2D and 3D Measurements

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VC.10 3D Surface Measurements
Basics

B.1) Sources, sinks, and Gauss's formula in 3D

□B.1.a)

Given a three-dimensional vector field
    Field[x, y, z] = {m[x, y, z], n[x, y, z], p[x, y, z]}:

```
Clear[Field, m, n, p, x, y, z]
Field[x_, y_, z_] = {m[x, y, z], n[x, y, z], p[x, y, z]}
```
{m[x, y, z], n[x, y, z], p[x, y, z]}

You calculate the divergence, divField[x, y, z], of Field[x, y, z] via the formula:

```
Clear[divField]
divField[x_, y_, z_] =
 D[m[x, y, z], x] + D[n[x, y, z], y] + D[p[x, y, z], z]
```
$p^{(0,0,1)}[x, y, z] + n^{(0,1,0)}[x, y, z] + m^{(1,0,0)}[x, y, z]$

Not much of surprise.
What do you do with this formula?

□**Answer:**

You use it the same way you use its 2D counterpart.

□B.1.b.i) Gauss's 3D formula

Gauss's 3D formula says that if you go with a 3D vector field
    Field[x, y, z] = {m[x, y, z], n[x, y, z], p[x, y, z]},
and if R is a solid body in three dimensions with boundary surface
(skin) C, then
    $\int\int\int_R$ divField[x, y] dx dy dz
    $= \int\int_C$ Field . outerunitnormal dA
    = flow of Field[x, y, z] across C.

Use Gauss's formula to measure the flow of the vector field
    Field[x, y, z] = {y − 0.5, x y, z^2}
across the surface of the box consisting of all points {x, y, z} with
    $-1 \le x \le 3$, $0 \le y \le 2$, and $-2 \le z \le 4$.

□**Answer:**

```
Clear[x, y, z, m, n, p, Field]
{m[x_, y_, z_], n[x_, y_, z_], p[x_, y_, z_]} = {y - 0.5, x y, z²};
Field[x_, y_, z_] = {m[x, y, z], n[x, y, z], p[x, y, z]}
```
{-0.5 + y, x y, z²}

Gauss's formula tells you that the net flow of Field[x, y, z] across the
surface of the box with
    $-1 \le x \le 3$, $0 \le y \le 2$, and $-2 \le z \le 4$
is
    $$\int_{-2}^{4}\int_{0}^{2}\int_{-1}^{3} \text{divField}[x, y, z]\, d\text{z}\, d\text{y}\, d\text{x}$$
Here you go:

```
Clear[divF]
divField[x_, y_, z_] =
 D[m[x, y, z], x] + D[n[x, y, z], y] + D[p[x, y, z], z];
∫₋₂⁴∫₀²∫₋₁³ divField[x, y, z] dz dy dx
```
144

Very positive.

Strong flow from inside the box to outside the box.

There must be lots of spigots turned on inside this box.

□B.1.b.ii) Sources and sinks

In three dimensions, if
    Field[x, y, z] = {m[x, y, z], n[x, y, z], p[x, y, z]},
then
    divField[x, y, z] =
    D[m[x, y, z], x] + D[n[x, y, z], y] + D[p[x, y, z], z]
tells you about sources and sinks the same way that divField[x, y] tells
you about sources and sinks in two dimensions:
→ The points {x, y, z} with divField[x, y, z] > 0 are sources of new
fluid.
→ The points {x, y, z} with divField[x, y, z] < 0 are sinks for old fluid.
Use Gauss's formula to explain why this interpretation is legitimate.

□**Answer:**

Here's how to see why:

Take a small sphere C centered at {x_0, y_0, z_0}. Calculate the flow

across C which, according to Gauss's formula, is

    $\int\int\int_R$ divField[x, y] dx dy dz

where R is the solid region consisting of the sphere C and everything

inside C.

Here's the kicker:

If you start out with

    divField[x_0, y_0, z_0] > 0,

then divField[x, y, z] is positive for all {x, y, z}'s close to {x_0, y_0, z_0},

so if C is small enough,

    divField[x_0, y_0, z_0] > 0

for all {x, y, z}'s inside C. This tells you that that for small spheres C

centered at {x_0, y_0, z_0},

    flow − across − C = $\int\int\int_R$ divField[x, y] dx dy dz > 0.

