization  $\{x[t], y[t]\}$  of the boundary  $\mathbb{R}$  with the extra feature that as  $t$  advances from  $a$  to  $b$ , the points  $\{x[t], y[t]\}$  advance around the boundary of  $\mathbb{R}$  exactly once in the counterclockwise way. Then you are guaranteed that when you set

$$n[x, y] = \int_0^x f[s, y] ds,$$

you are guaranteed that

$$\iint_{\mathbb{R}} f[x, y] dx dy = \int_a^b n[x[t], y[t]] y'[t] dt.$$

The beauty of this is that it allows you to calculate a 2 D integral by means of a single integral that you can usually calculate with NIntegrate.

Discuss why this works.

**□Answer:**

It's all a matter of the fundamental formula of calculus.

The fundamental formula of calculus tells you that when you start with a function  $f[x]$  and you put

$$n[x] = \int_0^x f[s] ds,$$

you are guaranteed that

$$D[n[x], x] = f[x].$$

Try it out.

```

Clear[f, n, s, x];
f[x_] = x Sin[Random[Real, {-5, 5}] x];
n[x_] = Integrate[f[s], {s, 0, x}]
0.128951 (-2.78476 x Cos[2.78476 x] + Sin[2.78476 x])
    
```

Compare  $D[n[x], x]$  and  $f[x]$ :

```

{D[n[x], x], f[x]}
{1. x Sin[2.78476 x], x Sin[2.78476 x]}
    
```

The same. Play with this by changing  $f[x]$  and rerunning.

The fundamental formula also says that when you start with a function  $f[x, y]$  and you put

$$n[x, y] = \int_0^x f[s, y] ds$$

you are guaranteed that

$$\partial_y n[x, y] = f[x, y].$$

Try it out.

```

Clear[f, n, s, x, y];
f[x_, y_] = x^2 Sin[y];
n[x_, y_] = Integrate[f[s, y], {s, 0, x}]
1/3 x^3 Sin[y]
    
```

Compare  $D[n[x, y], x]$  and  $f[x, y]$ :

```

{D[n[x, y], x], f[x, y]}
{x^2 Sin[y], x^2 Sin[y]}
    
```

The same. Play with this by changing  $f[x, y]$  and rerunning.

When you feel comfortable with this, then you are ready to use the

Gauss-Green formula comfortably.

Here is the Gauss-Green formula:

When you have a parameterization  $\{x[t], y[t]\}$  of the boundary of a region  $\mathbb{R}$  in the  $xy$ -plane with the extra feature that as  $t$  advances from  $a$  to  $b$ , the points  $\{x[t], y[t]\}$  advance around the boundary of  $\mathbb{R}$  exactly once in the counterclockwise way, then you are guaranteed that

$$\begin{aligned} \iint_{\mathbb{R}} \partial_x n[x, y] - \partial_y m[x, y] dx dy \\ = \int_a^b m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t] dt. \end{aligned}$$

To use this formula to calculate

$$\iint_{\mathbb{R}} f[x, y] dx dy$$

for a given function  $f[x, y]$ , you just set

$$m[x, y] = 0 \text{ for all } x\text{'s and } y\text{'s}$$

and you set

$$n[x, y] = \int_0^x f[s, y] ds.$$

This guarantees that

$$\partial_x n[x, y] = f[x, y] \text{ and } \partial_y m[x, y] = 0.$$

So

$$\begin{aligned} \iint_{\mathbb{R}} f[x, y] dx dy \\ = \iint_{\mathbb{R}} D[n[x, y], x] dx dy \\ = \iint_{\mathbb{R}} (D[n[x, y], x] - 0) dx dy \\ = \iint_{\mathbb{R}} (D[n[x, y], x] - D[m[x, y], y]) dx dy \\ = \int_a^b m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t] dt \\ = \int_a^b (0 + n[x[t], y[t]] y'[t]) dt \end{aligned}$$

$$= \int_a^b n[x[t], y[t]] y'[t] dt.$$

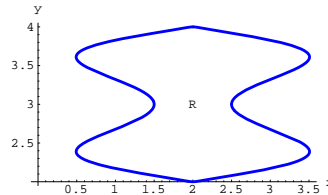
Explanation complete.

**□B.4.c)**

The region  $\mathbb{R}$  in the  $xy$ -plane consists of everything inside and on the curve you see below:

```

Clear[x, y, t];
{x[t_], y[t_]} = {2, 3} + {Cos[t] (2 - 1.5 Cos[2 t]^2), Sin[t]};
{a, b} = {0, 2 Pi};
ParametricPlot[{x[t], y[t]}, {t, a, b},
PlotStyle -> {{Thickness[0.01], Blue}}, AspectRatio -> Automatic,
AxesLabel -> {"x", "y"}, Epilog -> Text["R", {2, 3}]];
    
```



Use the Gauss-Green formula to calculate

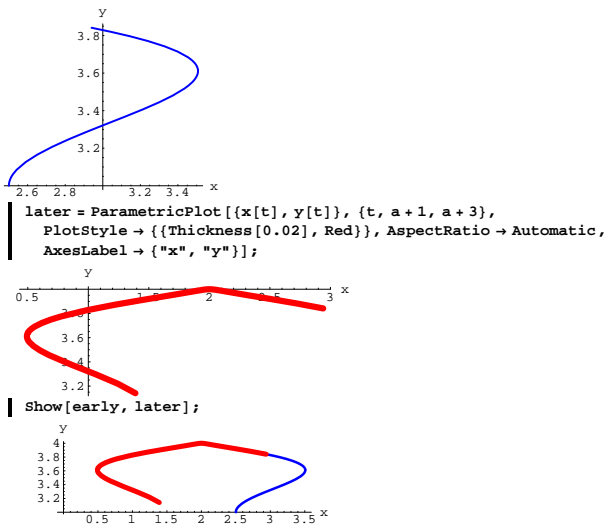
$$\iint_{\mathbb{R}} x^2 y dx dy.$$

**□Answer:**

First check to see whether the parameterization of the boundary of  $\mathbb{R}$  is counterclockwise:

```

early = ParametricPlot[{x[t], y[t]}, {t, a, a + 1},
PlotStyle -> {{Thickness[0.01], Blue}}, AspectRatio -> Automatic,
AxesLabel -> {"x", "y"}];
    
```



Yep; counterclockwise.

Now enter  $f[x, y] = x^2 y$ :

```

Clear[f];
f[x_, y_] = x^2 y
x^2 y

```

Set  $n[x, y] = \int_0^x f[s, y] ds$ :

```

Clear[n, s];
n[x_, y_] = Integrate[f[s, y], {s, 0, x}]
x^3 y / 3

```

Calculate

$$\iint_R f[x, y] dx dy = \iint_R D[n[x, y], x] dx dy = \int_a^b n[x[t], y[t]] y'[t] dt:$$

```

NIntegrate[n[x[t], y[t]] y'[t], {t, a, b}]
52.706

```

No sweat.

#### □B.4.d)

What else is the Gauss-Green formula good for?

□Answer:

If you're going on to vector calculus, you'll see that the Gauss-Green formula is at the base of a lot of the measurement procedures you'll learn.

Even if you're not going on to vector calculus, you'll see it used for a couple of snazzy things in this lesson.

Stay tuned.

### B.5) An indication of some of the ideas behind the Gauss-Green formula

#### □B.5.a)

What are some of the ideas behind the Gauss-Green formula?

□Answer:

The Gauss-Green formula says this:

When you have a parameterization  $\{x[t], y[t]\}$  of the boundary of a region  $R$  in the  $xy$ -plane with the extra feature that as  $t$  advances from  $a$  to  $b$ , the points  $\{x[t], y[t]\}$  advance around the boundary of a  $R$  exactly once in the counterclockwise way, then you are guaranteed that

$$\begin{aligned} & \iint_R \partial_x n[x, y] - \partial_y m[x, y] dx dy \\ &= \int_a^b m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t] dt. \end{aligned}$$

Actually the Gauss-Green formula comes from a pair of formulas:

$$\rightarrow \iint_R \partial_x n[x, y] dx dy = \int_a^b n[x[t], y[t]] y'[t] dt,$$

and

$$\rightarrow -\iint_R \partial_y m[x, y] dx dy = \int_a^b m[x[t], y[t]] x'[t] dt.$$

You might say both of them are manifestations of the fundamental formula.

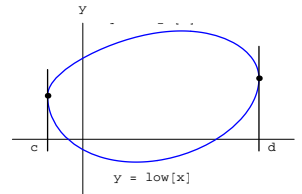
Here's the idea behind the formula

$$-\iint_R \partial_y m[x, y] dx dy = \int_a^b m[x[t], y[t]] x'[t] dt,$$

and it's going to be a hard pill to swallow:

Suppose  $R$  is the region between the curves  $y = \text{high}[x]$  and  $y = \text{low}[x]$  as follows:

Run the blank cell immediately below.



Take any function  $m[x, y]$  and calculate

$$\iint_R \partial_y m[x, y] dx dy$$

as follows:

$$\begin{aligned} & \iint_R \partial_y m[x, y] dx dy \\ &= \iint_R \partial_y m[x, y] dy dx \end{aligned}$$

$$\begin{aligned} &= \int_c^d \int_{\text{low}[x]}^{\text{high}[x]} \partial_y m[x, y] dy dx \\ &= \int_c^d m[x, \text{high}[x]] - m[x, \text{low}[x]] dx. \end{aligned}$$

Put this aside for a little while.

A way to parameterize the boundary curve of  $R$  in accordance with the restrictions laid down in the Gauss-Green formula is to put

$$\{x[t], y[t]\} = \{t, \text{low}[t]\} \text{ for } c \leq t \leq d.$$

This starts on the far left of the low curve and brings you over to the far right along the low curve IN THE COUNTERCLOCKWISE fashion.

Continue on the high curve from far right to far left by putting

$$\{x[t], y[t]\} = \{2d - t, \text{high}[2d - t]\}$$

for  $d \leq t \leq (2d - c)$ .

Put  $a = c$  and  $b = 2d - c$  and note that as  $t$  advances from  $a$  to  $b$ ,

$\{x[t], y[t]\}$  goes around the boundary curve exactly once and it goes in the counterclockwise fashion.