And this tells you that if divField[x_0, y_0, z_0] > 0, then the net flow of

Field[x, y, z] across small spheres centered at {x_0, y_0, z_0} is from

inside to outside.

The upshot:

If divField[x_0, y_0, z_0] > 0, then the point{x_0, y_0, z_0} is a source of new

fluid.

Similarly, if divField[x_0, y_0, z_0] < 0, then the net flow of Field[x, y, z]

across small spheres centered at {x_0, y_0, z_0} is from outside to inside.

Consequently, if divField[x_0, y_0, z_0] < 0, then the point {x_0, y_0, z_0} is a

sink (or drain) for old fluid.

□B.1.b.iii)

Given
    Field[x, y, z] = {y − 0.5, x y, z^2},
say how to identify the points {x, y, z} that are sources, and the points
{x, y, z} that are sinks.

□**Answer:**

```
Clear[x, y, z, m, n, p, Field]
{m[x_, y_, z_], n[x_, y_, z_], p[x_, y_, z_]} = {y - 0.5, x y, z²}
Field[x_, y_, z_] = {m[x, y, z], n[x, y, z], p[x, y, z]};
```
{-0.5 + y, x y, z²}

Calculate the divergence:

```
Clear[divField]
divField[x_, y_, z_] =
 D[m[x, y, z], x] + D[n[x, y, z], y] + D[p[x, y, z], z]
```
x + 2 z

Find out where divField[x, y, z] is 0:

```
Solve[divField[x, y, z] == 0]
{{x → -2 z}}
```

This says that the points {x, y, z} with x = −2 z are neither sources nor sinks.

Now look at:

```
Clear[a]
divField[x, y, z] /. x → -2 z + a
a
```

Food for thought.

Upon reflection, this tells you that if

x > −2 z, then divField[x, y, z] > 0.

and if

x < −2 z, then divField[x, y, z] < 0.

As a result, the points {x, y, z} that are sources are the points with

x > −2 z, and the points {x, y, z} that are sinks are the points with

x < −2 z.

□**B.1.b.iv)**

Given

Field[x, y, z] = {x − z, y − x, z − y},

say how to identify the points {x, y, z} that are sources, and the points {x, y, z} that are sinks.

Use your answer to determine whether the net flow of this vector field across the sphere of radius 2 centered at {4, 2, 1} is from inside to outside or from outside to inside.

□**Answer:**

```
Clear[m, n, p, Field, x, y, z]
{m[x_, y_, z_], n[x_, y_, z_], p[x_, y_, z_]} = {x - z, y - x, z - y};
Field[x_, y_, z_] = {m[x, y, z], n[x, y, z], p[x, y, z]}
{x - z, -x + y, -y + z}
```

divField[x, y, z] is given by:

```
Clear[divField]
divField[x_, y_, z_] = ∂_x m[x, y, z] + ∂_y n[x, y, z] + ∂_z p[x, y, z]
3
```

No matter what {x, y, z} happens to be,

divField[x, y, z] = 3.

So:

All points {x, y, z} are sources.

And now without further calculation, you know that the net flow of this vector field across sphere of radius 2 centered at {4, 2, 1} is from inside to outside.

Reason: There are no sinks inside the sphere to absorb excess outside-to-inside flow.

□**B.1.c)**

Explain the reasoning behind Gauss's formula

□**Answer:**

A heavy, long-winded explanation is possible, but leave it at this:

Once you have a good feeling for the ideas in B.2), B.3), and B.4), you can build your own explanation of Gauss's formula by copying and pasting the explanation of the two dimensional Gauss-Green formula from an earlier lesson and making technical adjustments.