Now

$$\begin{aligned} & \int_a^b m[x[t], y[t]] x'[t] dt = \int_c^{2d-c} m[x[t], y[t]] x'[t] dt \\ &= \int_c^d m[x[t], y[t]] x'[t] dt + \int_d^{2d-c} m[x[t], y[t]] x'[t] dt \\ &= \int_c^d m[t, \text{low}[t]] dt + \int_d^{2d-c} m[2d - t, \text{high}[2d - t]] (-1) dt. \end{aligned}$$

The substitution  $u = 2d - t$  transforms

$$\int_d^{2d-c} m[2d - t, \text{high}[2d - t]] (-1) dt$$

into

$$\int_d^c m[u, \text{high}[u]] du = -\int_c^d m[u, \text{high}[u]] du.$$

So

$$\begin{aligned} & \int_a^b m[x[t], y[t]] x'[t] dt \\ &= \int_c^d m[t, \text{low}[t]] dt - \int_c^d m[u, \text{high}[u]] du \end{aligned}$$



$$= \int_c^d m[x, \text{low}[x]] dx - \int_c^d m[x, \text{high}[x]] dx.$$

But from above,

$$\begin{aligned} & \iint_R D[m[x, y], y] dx dy \\ &= \int_c^d m[x, \text{high}[x]] - m[x, \text{low}[x]] dx. \end{aligned}$$

Shazam!

The inescapable conclusion is:

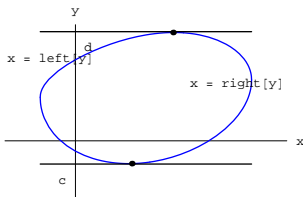
$$\begin{aligned} & \iint_R (-D[m[x, y], y]) dx dy \\ &= \int_a^b m[x[t], y[t]] x'[t] dt. \end{aligned}$$

To see the idea behind the companion formula

$$\begin{aligned} & \iint_R (D[n[x, y], x]) dx dy \\ &= \int_a^b n[x[t], y[t]] y'[t] dt, \end{aligned}$$

do the same thing as above but modify it at the start by going with left[y] and right[y] as in the plot below; then integrate first with respect to x and second with respect to y:

Run the blank cell immediately below.



At this point, you have the idea behind the formulas

$$\begin{aligned} & \iint_R (-\partial_y m[x, y]) dx dy \\ &= \int_a^b m[x[t], y[t]] x'[t] dt \end{aligned}$$

and

$$\begin{aligned} & \iint_R \partial_x n[x, y] dx dy \\ &= \int_a^b n[x[t], y[t]] y'[t] dt. \end{aligned}$$

Add both formulas together to get the full-blown Gauss-Green formula

$$\begin{aligned} & \iint_R \partial_x n[x, y] - \partial_y m[x, y] dx dy \\ &= \int_a^b m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t] dt. \end{aligned}$$

'Nuff said.

## 2.05 2D Integrals and the Gauss-Green Formula Tutorials

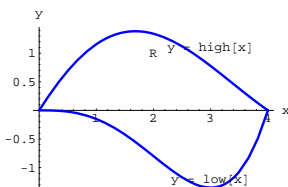
### T.1) Using a 2D integral to measure area

#### □T.1.a) Area measurements by double integrals

Here's a little region R in the xy-plane:

```
Clear[high, low, x];
high[x_] = 2 E^-0.2 x Sin[π x / 4];
low[x_] = 0.05 x^3 (x - 4);

regionplot = Plot[{high[x], low[x]}, {x, 0, 4},
  PlotStyle -> {{Blue, Thickness[0.01]}}, AspectRatio -> Automatic,
  AxesLabel -> {"x", "y"}, Epilog -> {Text["y = high[x]", {3, 1.1}],
  Text["y = low[x]", {3, -1.2}], Text["R", {2, 1}]}];
```



Use a single integral to measure the area of the region R between the two curves, and then use a 2 D integral to make the same measurement.

□Answer:

Without 2 D integrals, you just calculate

$$\int_0^4 \text{high}[x] - \text{low}[x] dx:$$

```
NIntegrate[high[x] - low[x], {x, 0, 4}]
6.02594
```

With 2 D integrals you set f[x, y] = 1 for all x's and y's and calculate

$$\iint_R f[x, y] dx dy = \int_0^4 \int_{\text{low}[x]}^{\text{high}[x]} f[x, y] dy dx:$$

```
Clear[y];
NIntegrate[1, {x, 0, 4}, {y, low[x], high[x]}]
6.02594
```

The reason this works is that when you go with

$$f[x, y] = 1$$

for all x's and y's, then  $\iint_R f[x, y] dx dy$  measures the volume of the flat solid one unit high whose vertical sides have the same shape as R:

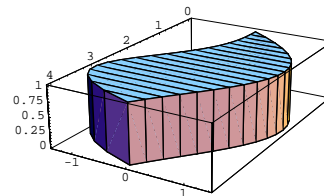
```
Clear[r];
top = ParametricPlot3D[{x, low[x] + r (high[x] - low[x]), 1}, {x, 0, 4},
  {r, 0, 1}, PlotPoints -> {Automatic, 2}, DisplayFunction -> Identity];

bottom =
ParametricPlot3D[{x, low[x] + r (high[x] - low[x]), 0}, {x, 0, 4},
  {r, 0, 1}, PlotPoints -> {Automatic, 2}, DisplayFunction -> Identity];

sidel = ParametricPlot3D[{x, low[x], r}, {x, 0, 4}, {r, 0, 1},
  PlotPoints -> {Automatic, 2}, DisplayFunction -> Identity];

side2 = ParametricPlot3D[{x, high[x], r}, {x, 0, 4}, {r, 0, 1},
  PlotPoints -> {Automatic, 2}, DisplayFunction -> Identity];

CMView = {2.7, 1.6, 1.2};
Show[top, bottom, sidel, side2, ViewPoint -> CMView,
  DisplayFunction -> $DisplayFunction];
```



Because this solid is one unit high, the volume measurement of the solid also gives you the area measurement of its base.

The upshot:

When you go with f[x, y] = 1 for all x's and y's, then

$$\iint_R f[x, y] dx dy$$

measures the area of R.

### T.2) Volume measurements with 2D integrals

#### □T.2.a.i)

Here's a plot of the surface

$$z = f[x, y] = 12 - x + 2 y$$

above the rectangle R in the xy-plane consisting of all points {x, y, 0} with

$$-1 \leq x \leq 3 \text{ and } -1 \leq y \leq 2:$$

```
Clear[f, x, y];
f[x_, y_] = 12 - x + 2 y;
{{a, b}, {c, d}} = {{-1, 3}, {-1, 2}};

surfaceplot = Plot3D[f[x, y], {x, a, b}, {y, c, d},
  DisplayFunction -> Identity];

Rplot = Graphics3D[
  Polygon[{{a, c, 0}, {a, d, 0}, {b, d, 0}, {b, c, 0}}]];

label = Graphics3D[Text["R", {2, 1.5, 0}]];

spacer = 0.2;
```



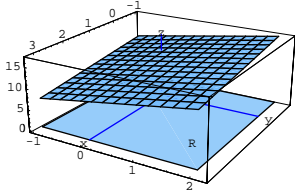
```

threedims =
Graphics3D[
{Blue, Line[{{a, 0, 0}, {b, 0, 0}}],
Text["x", {b + spacer, 0, 0}],
{Blue, Line[{{0, c, 0}, {0, d, 0}}],
Text["y", {0, d + spacer, 0}],
{Blue, Line[{{0, 0, 0}, {0, 0, 16}}],
Text["z", {0, 0, 16 + 3 spacer}]];

CMView = {2.7, 1.6, 1.2};

Show[surfaceplot, Rplot, label, threedims,
ViewPoint -> CMView, PlotRange -> All,
DisplayFunction -> $DisplayFunction];

```



Measure the volume of the solid consisting of all points between or on the rectangle and the surface plotted above.

□ Answer:

That's duck soup. The top surface is just a plot of  $f[x, y]$  for points  $\{x, y, 0\}$  in the rectangle  $R$  in the  $xy$ -plane. The rectangle  $R$  consists of those points  $\{x, y, 0\}$  with  $a \leq x \leq b$  and  $c \leq y \leq d$ . The volume measurement is just

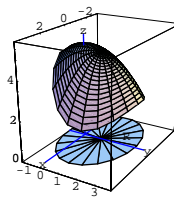
$$\int_c^d \int_a^b f[x, y] dx dy:$$

```

Integrate[f[x, y], {y, c, d}, {x, a, b}]
144

```

That's all there is to it.



Measure the volume of the solid consisting of all points between or on the ellipse and the surface plotted above.

□ Answer:

The top surface is just a plot of  $f[x, y]$  for points  $\{x, y, 0\}$  inside the ellipse  $R$  in the  $xy$ -plane. The volume measurement is just

$$\iint_R f[x, y] dx dy,$$

which you can easily calculate with the Gauss-Green formula once you remember the ellipse

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y-1}{2}\right)^2 = 1$$

(which is the boundary of  $R$ ) is parameterized in the counterclockwise way by:

```

Clear[x, y, t];
{x[t_], y[t_]} = {0, 1} + {3 Cos[t], 2 Sin[t]}
{3 Cos[t], 1 + 2 Sin[t]}

```

with  $t$  running from  $0$  to  $2\pi$ . Now you are ready to turn the winning team of *Mathematica* and the Gauss-Green formula loose:

The Gauss-Green formula says

$$\int \partial_x n[x, y] - \partial_y m[x, y] dx dy = \int_a^b m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t] dt$$

where  $a = 0$  and  $b = 2\pi$ .

To calculate

$$\iint_R f[x, y] dx dy,$$

you just go with

$$m[x, y] = 0 \text{ and}$$

$$n[x, y] = \int_0^x f[s, y] ds:$$

```

Clear[n, m, s];
m[x_, y_] = 0;
n[x_, y_] = Integrate[f[s, y], {s, 0, x}]
5. x - 0.1 x^3 - 0.3 x y^2

```

This gives you  $f[x, y] = \partial_x n[x, y] - \partial_y m[x, y]$ :

```

Expand[{D[n[x, y], x] - D[m[x, y], x], f[x, y]}]
{5. - 0.3 x^2 - 0.3 y^2, 5 - 0.3 x^2 - 0.3 y^2}

```

Now you know that

$$\iint_R f[x, y] dx dy = \int_a^b m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t] dt$$

where  $\{x[t], y[t]\}$  is the parameterization of the the boundary of  $R$  used above.

```

a = 0;
b = 2 Pi;
NIntegrate[m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t], {t, a, b}]
70.2146

```

The volume measures out to about 70.2 cubic units; that's all there is to it.

### T.3) Calculation strategies for minimum effort on your part

□ T.3.a) This situation signals integrating with respect to  $y$  first

Here's a region  $R$  plotted in the  $xy$ -plane.

□ T.2.a.ii)

Here's a plot of the surface

$$z = f[x, y] = 5 - 0.3(x^2 + y^2)$$

above region  $R$  in the  $xy$ -plane consisting of all points  $\{x, y, 0\}$  with  $\{x, y\}$  inside the ellipse

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y-1}{2}\right)^2 = 1:$$

```

Clear[f, x, y, r, t];
{x[r_-, t_], y[r_-, t_]} = {0, 1} + r {3 Cos[t], 2 Sin[t]};
f[x_-, y_] = 5 - 0.3 (x^2 + y^2);

surfaceplot = ParametricPlot3D[{x[r, t], y[r, t], f[x[r, t], y[r, t]]},
{t, 0, 2 Pi}, {r, 0, 1},
DisplayFunction -> Identity];

Rplot = ParametricPlot3D[{x[r, t], y[r, t], 0},
{t, 0, 2 Pi}, {r, 0, 1}, PlotPoints -> {Automatic, 2},
DisplayFunction -> Identity];

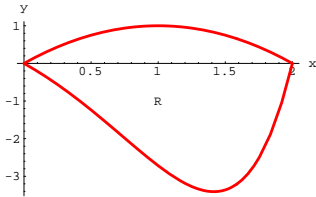
label = Graphics3D[Text["R", {-1, 2, 0}]];

spacer = 0.2;
threedims =
Graphics3D[
{Blue, Line[{{-3, 0, 0}, {3.5, 0, 0}}],
Text["x", {3.5 + spacer, 0, 0}],
{Blue, Line[{{0, -1, 0}, {0, 3.5, 0}}],
Text["y", {0, 3.5 + spacer, 0}],
{Blue, Line[{{0, 0, 0}, {0, 0, 5.5}}],
Text["z", {0, 0, 5.5 + spacer}]];
CMView = {2.7, 1.6, 1.2};

Show[surfaceplot, Rplot, label, threedims,
ViewPoint -> CMView, PlotRange -> All,
DisplayFunction -> $DisplayFunction];

```

```
Clear[x];
Rplot = Plot[{x (x - 2) E^x, x (2 - x)}, {x, 0, 2},
  PlotStyle -> {{Red, Thickness[0.01]}}, AxesLabel -> {"x", "y"},
  Epilog -> Text["R", {1, -1}]]];
```



How does this plot and its coding signal to you the minimum effort way of calculating

$$\iint_R f[x, y] dx dy$$

for a given function  $f[x, y]$ ?

□ Answer:

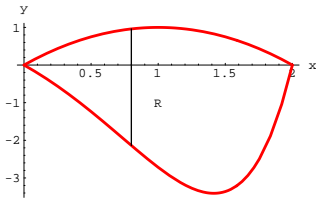
Things go best when you slice perpendicularly to the x-axis. In fact, when you look at the code above you can see that when you slice perpendicularly to the x-axis at a fixed x, then

$$x(x-2)E^x \leq y \leq x(2-x):$$

```
Clear[yhigh, ylow, x, y];
yhigh[x_] = x (2 - x);
ylow[x_] = x (x - 2) E^x;

Clear[xslicer];
xslicer[x_] = Graphics[Line[{{x, ylow[x]}, {x, yhigh[x]}]};

Show[Rplot, xslicer[0.8]]];
```



This tells you to integrate with respect to y first to get

$$\begin{aligned} \iint_R f[x, y] dx dy &= \int_0^2 \int_{y_{low}[x]}^{y_{high}[x]} f[x, y] dy dx \\ &= \int_0^2 \int_{y_{low}[x]}^{y_{high}[x]} f[x, y] dy dx \end{aligned}$$

because x runs from 0 to 2.

Here is the calculation in the case  $f[x, y] = 5x + y$ :

```
Clear[f, x, y];
f[x_, y_] = 5 x + y;

first = Integrate[f[x, y], {y, ylow[x], yhigh[x]}]
5 (2 - x) x^2 + 1/2 (2 - x)^2 x^2 - 5 E^x (-2 + x) x^2 - 1/2 E^2 x (-2 + x)^2 x^2
second = NIntegrate[first, {x, 0, 2}]
25.8908
```

Or you can try to get it with one instruction:

```
N[Integrate[Integrate[f[x, y], {y, ylow[x], yhigh[x]}], {x, 0, 2}]]
25.8908
```

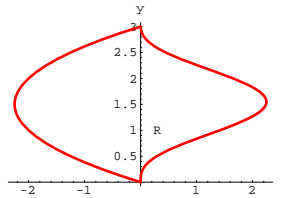
Slicing perpendicularly to the y-axis would have resulted in a calculational nightmare; so you wouldn't want to try to integrate with respect to x first.

Using the Gauss-Green formula would have been somewhat inconvenient because you would have to do quite a bit of thinking to come up with the required counterclockwise parameterization of the boundary.

□ T.3.b) This situation signals integrating with respect to x first

Here's another region R plotted in the xy-plane.

```
Clear[y];
Rplot = ParametricPlot[{{y (y - 3), y}, {y (3 - y) Sin[y]^2, y}}, {y, 0, 3},
  PlotStyle -> {{Red, Thickness[0.01]}}, AxesLabel -> {"x", "y"},
  Epilog -> Text["R", {0.3, 1}]]];
```



How does this plot and its coding signal to you the minimum effort way of calculating  $\iint_R f[x, y] dx dy$  for a given function  $f[x, y]$ ?

□ Answer:

You get a calculational clean answer when you slice perpendicularly to the y-axis. In fact, when you look at the code above you can see that when you slice perpendicularly to the y-axis at a fixed y, then

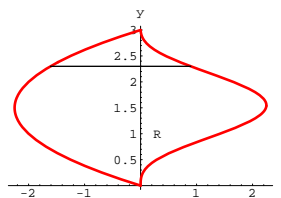
$$y(y-3) \leq x \leq y(3-y) \sin^2[y].$$

Take a look:

```
Clear[xlow, xhigh, y];
xhigh[y_] = y (3 - y) Sin[y]^2;
xlow[y_] = y (y - 3);

Clear[yslicer];
yslicer[y_] = Graphics[Line[{{xlow[y], y}, {xhigh[y], y]}]};

Show[Rplot, yslicer[2.3]]];
```



This tells you to integrate with respect to x first to get

$$\begin{aligned} \iint_R f[x, y] dx dy &= \int_0^3 \int_{x_{low}[y]}^{x_{high}[y]} f[x, y] dx dy \end{aligned}$$

because y runs from 0 to 3.

Here is the calculation in the case  $f[x, y] = \sin[x + y]$ :

```
Clear[f, x, y];
f[x_, y_] = Sin[x + y];

first = Integrate[f[x, y], {x, xlow[y], xhigh[y]}]
Cos[y + (-3 + y) y] - Cos[y + (3 - y) y Sin[y]^2]
second = NIntegrate[first, {y, 0, 3}]
3.02982
```

Or you can try a single instruction:

```
N[Integrate[Integrate[f[x, y], {x, xlow[y], xhigh[y]}], {y, 0, 3}]]
3.02982
```

In this case, running the two integrals takes less time than running the single instruction.

Slicing perpendicularly to the x-axis would have resulted in a calculational nightmare; so you wouldn't want to try to integrate with respect to y first.

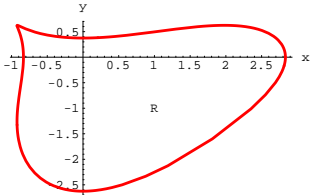
Using the Gauss-Green formula would have been inconvenient because

you would have to do a bit of thinking to come up with the required counterclockwise parameterization of the boundary.

### □T.3.c) This situation signals using the Gauss-Green formula

Here's another region R plotted in the xy-plane.

```
Clear[x, y, t];
x[t_] = Cos[t] (2 + Cos[t] - 1/6 Cos[2 t]);
y[t_] = Sin[t] (1.5 - Sin[t] + 1/8 Sin[3 t]);
Rplot = ParametricPlot[{x[t], y[t]}, {t, 0, 2 π},
  PlotRange → All, PlotStyle → {{Red, Thickness[0.01]}},
  AxesLabel → {"x", "y"}, Epilog → Text["R", {1, -1}];
```

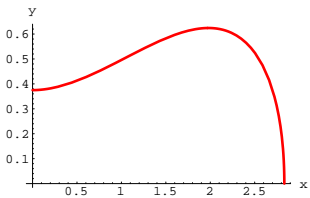


How does this plot and its code signal to you the minimum effort way of calculating  $\iint_R f[x, y] dx dy$  for a given function  $f[x, y]$ ?

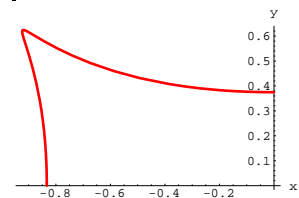
□Answer:

The boundary of R is given parametrically, and that's what Gauss-Green is set-up to use. Before you use it, you have to check that the boundary curve is swept out exactly once in the counterclockwise direction.

```
ParametricPlot[{x[t], y[t]}, {t, 0, π/2},
  PlotRange → All, PlotStyle → {{Red, Thickness[0.01]}},
  AxesLabel → {"x", "y"}];
```



```
ParametricPlot[{x[t], y[t]}, {t, π/2, π},
  PlotRange → All, PlotStyle → {{Red, Thickness[0.01]}},
  AxesLabel → {"x", "y"}];
```



Yep, exactly once and counterclockwise.

Here's how to use the Gauss-Green formula to calculate

$$\iint_R f[x, y] dx dy$$

in the case that  $f[x, y] = x - 2y$ :

The Gauss-Green formula says

$$\iint_R \partial_x n[x, y] - \partial_y m[x, y] dx dy = \int_a^b m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t] dt$$

where  $a = 0$  and  $b = 2\pi$ .