If you cannot find satisfaction without seeing some more details, see:

W. Kaplan, Advanced Calculus, 1972, Addison-Wesley,
Reading, Masschusetts, page 338.

B.2) Measuring area on surfaces

□**B.2.a) Area of a parallelogram in three dimensions**

Here's a parallelogram in 3 dimensions:

```
basepoint = {1, 2, 0};
X = {-2, 1, 1};
Y = {1, -2, 1};
parallelogram = Show[Graphics3D[
   Polygon[{basepoint, basepoint + X, basepoint + X + Y, basepoint + Y}]]],
   ViewPoint → CMView, Axes → True, AxesLabel → {"x", "y", "z"}];
```
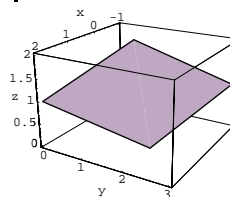

Because of its unfortunate position, it might seem hard to measure the area of this parallelogram, but the cross product can bail you out. How?

□**Answer:**

Throw in the vectors that define this parallelogram:

```
Show[parallelogram, Arrow[X, Tail → basepoint, VectorColor → Blue],
   Arrow[Y, Tail → basepoint, VectorColor → Blue],
   Graphics3D[{PointSize[0.03], Point[basepoint]}],
   Graphics3D[Text["X", basepoint + X/2 + {0, 0.2, 0}]], Graphics3D[
   Text["Y", basepoint + Y/2 + {0.2, 0, 0}]], ViewPoint → CMView,
   PlotRange → All, Axes → True, AxesLabel → {"x", "y", "z"}];
```
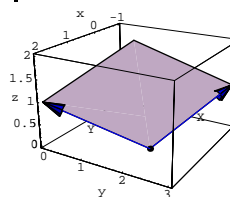

As you probably remember,

‖X×Y‖ = ‖X‖ ‖Y‖ |Sin[angle between]|.

This quantity is also equal to the area of this parallelogram; so the area in square units of this unfortunately positioned parallelogram is given by:

```
XcrossY = X × Y;
√XcrossY . XcrossY
3√3
```
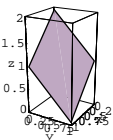
Easy.

Rerun for other choices of X and Y to get the hang of it.

□**B.2.b.i) Measuring area on planes**

Here's a look at part of the plane

x + y + z = 2:

```
Clear[x, y, z, u, v]
x[u_, v_] = u;
y[u_, v_] = v;
z[u_, v_] = 2 - u - v;
surface = ParametricPlot3D[{x[u, v], y[u, v], z[u, v]},
   {u, 0, 1}, {v, 0, 1}, PlotPoints → {2, 2}, Axes → Automatic,
   AxesLabel → {"x", "y", "z"}, ViewPoint → CMView];
```


Here is another way of getting a plot of the same surface:
This plot also includes the vectors that generate the parallelogram.

```
{a, b} = {0, 0};
h = 1;
Show[Graphics3D[Polygon[
   {{x[a, b], y[a, b], z[a, b]}, {x[a + h, b], y[a + h, b], z[a + h, b]},
   {x[a + h, b + h], y[a + h, b + h], z[a + h, b + h]},
   {x[a, b + h], y[a, b + h], z[a, b + h]}}]], Arrow[
   {x[a + h, b], y[a + h, b], z[a + h, b]} - {x[a, b], y[a, b], z[a, b]},
   Tail → {x[a, b], y[a, b], z[a, b]}, VectorColor → Blue], Arrow[
```

```
   {x[a, b + h], y[a, b + h], z[a, b + h]} - {x[a, b], y[a, b], z[a, b]},
   Tail → {x[a, b], y[a, b], z[a, b]}, VectorColor → Blue],
   PlotRange → All, Axes → Automatic, AxesLabel → {"x", "y", "z"},
   ViewPoint → CMView];
```
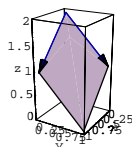

For this choice of a, b, and h, the corresponding parallelogram has area measurement:

```
X = {x[a + h, b], y[a + h, b], z[a + h, b]} - {x[a, b], y[a, b], z[a, b]};
Y = {x[a, b + h], y[a, b + h], z[a, b + h]} - {x[a, b], y[a, b], z[a, b]};
XcrossY = X × Y;
√XcrossY . XcrossY
```
$\sqrt{3}$

For unspecified choices of a, b, and h, the corresponding parallelogram has area measured by:

```
Clear[a, b, h]
X = {x[a + h, b], y[a + h, b], z[a + h, b]} - {x[a, b], y[a, b], z[a, b]};
Y = {x[a, b + h], y[a, b + h], z[a, b + h]} - {x[a, b], y[a, b], z[a, b]};
XcrossY = X × Y;
√XcrossY . XcrossY
```
$\sqrt{3}\sqrt{h^4}$

What is the area conversion factor
$$SA_{xyz}[u, v]$$
that you use to convert uv-paper area measurement into xyz-surface area measured on the plane $x + y + z = 2$?