To calculate

$$\iint_R (x - 2y) dx dy,$$

you just say

$$f[x, y] = x - 2y$$

$$m[x, y] = 0 \text{ and}$$

$$n[x, y] = \int_0^x f[s, y] ds:$$

```
Clear[x, y, n, m, s];
f[x_, y_] = x - 2 y;
m[x_, y_] = 0;
n[x_, y_] = Integrate[f[s, y], {s, 0, x}]
x^2/2 - 2 x y
```

This gives you  $f[x, y] = \partial_x n[x, y] - \partial_y m[x, y]$ :

```
{∂x n[x, y] - ∂y m[x, y], f[x, y]}
{x - 2 y, x - 2 y}
```

Now you know that

$$\iint_R f[x, y] dx dy = \int_a^b m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t] dt$$

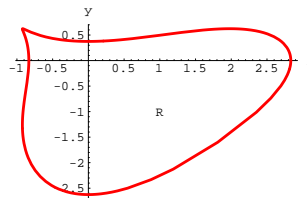
where  $\{x[t], y[t]\}$  is a parameterization of the boundary of R used above.

```
a = 0;
b = 2 π;
Clear[x, y, t];
x[t_] = Cos[t] (2 + Cos[t] - 1/6 Cos[2 t]);
y[t_] = Sin[t] (1.5 - Sin[t] + 1/8 Sin[3 t]);
NIntegrate[m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t], {t, a, b}]
21.5248
```

That's it.

If you don't go with the Gauss-Green formula, you've got to slice perpendicularly to the x-axis or the y-axis. Either way, you're going to have trouble because of the bulges and hollows on the boundary of R.

```
Show[Rplot];
```



### □T.3.d) This situation signals using the Gauss-Green formula

Given that the region R consists of everything inside the ellipse

$$\left(\frac{x-1}{5}\right)^2 + \left(\frac{y+2}{3}\right)^2 = 1$$

in the xy-plane and

$$f[x, y] = y \sin[x],$$

how does the set up signal to you the minimum effort way of calculating

$$\iint_R f[x, y] dx dy?$$

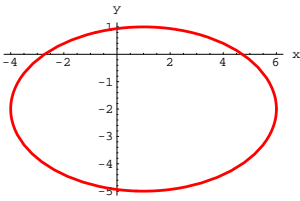
□Answer:

This is a natural for the Gauss-Green formula because you can write down a quick parameterization of the ellipse

$$\left(\frac{x-1}{5}\right)^2 + \left(\frac{y+2}{3}\right)^2 = 1.$$

Just go with the parameterization below and see the ellipse unfold right before your eager eyes:

```
Clear[x, y, t];
{x[t], y[t]} = {1 + 5 Cos[t], -2 + 3 Sin[t]};
Rplot = ParametricPlot[{x[t], y[t]}, {t, 0, 2 π},
  PlotStyle → {{Red, Thickness[0.01]}}, AxesLabel → {"x", "y"}];
```



Here's how to use the Gauss-Green formula to calculate

$$\iint_R f[x, y] dx dy$$

in the case that  $f[x, y] = y \sin[x]$ :

The Gauss-Green formula says

$$\begin{aligned} \iint_R \partial_x n[x, y] - \partial_y m[x, y] dx dy \\ = \int_a^b m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t] dt \end{aligned}$$

where  $a = 0$  and  $b = 2\pi$ .

To calculate

$$\iint_R y \sin[x] dx dy$$

You just say

$$f[x, y] = y \sin[x]$$

$$m[x, y] = 0 \text{ and}$$

$$n[x, y] = \int_0^x f[s, y] ds:$$

```
Clear[x, y, n, m, t];
f[x_, y_] = y Sin[x];
m[x_, y_] = 0;
n[x_, y_] = Integrate[f[t, y], {t, 0, x}]
y - y Cos[x]
```

This gives you  $f[x, y] = \partial_x n[x, y] - \partial_y m[x, y]$ :

```
{D[n[x, y], x] - D[m[x, y], y], f[x, y]}
{y Sin[x], y Sin[x]}
```

Now you know that

$$\begin{aligned} \iint_R f[x, y] dx dy \\ = \int_a^b m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t] dt \end{aligned}$$

where  $\{x[t], y[t]\}$  is the parameterization of the boundary of  $R$  used above.

```
a = 0;
b = 2 Pi;
Clear[x, y, t];
{x[t_], y[t_]} = {1 + 5 Cos[t], -2 + 3 Sin[t]};

NIntegrate[
m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t], {t, a, b}, AccuracyGoal -> 3]
10.3917
```

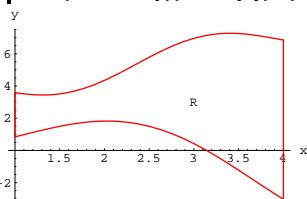
Quick and sweet.

#### □T.3.e) This situation signals integrating with respect to $y$ first

Here's another region  $R$  plotted in the  $xy$ -plane.

```
Clear[high, low, x];
high[x_] = 3 + x + Cos[2 x];
low[x_] = x Sin[x];

Rplot = Plot[{high[x], low[x]}, {x, 1, 4},
PlotStyle -> Red, AxesLabel -> {"x", "y"},
Epilog -> {{Red, Line[{1, low[1]}, {1, high[1]}]},
{Red, Line[{4, low[4]}, {4, high[4]}]}, Text["R", {3, 3}]}];
```



How does this plot and its coding signal to you the minimum effort way of calculating  $\iint_R f[x, y] dx dy$  for a given function  $f[x, y]$ ?

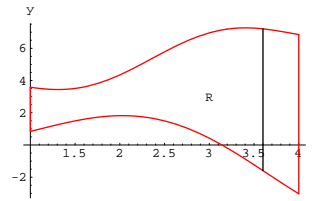
□Answer:

You get a computationally clean answer when you slice perpendicularly to the  $x$ -axis. In fact, when you look at the code above you can see that when you slice perpendicularly to the  $y$ -axis at a fixed  $x$ , then

$$\text{low}[x] \leq x \leq \text{high}[x].$$

Take a look:

```
Clear[xslicer];
xslicer[x_] = Graphics[Line[{{x, low[x]}, {x, high[x]}]]];
Show[Rplot, xslicer[3.6]];
```



This tells you to integrate with respect to  $y$  first to get

$$\begin{aligned} \iint_R f[x, y] dx dy \\ = \int_1^4 \int_{\text{low}[x]}^{\text{high}[x]} f[x, y] dy dx \\ = \int_1^4 \int_{\text{low}[x]}^{\text{high}[x]} f[x, y] dy dx \end{aligned}$$

because  $x$  runs from 1 to 4.

Here is the calculation in the sample case  $f[x, y] = E^{2x-y}$ :

```
Clear[f];
f[x_, y_] = E^{2 x - y};
first = Integrate[f[x, y], {y, low[x], high[x]}]
-E^{-3 + x - Cos[2 x]} + E^{-x (-2 + Sin[x])}
second = NIntegrate[first, {x, 1, 4}]
11140.6
```

Or you can try one instruction:

```
Clear[f, x, y];
f[x_, y_] = E^{2 x - y};
N[Integrate[Integrate[f[x, y], {y, low[x], high[x]}], {x, 1, 4}]]
11140.6
```

Slicing perpendicularly to the  $y$ -axis would have resulted in a calculational nightmare; so you wouldn't want to try to integrate with respect to  $x$  first.

Using the Gauss-Green formula would have been inconvenient because you would have to do quite a bit of work to come up with the required counterclockwise parameterization of the boundary.

#### T.4) Gauss-Green when you have a clockwise parameterization

##### □T.4.a)

What happens when you try to use the Gauss-Green formula but you find that your parameterization moves in the clockwise way rather than the required counterclockwise way?

□Answer:

Go ahead and use the clockwise parameterization to make the calculation, but adjust your answer by multiplying it by  $-1$ .

The reason this works is that

$$\int_a^b g[t] dt = - \int_b^a g[t] dt.$$

## T.5) Gauss's normal law in 2D and using it to plan bombing runs

runs

Think of it!  
The giant scientist Karl Friedrich Gauss  
had his fingers in nearly every part  
of the mathematics in this lesson!

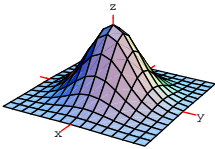
Look at this:

```
Clear[x, bell, y];
bell[x_] =  $\frac{E^{-x^2}}{\sqrt{2\pi}}$ ;
bellshaped = Plot3D[bell[x] bell[y], {x, -3, 3}, {y, -3, 3},
DisplayFunction -> Identity];

spacer = 0.2;
h = 3.5;
threedims =
Graphics3D[{{
{Red, Line[{{-h, 0, 0}, {h, 0, 0}}]},
Text["x", {h + spacer, 0, 0}],
{Red, Line[{{0, -h, 0}, {0, h, 0}}]},
Text["y", {0, h + spacer, 0}],
{Red, Line[{{0, 0, 0}, {0, 0, 0.18}}]},
Text["z", {0, 0, 0.18 + spacer/10}]}];

CMView = {2.7, 1.6, 1.2};

Show[threedims, bellshaped, PlotRange -> All,
ViewPoint -> CMView, Boxed -> False, BoxRatios -> {1, 1, 1/2},
DisplayFunction -> $DisplayFunction];
```



### □T.5.a.i)

What is this?

□Answer:

It's the bell-shaped surface!

You get it by taking the bell-shaped curve function

$$\text{bell}[x] = \left(\frac{1}{\sqrt{2\pi}}\right) E^{-\frac{x^2}{2}}$$

and plotting the surface

$$z = f[x, y] = \text{bell}[x] \text{bell}[y].$$

### □T.5.a.ii)

What's the version of Gauss's normal law corresponding to the bell-shaped surface?

□Answer:

When a numerical measurement is normally distributed, then the fraction of the time that this measurement comes out between given numbers a and b is estimated by

$$\int_a^b \text{normal}[x, \text{mean}, \text{dev}] dx$$

where the parameters mean and dev (= standard deviation) come from the data.

Here

$$\text{normal}[x, \text{mean}, \text{dev}] = \frac{\text{bell}\left[\frac{x-\text{mean}}{\text{dev}}\right]}{\text{dev}}$$

For a review of Gauss's normal law, see the lesson on transforming integrals.

Some measurements come in pairs of numbers {x, y}.

The version of Gauss's normal law corresponding to bell-shaped surfaces is this:

If you know that

→ the number in the x slot is normally distributed with mean = xmean and standard deviation = xdev and it's known that

→ the number in the y slot is normally distributed with mean = ymean and standard deviation = ydev, and

→ the measurements x and y have absolutely no influence on each other,

then you can estimate the fraction of these measurements that come out in a given region R in the xy-plane by calculating the double integral

$$\iint_R f[x, y] dy dx$$

where

$$f[x, y] = \text{normal}[x, \text{xmean}, \text{xdev}] \text{normal}[y, \text{ymean}, \text{ydev}].$$

### □T.5.b.i) Gauss's normal law and the bomb damage assessment

Calculus&Mathematica thanks Arthur Zaminokis, formerly of the Defense Intelligence Agency, for help on this problem.

You've got a target sitting at {0, 0} and you want to knock it out with an iron bomb. The Northrup bombing sights are pretty good. In fact, if you set the sight to hit the target at {0, 0} and examine the point {x, y} at which an individual bomb lands, you will find that the resulting coordinates x and y

→ have no influence on each other;

→ are both normally distributed; and

→ both average out to {0, 0}.

You will also find that the standard deviation of the x measurement is not the same as the standard deviation of the y measurement.

For a specific bomb and airplane and altitude, it turns out that

$$xdev = 100 \text{ feet and } ydev = 300 \text{ ft.}$$

And to destroy the target you are trying to knock out, the bomb must hit within 150 feet of the target.

Estimate the fraction of these bombs dropped on this target that actually destroy the target.

□Answer:

Load up *Mathematica*:

```
{xmean, ymean} = {0, 0}
{0, 0}
{xdev, ydev} = {100, 300}
{100, 300}
Clear[x, bell];
bell[x_] =  $\frac{E^{-x^2}}{\sqrt{2\pi}}$ 
 $\frac{E^{-x^2}}{\sqrt{2\pi}}$ 
Clear[normal, mean, dev];
normal[x_, mean_, dev_] =  $\frac{\text{bell}\left[\frac{x-\text{mean}}{\text{dev}}\right]}{\text{dev}}$ 
 $\frac{E^{-\frac{(x-\text{mean})^2}{\text{dev}^2}}}{\text{dev} \sqrt{2\pi}}$ 
```

In order for a bomb to destroy the target at {0, 0}, the bomb has to land inside the region R inside the circle

$$x^2 + y^2 = 150^2.$$

To estimate the fraction of bombs that do this you calculate

$$\iint_R f[x, y] dy dx$$

where:

```
Clear[f];
f[x_, y_] = normal[x, xmean, xdev] normal[y, ymean, ydev]
 $\frac{E^{-\frac{x^2}{100^2}} E^{-\frac{y^2}{300^2}}}{60000 \pi}$ 
```

It is very easy to parameterize the circle  $x^2 + y^2 = 150^2$  in accordance with the restrictions imposed by the Gauss-Green formula:

```
Clear[t];
{x[t_], y[t_]} = 150 {Cos[t], Sin[t]}
{150 Cos[t], 150 Sin[t]}
```

with  $0 \leq t \leq 2\pi$ .

The Gauss-Green formula says

$$\begin{aligned} & \int \int_R \partial_x n[x, y] - \partial_y m[x, y] dx dy \\ &= \int_a^b m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t] dt \end{aligned}$$

where  $a = 0$  and  $b = 2\pi$ .

To calculate

$$\int \int_R f[x, y] dx dy,$$

you just say

$$m[x, y] = 0 \text{ and}$$

$$n[x, y] = \int_0^x f[s, y] ds:$$

```
Clear[n, m, s];
m[x_, y_] = 0;
n[x_, y_] = Integrate[f[s, y], {s, 0, x}];
E^-100000 Erf[100000/(100000*sqrt(2))]
600*sqrt(2)*pi
```

This gives you

$$f[x, y] = \partial_x n[x, y] - \partial_y m[x, y]:$$

```
{D[n[x, y], x] - D[m[x, y], y], f[x, y]}
{E^-100000/(60000*pi), E^-100000/(60000*pi)}
```

Now you know that

$$\begin{aligned} & \int \int_R f[x, y] dx dy \\ &= \int \int_R (\partial_x n[x, y] - \partial_y m[x, y]) dx dy \end{aligned}$$

$$\text{(because } m[x, y] = 0)$$

$$= \int_a^b m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t] dt$$

where  $\{x[t], y[t]\}$  is a parameterization of the the boundary of  $R$  used above.

```
a = 0;
b = 2*pi;
onehit = NIntegrate[m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t],
  {t, a, b}, AccuracyGoal -> 3]
0.283813
```

A little more than 1 out of 4 bombs dropped on the target will actually destroy the target.

#### □T.5.b.ii)

In part i) above, what are your chances of destroying the target if you bomb it twice?

How many times do you have to bomb the target to get 95 % confidence of destroying it?

□Answer:

The fraction of the time that you expect to hit the target with one bomb is:

```
onehit
0.283813
```

The fraction of the time you expect to miss with one bomb is:

```
onemiss = 1 - onehit
0.716187
```

The fraction of the time you expect to miss with two bombs is:

```
onemiss^2
0.512924
```

The fraction of the time you expect to miss with three bombs is:

```
onemiss^3
0.367349
```

The fraction of the time you expect to destroy the target with  $k$  bombs is:

is:

```
Clear[destroy, k];
destroy[k_] = 1 - onemiss^k
1 - 0.716187^k
```

To find out how many bombs you have to drop to get 95 % confidence of destroying the target, look at the following table:

```
Table[{k "bombs", destroy[k]}, {k, 1, 12}]
{{bombs, 0.283813}, {2 bombs, 0.487076}, {3 bombs, 0.632651},
{4 bombs, 0.736909}, {5 bombs, 0.811578}, {6 bombs, 0.865055},
{7 bombs, 0.903354}, {8 bombs, 0.930783}, {9 bombs, 0.950428},
{10 bombs, 0.964497}, {11 bombs, 0.974573}, {12 bombs, 0.98179}}
```

It will take 9 bombs to be 95 % sure that the target is destroyed.

## 2.05 2D Integrals and the Gauss-Green Formula Give it a try!

Experience with the starred problems is essential if you're going on to study vector calculus.

### G.1) Volume measurements\*

#### □G.1.a)

Here is the surface

$$z = f[x, y] = 2 \sin[x + y] + 2.5$$

plotted over the rectangle  $R$  in the  $xy$ -plane consisting of the points  $\{x, y, 0\}$  with  $-2 \leq x \leq 2$  and  $-3 \leq y \leq 3$ :

```
Clear[x, y, z, r, t, f];
f[x_, y_] = 2 Sin[x + y] + 2.5;

surfaceplot = Plot3D[f[x, y], {x, -2, 2}, {y, -3, 3},
  DisplayFunction -> Identity];

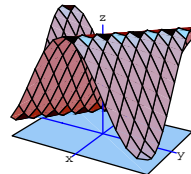
Rplot = Graphics3D[
  Polygon[{{-2, -3, 0}, {2, -3, 0}, {2, 3, 0}, {-2, 3, 0}}]];

spacer = 0.2;

threedims =
  Graphics3D[
    {Blue, Line[{{-2.2, 0, 0}, {2.2, 0, 0}}]},
    Text["x", {2.2 + spacer, 0, 0}],
    {Blue, Line[{{0, -3.2, 0}, {0, 3.2, 0}}]},
    Text["y", {0, 3.2 + spacer, 0}],
    {Blue, Line[{{0, 0, 0}, {0, 0, 4.5}}]},
    Text["z", {0, 0, 4.5 + spacer + 0.1}]]];

CMView = {2.7, 1.6, 1.2};

Show[threedims, Rplot, surfaceplot, PlotRange -> All,
  ViewPoint -> CMView, Boxed -> False,
  DisplayFunction -> $DisplayFunction];
```



Use a double integral to measure the volume of the solid consisting of everything between the surface and the rectangle.

#### □G.1.b)

Here is the surface

$$z = f[x, y] = x y + 3$$

plotted above the region  $R$  inside the ellipse

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

in the  $xy$ -plane.

```
Clear[x, y, z, r, t, f];
f[x_, y_] = x y + 3;

surfaceplot = ParametricPlot3D[
  {2 r Cos[t], 3 r Sin[t],
  f[2 r Cos[t], 3 r Sin[t]]},
  {r, 0, 1}, {t, 0, 2 Pi},
  DisplayFunction -> Identity];

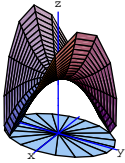
Rplot = ParametricPlot3D[
  {2 r Cos[t], 3 r Sin[t], 0},
  {r, 0, 1}, {t, 0, 2 Pi},
  PlotPoints -> {2, Automatic},
  DisplayFunction -> Identity];
```

```

threeDims =
Graphics3D[
{Blue, Line[{{-2.2, 0, 0}, {2.2, 0, 0}}]},
Text["x", {2.2 + spacer, 0, 0}],
{Blue, Line[{{0, -3.2, 0}, {0, 3.2, 0}}]},
Text["y", {0, 3.2 + spacer, 0}],
{Blue, Line[{{0, 0, 0}, {0, 0, 6.2}}]},
Text["z", {0, 0, 6.2 + spacer + 0.2}]}];

CMView = {2.7, 1.6, 1.2};

Show[threeDims, surfaceplot, Rplot, PlotRange -> All,
ViewPoint -> CMView, Boxed -> False,
DisplayFunction -> $DisplayFunction];
    
```



Think of the solid that consists of everything below the surface and above the ellipse R, and use the Gauss-Green formula to help you measure the volume of this solid.

□G.1.c.i)

Use the idea of volume calculation to explain the statement:  
 If a function  $f[x, y]$  takes only positive values on a region R in the xy-plane, then  $\iint_R f[x, y] dx dy > 0$ .

□G.1.c.ii)

Explain the statement:  
 If for a given function  $f[x, y]$  and a given region R in the xy-plane, it turns out that  $\iint_R f[x, y] dx dy < 0$ , then you are guaranteed that there are some points  $\{x, y\}$  in R with  $f[x, y] < 0$ .

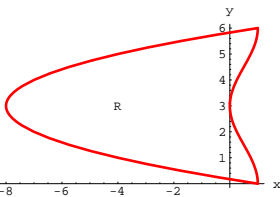
G.2) Calculating double integrals\*

□G.2.a)

Here's a region R plotted in the xy-plane.

```

Clear[y];
Rplot = ParametricPlot[{{(y (y - 6) + 1, y), {Cos[π y / 6]^2, y}}, {y, 0, 6},
PlotStyle -> {{Red, Thickness[0.01]}}, AxesLabel -> {"x", "y"},
Epilog -> Text["R", {-4, 3}]];
    
```



Go with

$$f[x, y] = y - x + 3$$

and calculate

$$\iint_R f[x, y] dx dy$$

by the method of least labor on your part.