□Answer:

The uv-paper rectangle with corners at

{a, b}, {a + h, b}, {a + h, b + h}, and {a, b + h}

has uv-paper area given by:

```
Clear[h]
uvarea = h²
```
h^2

This uv-paper rectangle plots out in xyz-coordinates as the parallelogram with corners at

{x[a, b], y[a, b], z[a, b]},

{x[a + h, b], y[a + h, b], z[a + h, b]},

{x[a + h, b + h], y[a + h, b + h], z[a + h, b + h]},

and

{x[a, b + h], y[a, b + h], z[a, b + h]}.

The area of this parallelogram is given by:

```
Clear[a, b, h]
X = {x[a + h, b], y[a + h, b], z[a + h, b]} - {x[a, b], y[a, b], z[a, b]};
Y = {x[a, b + h], y[a, b + h], z[a, b + h]} - {x[a, b], y[a, b], z[a, b]};
XcrossY = X × Y;
planararea = √XcrossY . XcrossY
```
$\sqrt{3}\sqrt{h^4}$

The area conversion factor $SA_{xyz}[u, v]$ that you integrate to convert uv-paper area into xyz-surface area measured on the plane

$x + y + z = 2$ is:

```
Clear[SAxyz]
SAxyz[u_, v_] = PowerExpand[ planararea / uvarea ]
```
$\sqrt{3}$

Still pretty easy.

□**B.2.b.ii) Measuring planar area**

Here is a plot of the uv-paper circle
$$u^2 + v^2 = 4$$
on the plane
$$x + y + z = 2$$
under the parameterization
$$x[u, v] = u,$$
$$y[u, v] = v, \text{ and}$$
$$z[u, v] = 2 - u - v$$
used above.

```
Clear[x, y, z, u, v, t]
x[u_, v_] = u;
y[u_, v_] = v;
z[u_, v_] = 2 - u - v;
u[t_] = 2 Cos[t];
v[t_] = 2 Sin[t];
curve = ParametricPlot3D[{x[u[t], v[t]], y[u[t], v[t]], z[u[t], v[t]]},
   {t, 0, 2 π}, DisplayFunction → Identity];
{a, b} = {-4, -4};
h = 8;
plane = Graphics3D[Polygon[
   {{x[a, b], y[a, b], z[a, b]}, {x[a + h, b], y[a + h, b], z[a + h, b]},
    {x[a + h, b + h], y[a + h, b + h], z[a + h, b + h]},
    {x[a, b + h], y[a, b + h], z[a, b + h]}}]];

Show[curve, plane, Axes → Automatic, AxesLabel → {"x", "y", "z"},
 ViewPoint → CMView, DisplayFunction → $DisplayFunction];
```
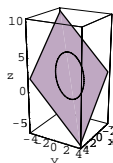

Come up with a measurement of area enclosed by this curve as measured on the plane
$$x + y + z = 2.$$

□Answer:

Area on the plane is $SA_{xyz}[u, v]$ times uv-paper area as calculated above in part i):

```
SAxyz[u, v]
```
$\sqrt{3}$

The circle $u^2 + v^2 = 4$ is a circle of radius 2 on uv-paper. It encloses an area of $2^2 \pi = 4\pi$ square units measured on uv-paper.

Its plot on the plane $x + y + z = 2$ encloses a total of:

```
SAxyz[u, v] 4 π
```
$4\sqrt{3}\pi$

in square units.