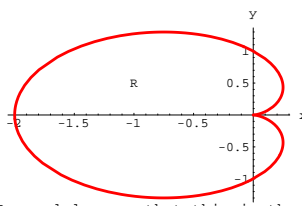
□G.2.b)

Here's another region R plotted in the xy-plane.

```

Clear[x, y, t];
x[t_] = Cos[t] (1 - Cos[t]);
y[t_] = Sin[t] (1 - Cos[t]);

Rplot = ParametricPlot[{x[t], y[t]}, {t, 0, 2π},
PlotRange -> All, PlotStyle -> {{Red, Thickness[0.01]}},
AxesLabel -> {"x", "y"}, Epilog -> Text["R", {-1, 0.5}]];
    
```



Fancy dudes say that this is the cardioid described in polar coordinates by the polar equation  $r[t] = 1 - \text{Cos}[t]$ .

Go with

$$f[x, y] = 3 + y - x$$

and calculate

$$\iint_R f[x, y] dx dy$$

by the method of least labor on your part.

Next, go with

$$g[x, y] = 3 - x$$

and calculate

$$\iint_R g[x, y] dx dy$$

by the method of least labor on your part.

Compare the values of

$$\iint_R f[x, y] dx dy \text{ and } \iint_R g[x, y] dx dy.$$

Discuss how you think that the shape and position of R affect the relationship between these two numbers.

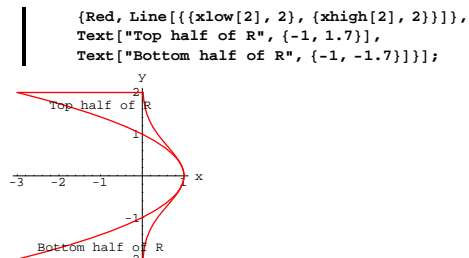
□G.2.c)

Here is another region R plotted in the xy-plane.

```

Clear[high, low, y];
xhigh[y_] = E^-y^2;
xlow[y_] = 1 - y^2;

Rplot = ParametricPlot[{{xhigh[y], y}, {xlow[y], y}}, {y, -2, 2},
PlotStyle -> Red,
AxesLabel -> {"x", "y"}, PlotRange -> All, AspectRatio -> Automatic,
Epilog -> {{Red, Line[{{xlow[-2], -2}, {xhigh[-2], -2}}]},
{Red, Line[{{xlow[2], 2}, {xhigh[2], 2}}]},
Text["Top half of R", {-1, 1.7}],
Text["Bottom half of R", {-1, -1.7}]}];
    
```



Go with

$$f[x, y] = y E^{-x}$$

$$g[x, y] = y^2 E^{-x} \text{ and}$$

$$h[x, y] = y^3 E^{-x}$$

and calculate

$$\iint_R f[x, y] dx dy,$$

$$\iint_R g[x, y] dx dy, \text{ and}$$

$$\iint_R h[x, y] dx dy$$

by the method of least labor on your part.

Look at the formulas for  $f[x, y]$  and  $g[x, y]$  and then look at the shape and position of R and explain why it was no surprise that

$$\iint_R f[x, y] dx dy \text{ and } \iint_R h[x, y] dx dy$$

came out the way they did.

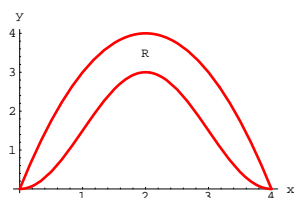
□G.2.d)

Here is another region R plotted in the xy-plane.

```

Clear[x, y, Rplot];
Rplot = Plot[{{3 Sin[π x / 4]^2, x (4 - x)}, {x, 0, 4},
PlotStyle -> {{Red, Thickness[0.01]}}, AxesLabel -> {"x", "y"},
Epilog -> Text["R", {2, 3.5}]];
    
```





Go with

$$f[x, y] = 4x - y$$

and calculate

$$\int \int_R f[x, y] dx dy$$

by the method of least labor on your part.

### □G.2.e)

Given that the region R consists of everything inside the ellipse

$$\left(\frac{x+3}{7}\right)^2 + \left(\frac{y-1}{2}\right)^2 = 1,$$

go with

$$f[x, y] = x^2 - 3y^2$$

and calculate

$$\int \int_R f[x, y] dx dy$$

by the method of least labor on your part.

## G.3) Area and volume measurements via the Gauss-Green formula\*

This problem has lots of overlap with some of the work in the lesson on transformations of integrals.

The Gauss-Green formula says

$$\int \int_R \partial_x n[x, y] - \partial_y m[x, y] dx dy$$

$$= \int_a^b m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t] dt$$

provided that  $\{x[t], y[t]\}$  sweeps out the boundary of the region R

exactly one time in the counterclockwise fashion as  $t$  advances from  $a$  to  $b$ .

If you take  $m[x, y] = y$  and  $n[x, y] = 0$  and plug into the Gauss-Green

formula, then you get

$$- \int \int_R 1 dx dy$$

$$= \int \int_R (0 - 1) dx dy$$

$$= \int_a^b (y[t] x'[t] + 0 y'[t]) dt$$

$$= \int_a^b y[t] x'[t] dt.$$

Consequently,

$$- \int_a^b y[t] x'[t] dt$$

measures the area of R provided that  $\{x[t], y[t]\}$  sweeps out the boundary of R exactly one time in the counterclockwise fashion as  $t$  advances from  $a$  to  $b$ .

### □G.3.a.i)

Adapt the discussion above to explain why

$$\int_a^b x[t] y'[t] dt$$

also measures the area of R provided that  $\{x[t], y[t]\}$  sweeps out the boundary of R exactly one time in the counterclockwise fashion as  $t$  advances from  $a$  to  $b$ .

### □G.3.a.ii)

Explain why

$$\left(\frac{1}{2}\right) \int_a^b (x[t] y'[t] - y[t] x'[t]) dt$$

measures the area of R provided that  $\{x[t], y[t]\}$  sweeps out the boundary of R exactly one time in the counterclockwise fashion as  $t$  advances from  $a$  to  $b$ .

□Tip:

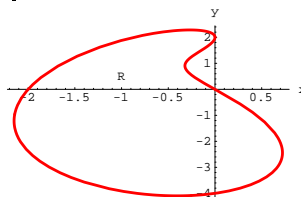
What happens when you average two correct measurements?

### □G.3.a.iii)

Here's the plot of a whimsical region R in the xy-plane.

```
Clear[x, y, t];
x[t_] = Cos[t] (1 - Sin[t] - Cos[t]);
y[t_] = Sin[t] (3 - Cos[t] - Sin[t]);
```

```
Rplot = ParametricPlot[{x[t], y[t]}, {t, 0, 2 Pi},
  PlotRange -> All, PlotStyle -> {{Red, Thickness[0.01]}},
  AxesLabel -> {"x", "y"}, Epilog -> Text["R", {-1, 1/2}]]];
```



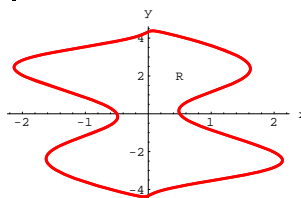
Choose any of the three formulas explained above to measure the area of this region.

### □G.3.a.iv)

Here's the plot of another region R in the xy-plane.

```
Clear[x, y, t];
x[t_] = Cos[2 t] (3/2 - 1/3 Sin[4 t] - Cos[8 t]);
y[t_] = Sin[2 t] (4 - 1/3 Cos[4 t] - 1/5 Sin[8 t]);

Rplot = ParametricPlot[{x[t], y[t]}, {t, 0, 2 Pi},
  PlotRange -> All, PlotStyle -> {{Red, Thickness[0.01]}},
  AxesLabel -> {"x", "y"}, Epilog -> Text["R", {1/2, 2}]]];
```



Using the formula  $\int_a^b x[t] y'[t] dt$  with  $a = 0$  and  $b = 2\pi$  to measure the area inside this region, you get the area measurement:

```
NIntegrate[x[t] y'[t], {t, 0, 2 Pi}, AccuracyGoal -> 2]
40.5265
```

Unfortunately, this measurement is way off.

Find the error and come up with the correct measurement.

### □G.3.b) Using $m[x, y]$ instead of $n[x, y]$ in Gauss-Green calculations

Here's a plot of the surface

$$z = f[x, y] = 5 - 0.2(x^2 + y^2)$$

above the region R in the xy-plane consisting of all points  $\{x, y, 0\}$  with  $\{x, y\}$  inside the ellipse

$$\left(\frac{x+1}{2}\right)^2 + \left(\frac{y+1}{3}\right)^2 = 1:$$

```
Clear[f, x, y, r, t];
{x[r_, t_], y[r_, t_]} = {-1, -1} + r {2 Cos[t], 3 Sin[t]};
f[x_, y_] = 5 - 0.2 (x^2 + y^2);
```

```
surfaceplot = ParametricPlot3D[
  Evaluate[{x[r, t], y[r, t], f[x[r, t], y[r, t]]}],
  {t, 0, 2 Pi}, {r, 0, 1},
  DisplayFunction -> Identity];
```

```
Rplot = ParametricPlot3D[Evaluate[{x[r, t], y[r, t], 0}],
  {t, 0, 2 Pi}, {r, 0, 1}, PlotPoints -> {Automatic, 2},
  DisplayFunction -> Identity];
```

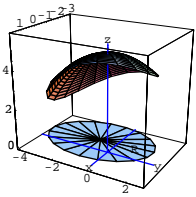
```
label = Graphics3D[Text["R", {-1, 1, 0}]];
```

```
spacer = 0.2;
```

```
threedims =
```

```
Graphics3D[{
  {Blue, Line[{{-3, 0, 0}, {1.5, 0, 0}}]},
  Text["x", {1.5 + spacer, 0, 0}],
  {Blue, Line[{{0, -4, 0}, {0, 2.5, 0}}]},
  Text["y", {0, 2.5 + spacer, 0}],
  {Blue, Line[{{0, 0, 0}, {0, 0, 5.5}}]},
  Text["z", {0, 0, 5.5 + spacer}]}];
CMView = {2.7, 1.6, 1.2};
```

```
Show[surfaceplot, Rplot, label, threedims,
  ViewPoint -> CMView, PlotRange -> All,
  DisplayFunction -> $DisplayFunction];
```



One way of measuring the volume of the solid consisting of all points between or on the ellipse and the surface plotted above is to realize that the top surface is just a plot of  $f[x, y]$  for points  $\{x, y, 0\}$  inside the ellipse  $R$  in the  $xy$ -plane. The volume measurement is just

$$\iint_R f[x, y] \, dx \, dy,$$

which you can easily calculate with the Gauss-Green formula once you remember the ellipse

$$\left(\frac{x+1}{2}\right)^2 + \left(\frac{y+1}{3}\right)^2 = 1$$

(which is the boundary of  $R$ ) is parameterized in the counterclockwise way by

$$\{x[t], y[t]\} = \{-1, -1\} + \{2 \cos[t], 3 \sin[t]\}$$

with  $0 \leq t \leq 2\pi$ .

Now you can turn the winning team of *Mathematica* and the Gauss-Green formula loose:

```
Clear[n, m, x, y, s, t];
m[x_, y_] = 0;
n[x_, y_] = Integrate[f[s, y] ds, {s, 0, x}];

a = 0;
b = 2 Pi;

{x[t_], y[t_]} = {-1, -1} + {2 Cos[t], 3 Sin[t]};

volume = NIntegrate[m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t], {t, a, b}]
74.4557
```

Now look at this alternate calculation:

```
Clear[n, m, x, y, s, t];
m[x_, y_] = Integrate[-f[x, s] ds, {s, 0, y}];
```

```
n[x_, y_] = 0;

a = 0;
b = 2 Pi;

{x[t_], y[t_]} = {-1, -1} + {2 Cos[t], 3 Sin[t]};

volume = NIntegrate[m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t], {t, a, b}]
74.4557
```

Use the Gauss-Green formula

$$\iint_R D[n[x, y], x] - D[m[x, y], y] \, dx \, dy = \int_a^b m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t] \, dt$$

and the fundamental formula of calculus to explain why the alternate calculation was guaranteed to work.

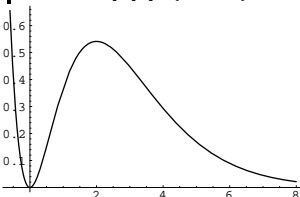
**G.4) Plot3D versus ParametricPlot3D\***

**G.4.a.i)**

Go with  $f[x] = x E^{-x}$  and explain with words and pictures what you think that this plot represents:

```
Clear[f, x];
{a, b} = {-0.6, 8};
f[x_] = x^2 E^-x;

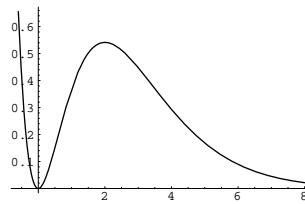
raw = Plot[f[x], {x, a, b}, PlotRange -> All];
```



**G.4.a.ii)**

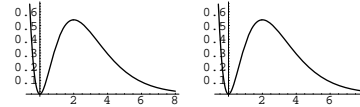
Stay with the same function  $f[x]$  and the same values for  $a$  and  $b$  and look at:

```
parametric = ParametricPlot[{x, f[x]}, {x, a, b}, PlotRange -> All];
```



See the two plots side-by-side:

```
Show[GraphicsArray[{raw, parametric}]];
```



Say what happened, and try to explain why it will happen no matter what function  $f[x]$  you go with.

**G.4.b)**

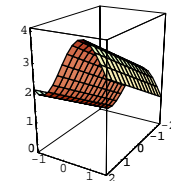
Go with  $f[x, y] = 2 + \sin[y] - x$  and explain with words and pictures what you think this plot represents:

```
Clear[f, x, y];
{{a, b}, {c, d}} = {{-2, 2}, {-Pi/3, Pi/2}};

f[x_, y_] = 2 + Sin[2 y] + x/2;

CMView = {2.7, 1.6, 1.2};

raw = Plot3D[f[x, y], {x, a, b}, {y, c, d}, ViewPoint -> CMView, BoxRatios -> Automatic];
```

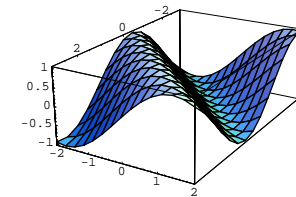


**G.4.c)**

Look at this plot:

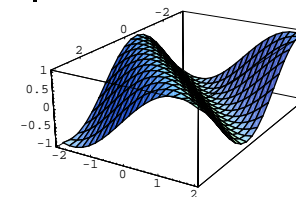
```
Clear[f, x, y];
{a, b} = {3, 2};
f[x_, y_] = Sin[x - y];
CMView = {2.7, 1.6, 1.2};

regular = Plot3D[f[x, y], {x, -a, a}, {y, -b, b}, ViewPoint -> CMView, BoxRatios -> Automatic];
```



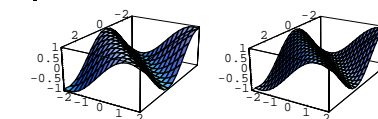
Stay with the same function  $f[x, y]$  and the same  $a$  and  $b$ , and look at this parametric plot:

```
parametric = ParametricPlot3D[{x, y, f[x, y]}, {x, -a, a}, {y, -b, b}, ViewPoint -> CMView, BoxRatios -> Automatic];
```



See them side-by-side:

```
Show[GraphicsArray[{regular, parametric}]];
```



Say what happened, and try to explain why it will happen no matter what function  $f[x, y]$  you go with.

## G.5) Big league plotting and measuring

### □G.5.a.i)

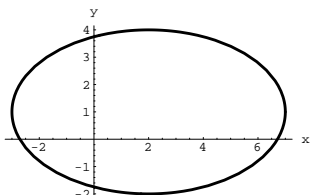
When you want to plot the ellipse

$$\left(\frac{x-2}{5}\right)^2 + \left(\frac{y-1}{3}\right)^2 = 1$$

in two dimensions in true scale with a counterclockwise parameterization, you can go with:

```
Clear[x2D, y2D, t];
{x2D[t_], y2D[t_]} = {2, 1} + {5 Cos[t], 3 Sin[t]};

ParametricPlot[{x2D[t], y2D[t]}, {t, 0, 2 Pi},
PlotStyle -> Thickness[0.01], AspectRatio -> Automatic,
AxesLabel -> {"x", "y"}];
```



When you want to plot the same ellipse

$$\left(\frac{x-2}{5}\right)^2 + \left(\frac{y-1}{3}\right)^2 = 1$$

and everything inside it on the xy-plane in three dimensions with a counterclockwise parameterization, you can go with:

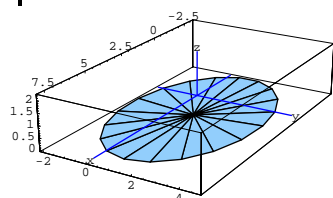
```
Clear[x, y, z, t];
{x[r_ , t_], y[r_ , t_], z[r_ , t_]} =
{2, 1, 0} + r {5 Cos[t], 3 Sin[t], 0};

Rplot = ParametricPlot3D[{x[r, t], y[r, t], z[r, t]},
{r, 0, 1}, {t, 0, 2 Pi},
PlotPoints -> {2, Automatic}, DisplayFunction -> Identity];

spacer = 0.2;
threedims =
```

```
Graphics3D[{
Blue, Line[{{{-3, 0, 0}, {7.5, 0, 0}}]},
Text["x", {7.5 + spacer, 0, 0}],
Blue, Line[{{{0, -2, 0}, {0, 4.5, 0}}]},
Text["y", {0, 4.5 + spacer, 0}],
Blue, Line[{{{0, 0, 0}, {0, 0, 2}}]},
Text["z", {0, 0, 2 + spacer}]}];
CMView = {2.7, 1.6, 1.2};

threeDplot = Show[Rplot, threedims, BoxRatios -> Automatic,
ViewPoint -> CMView, PlotRange -> All,
DisplayFunction -> $DisplayFunction];
```



Now look at this:

```
{ {x2D[t], y2D[t]}, {x[r, t], y[r, t], z[r, t]} }
{{2 + 5 Cos[t], 1 + 3 Sin[t]}, {2 + 5 r Cos[t], 1 + 3 r Sin[t], 0}}
```

Where do the points

$\{x[r, t], y[r, t], z[r, t]\}$

land when you go with  $r = 0$ ?

Where do the points

$\{x[r, t], y[r, t], z[r, t]\}$

land when you go with  $0 < r < 1$ ?

Where do the points

$\{x[r, t], y[r, t], z[r, t]\}$

land when you go with  $r = 1$ ?

### □G.5.a.ii)

Give a plot of the part of the surface

$$z = f[x, y] = 1 + E^{\frac{3-x+y}{5}}$$

consisting of the points  $\{x, y, z\}$  such that  $\{x, y, 0\}$  is in the elliptical region in the xy-plane plotted above. Show the surface plot together with the plot of the elliptical region in the xy-plane.

Measure the volume of the solid whose top skin is the part of the surface you plotted and whose bottom skin is the elliptical region in the xy-plane plotted above.

□Tip:

When you are plotting the surface, don't use the plotting option

`PlotPoints->{2, Automatic}`.

When you make the volume measurement, use the Gauss-Green formula and `NIntegrate`.

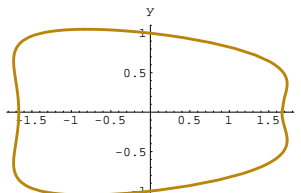
### □G.5.b.i)

Here's a plot of a corny curve in two dimensions:

What do you expect from a calculus course laid out in Illinois and Ohio?

```
Clear[x2D, y2D, t];
{x2D[t_], y2D[t_]} = {2 Cos[t] - 1/3 Cos[3 t], Sin[t] - 1/6 Sin[2 t]};

corn = ParametricPlot[{x2D[t], y2D[t]},
{t, 0, 2 Pi}, PlotStyle -> {{Thickness[0.01], DarkGoldenrod}},
AspectRatio -> Automatic, AxesLabel -> {"x", "y"}];
```



Plot the same curve and everything inside it on the xy-plane in three dimensions with a counterclockwise parameterization.