□**B.2.c.i) Measuring area on curved surfaces**

Here's a portion of the surface whose parametric equations are:

```
Clear[x, y, z, u, v]
x[u_, v_] = u;
y[u_, v_] = v;
z[u_, v_] = u v;
{ulow, uhigh} = {0, 1};
{vlow, vhigh} = {0, 2};
surface = ParametricPlot3D[Evaluate[{x[u, v], y[u, v], z[u, v]}],
   {u, ulow, uhigh}, {v, vlow, vhigh}, ViewPoint → CMView,
   BoxRatios → Automatic, Axes → Automatic, AxesLabel → {"x", "y", "z"}];
```
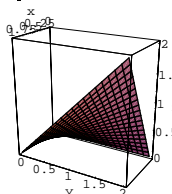

Measure the surface area of the plotted portion of this surface.

□Answer:

Mathematica plots this surface by approximating the true surface with a whole herd of little patches that resemble parallelograms.

The message is clear.

You can use what you know about measuring the area of parallelograms to help to measure the area of curved surfaces.

See what some parallelograms will do:

```
Clear[h, basepoint, leg1, leg2, parallelogram]
basepoint[u_, v_] = {x[u, v], y[u, v], z[u, v]};
leg1[u_, v_, h_] = {x[u + h, v], y[u + h, v], z[u + h, v]} - basepoint[u, v];
leg2[u_, v_, h_] = {x[u, v + h], y[u, v + h], z[u, v + h]} - basepoint[u, v];
parallelogram[u_, v_, h_] = Graphics3D[
   {Black, Polygon[{basepoint[u, v], basepoint[u, v] + leg1[u, v, h],
      basepoint[u, v] + leg1[u, v, h] + leg2[u, v, h],
      basepoint[u, v] + leg2[u, v, h]}]}];
jump = 0.3;
h = 0.25;
parallelograms = Table[parallelogram[u, v, h],
   {u, ulow, uhigh - jump, jump}, {v, vlow, vhigh - jump, jump}];

Show[surface, parallelograms, ViewPoint → CMView, Axes → Automatic,
 AxesLabel → {"x", "y", "z"}];
```
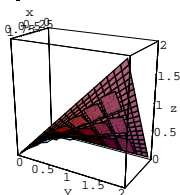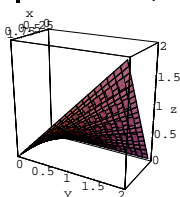

Hug that surface, mama!

The smart money says:

The smaller the parallelograms are, the closer they hug the surface.

Check it out:

```
jump = 0.2;
h = 0.1;
parallelograms = Table[parallelogram[u, v, h],
   {u, ulow, uhigh - jump, jump}, {v, vlow, vhigh - jump, jump}];

Show[surface, parallelograms, ViewPoint → CMView, Axes → Automatic,
 AxesLabel → {"x", "y", "z"}];
```


The smart money wins.

These smaller parallelograms are really glued to the surface. Even smaller parallelograms will share even more ink with the surface.

Here's how to get the area conversion factor $SA_{xyz}[u, v]$ for going from uv-paper area to xyz-area on the surface at a uv-point $\{u, v\}$:

Notice that for small positive h, the area conversion factors at a uv-point $\{u, v\}$ are nearly the same for area on the surface and for area on the parallelograms plotted above which are determined by the vectors:

```
Clear[h]
leg1[u, v, h]
leg2[u, v, h]
```
$\{h, 0, -u v + (h + u) v\}$
$\{0, h, -u v + u (h + v)\}$

For fixed $\{u, v\}$ and h, the area conversion factor for converting uv-area to xyz-area on the parallelogram determined by leg1[u, v, h] and leg2[u, v, h] is given by:

```
cross = leg1[u, v, h] × leg2[u, v, h];
Clear[paraareafactor]

paraareafactor[u_, v_, h_] = √(cross . cross) / h²
```
$\frac{\sqrt{h^4 + h^4 u^2 + h^4 v^2}}{h^2}$

The area conversion factor $SA_{xyz}[u, v]$ for the surface at $\{u, v\}$ is the limiting case of the above as $h \to 0$:

```
Clear[SAxyz]
SAxyz[u_, v_] = Limit[paraareafactor[u, v, h], h → 0]
```
$\sqrt{1 + u^2 + v^2}$

The total surface area (in square units) of the surface plotted above is $\int_{vlow}^{vhigh} \int_{ulow}^{uhigh} SA_{xyz}[u, v] \, du \, dv$:

```
surfarea = NIntegrate[SAxyz[u, v], {u, ulow, uhigh}, {v, vlow, vhigh}]
```
3.18041

And once you've done one of these, you've done them all.