### □G.5.b.ii)

Give a plot of the part of the surface

$$z = f[x, y] = 1 + 0.2 x^2 - 0.2 y^2$$

consisting of the points  $\{x, y, z\}$  such that  $\{x, y, 0\}$  is in the region in the xy-plane that you plotted above. Show the surface plot together with the plot of the region in the xy-plane. Measure the volume of the solid whose top skin is the part of the surface you plotted and whose bottom skin is the region in the xy-plane you plotted in part i).

□Tip:

When you are plotting the surface, don't use the plotting option

`PlotPoints->{2, Automatic}`.

## G.6) Average value and centroids

Most math folks say that the average value of a function  $f[x]$  on an interval  $[a, b]$  is the number  $A$  satisfying

$$\int_a^b f[x] dx = A (b - a) = A \int_a^b dx.$$

Going with this formula, you see that the average value of  $\text{Cos}[x]$  on  $[1, 3]$  is given by:

$$A = \frac{\text{NIntegrate}[\text{Cos}[x], \{x, 1, 3\}]}{3 - 1}$$

-0.350175

The following routine calculates  $\text{Cos}[x]$  for 50 random points  $x$  chosen out of  $[1, 3]$  by *Mathematica*:

```
Clear[k];
randoms = Table[N[Cos[Random[Real, {1, 3}]]], {k, 1, 50}]
{0.449546, -0.375018, -0.780356, -0.989188,
0.133432, -0.451138, -0.98533, -0.959184, 0.201714, -0.507538,
-0.52544, -0.925744, -0.342787, -0.586337, -0.835196, -0.113155,
-0.351793, -0.00204985, -0.957588, -0.396645, -0.103716,
0.27495, -0.0165386, -0.751879, 0.000636401, -0.693438,
-0.52312, -0.755552, 0.423769, 0.269223, -0.548369, -0.842333,
-0.930109, -0.584523, 0.517234, -0.94829, -0.270614, -0.989674,
0.10272, -0.534241, -0.974335, -0.753994, -0.0477734, 0.403232,
-0.620241, -0.532763, 0.513745, -0.162527, -0.0987289, -0.940743}
```

Average these values:

```
average =
Sum[randoms[[k]],
{k, 1, Length[randoms]}] / Length[randoms]
-0.388356
```

Do it again for 100 randomly chosen points from [1, 3]:

```
Clear[k];
randoms = Table[N[Cos[Random[Real, {1, 3}]]], {k, 1, 100}];

average =
Sum[randoms[[k]],
{k, 1, Length[randoms]}] / Length[randoms]
-0.407316
```

Do it again for 300 randomly chosen points from [1, 3]:

```
randoms = Table[N[Cos[Random[Real, {1, 3}]]], {k, 1, 300}];

average =
Sum[randoms[[k]],
{k, 1, Length[randoms]}] / Length[randoms]
-0.353316
```

### □G.6.a)

In your opinion, how is the average value of  $\text{Cos}[x]$  on [1, 3] calculated through the formula

$$\frac{\int_1^3 \text{Cos}[x] dx}{3-1},$$

as above, related to the outcomes of the numerical experiments?

### □G.6.b.i)

Calculate the average value of  $\text{Sin}[x]$  on  $[0, \pi]$  in decimals.

### □G.6.b.ii)

Look at:

```
randoms = Table[N[Sin[Random[Real, {0, N[Pi]}]]], {k, 1, 300}];

average =
Sum[randoms[[k]],
{k, 1, Length[randoms]}] / Length[randoms]
0.629421

randoms = Table[N[Sin[Random[Real, {0, N[Pi]}]]], {k, 1, 300}];
```

```
average =
Sum[randoms[[k]],
{k, 1, Length[randoms]}] / Length[randoms]
0.641335

randoms = Table[N[Sin[Random[Real, {0, N[Pi]}]]], {k, 1, 300}];

average =
Sum[randoms[[k]],
{k, 1, Length[randoms]}] / Length[randoms]
0.606149

randoms = Table[N[Sin[Random[Real, {0, N[Pi]}]]], {k, 1, 300}];

average =
Sum[randoms[[k]],
{k, 1, Length[randoms]}] / Length[randoms]
0.64026
```

Discuss the results.

### □G.6.c) Average value of $f[x, y]$ on a region R

Above, you saw that the average value of a function  $f[x]$  on an interval  $[a, b]$  on the  $x$ -axis is the number  $A$  satisfying

$$\int_a^b f[x] dx = A(b-a) = A \int_a^b dx.$$

Math folks have a similar notion for functions of two variables; they say:

The average value of a function  $f[x, y]$  on a region  $R$  in the  $xy$ -plane is the number  $A$  satisfying

$$\iint_R f[x, y] dx dy = A \iint_R dx dy.$$

Calculate the average value of  $f[x, y] = x^2 + y^2$  on the rectangle  $R$  consisting of the the points with  $-3 \leq x \leq 3$  and  $-2 \leq y \leq 2$ .

### □G.6.d.i) Centroid of a region

The centroid of a region  $R$  in the  $xy$ -plane is the point  $\{x^*, y^*\}$  where  $x^*$  is the average value of  $f[x, y] = x$  on  $R$  and  $y^*$  is the average value of  $g[x, y] = y$  on  $R$ .

Calculate the centroid of the rectangle  $R$  consisting of the the points with  $-3 \leq x \leq 3$  and  $-2 \leq y \leq 2$ .

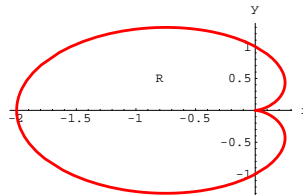
Are you surprised by the result?

### □G.6.d.ii)

Here's region  $R$  plotted in the  $xy$ -plane:

```
Clear[x, y, t];
x[t_] = Cos[t] (1 - Cos[t]);
y[t_] = Sin[t] (1 - Cos[t]);

Rplot = ParametricPlot[{x[t], y[t]}, {t, 0, 2 Pi},
PlotRange -> All, PlotStyle -> {{Red, Thickness[0.01]}},
AxesLabel -> {"x", "y"}, Epilog -> Text["R", {-0.8, 0.5}]];
```



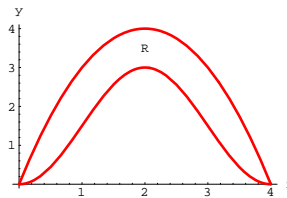
Fancy dudes from Ivy League schools say that this is the cardioid described in polar coordinates by the polar equation  $r[t] = 1 - \text{Cos}[t]$ .

Use the Gauss-Green formula to help you to calculate the centroid of this region.

### □G.6.d.iii)

Here's another region  $R$  plotted in the  $xy$ -plane:

```
Clear[x, Rplot];
Rplot = Plot[{3 Sin[Pi x / 4]^2, x (4 - x)}, {x, 0, 4},
PlotStyle -> {{Red, Thickness[0.01]}}, AxesLabel -> {"x", "y"},
Epilog -> Text["R", {2, 3.5}]];
```



Calculate the centroid of this region.

Is the centroid inside or outside the region?

### □G.6.d.iv)

Sometimes you can look at a plot of a region  $R$  and know before you do any calculation that the centroid of  $R$  will calculate out to be inside  $R$ .

What is it about the bulges and hollows of the plot of the boundary of  $R$  that helps you to do this?

## G.7) Bombing runs

### □G.7.a.i)

You've got a target sitting at  $\{0, 0\}$  and you want to knock it out with an iron bomb. If you set the Northrup bomb sight to hit the target at  $\{0, 0\}$  and examine the point  $\{x, y\}$  at which an individual bomb lands, you will find that the resulting coordinates  $x$  and  $y$

- have no influence on each other;
- are both normally distributed; and
- both average out to  $\{0, 0\}$ .

You will also find that the standard deviation of the  $x$  measurement is not the same as the standard deviation of the  $y$  measurement.

For a specific bomb and airplane, it turns out that  $x_{\text{dev}} = 50$  feet and  $y_{\text{dev}} = 100$  feet.

And to destroy the target, this type of bomb must hit within 100 feet of the target.

Estimate the fraction of the bombs of this type, aimed at this target, that actually destroy the target.

### □G.7.a.ii)

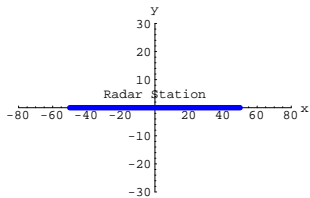
In part i) above, what are your chances of destroying the target if you bomb it twice?

How many times do you have to bomb the target to get 95% confidence of destroying it?

## □G.7.b.i)

This time your target is a long, thin radar installation 100 feet long centered on  $\{0, 0\}$  and visualized as the line segment running from  $\{-50, 0\}$  to  $\{50, 0\}$ :

```
radarstation =
Show[Graphics[{Blue, Thickness[0.02], Line[{{-50, 0}, {50, 0}}]},
Graphics[Text["Radar Station", {0, 5}]],
PlotRange -> {{-80, 80}, {-30, 30}}, Axes -> True,
AxesLabel -> {"x", "y"}];
```



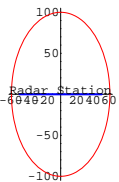
The Air Force bomb damage assessment analysts have determined that to knock out this radar station, you must place your iron bomb within the ellipse

$$\left(\frac{x}{60}\right)^2 + \left(\frac{y}{100}\right)^2 = 1:$$

```
Clear[x, y, t];
x[t_] = 60 Cos[t];
y[t_] = 100 Sin[t];

ellipse = ParametricPlot[{x[t], y[t]}, {t, 0, 2 Pi},
PlotStyle -> {{Thickness[0.01], Red}}, DisplayFunction -> Identity];

Show[ellipse, radarstation,
AspectRatio -> Automatic, PlotRange -> All,
DisplayFunction -> $DisplayFunction];
```



Here are two reasons for the shape of the ellipse:

→ The long, flat nature of the radar station makes it very vulnerable to bombs that land to its front or rear.

→ The same long, flat nature of the radar station makes it almost invulnerable to bombs that land to its left or right.

The bombs you are going to use have

$$x_{dev} = 40 \text{ feet and } y_{dev} = 80 \text{ feet.}$$

Estimate the fraction of the bombs aimed at  $\{0, 0\}$  that actually destroy the radar station.

## □G.7.b.ii)

In part i) above, what are your chances of destroying the radar station if you bomb it twice aiming the bombs at  $\{0, 0\}$ ?

How many times do you have to bomb the radar station, aiming the bombs at  $\{0, 0\}$ , to get 95 % confidence of destroying the radar station?