□B.2.c.ii)

Here is a direct formula for the area conversion factor $SA_{xyz}[u, v]$ that converts uv-paper area measurements into area measurements on the surface whose parametric formulas are $\{x[u, v], y[u, v], z[u, v]\}$:

```
Clear[x, y, z, u, v, cross, SAxyz]
cross[u_, v_] =
 D[{x[u, v], y[u, v], z[u, v]}, u] × D[{x[u, v], y[u, v], z[u, v]}, v];
SAxyz[u_, v_] = √(cross[u, v] . cross[u, v])
```
$\sqrt{\big((y^{(0,1)}[u, v] \, x^{(1,0)}[u, v] - x^{(0,1)}[u, v] \, y^{(1,0)}[u, v])^2 + }$
$\quad (-z^{(0,1)}[u, v] \, x^{(1,0)}[u, v] + x^{(0,1)}[u, v] \, z^{(1,0)}[u, v])^2 +$
$\quad (z^{(0,1)}[u, v] \, y^{(1,0)}[u, v] - y^{(0,1)}[u, v] \, z^{(1,0)}[u, v])^2 \big)}$

Nasty looking, but useful.

Explain where this formula for $SA_{xyz}[u, v]$ comes from.

□**Answer:**

You can make a good run at the formula by clearing the parametric functions x[u, v], y[u, v], and z[u, v], and doing what was done in the last part above:

```
Clear[x, y, z, u, v, h, basepoint, leg1, leg2, parallelogram]
basepoint[u_, v_] = {x[u, v], y[u, v], z[u, v]}
```
$\{x[u, v], y[u, v], z[u, v]\}$
```
leg1[u_, v_, h_] = {x[u + h, v], y[u + h, v], z[u + h, v]} - basepoint[u, v]
```
$\{-x[u, v] + x[h + u, v], -y[u, v] + y[h + u, v], -z[u, v] + z[h + u, v]\}$
```
leg2[u_, v_, h_] = {x[u, v + h], y[u, v + h], z[u, v + h]} - basepoint[u, v]
```
$\{-x[u, v] + x[u, h + v], -y[u, v] + y[u, h + v], -z[u, v] + z[u, h + v]\}$

As h closes in on 0, the area conversion factor for the parallelogram determined by the vectors:

```
leg1[u, v, h]
```
$\{-x[u, v] + x[h + u, v], -y[u, v] + y[h + u, v], -z[u, v] + z[h + u, v]\}$
and
```
leg2[u, v, h]
```
$\{-x[u, v] + x[u, h + v], -y[u, v] + y[u, h + v], -z[u, v] + z[u, h + v]\}$
with tails at
```
basepoint[u, v]
```
$\{x[u, v], y[u, v], z[u, v]\}$

closes in on the area conversion factor $SA_{xyz}[u, v]$ for the surface at $\{u, v\}$.

The area conversion factor for this parallelogram is given by $\frac{\|leg1[u,v,h] \times leg2[u,v,h]\|}{h^2}$.

As a result, the formula for the area conversion factor surface at $\{u, v\}$ is $\lim_{h \to 0} \frac{\|leg1[u,v,h] \times leg2[u,v,h]\|}{h^2}$.

This is the same as $\lim_{h \to 0} \| \frac{leg1[u,v,h]}{h} \times \frac{leg2[u,v,h]}{h} \|$.

This is the same as

$$\left\|\lim_{h\to 0}\left(\frac{\text{leg1}[u,v,h]}{h}\right) \times \lim_{h\to 0}\left(\frac{\text{leg2}[u,v,h]}{h}\right)\right\|.$$

Look at:

```
leg1[u, v, h]
     h
{ -x[u, v] + x[h + u, v] ,  -y[u, v] + y[h + u, v] ,  -z[u, v] + z[h + u, v] }
        h                        h                        h
leg2[u, v, h]
     h
{ -x[u, v] + x[u, h + v] ,  -y[u, v] + y[u, h + v] ,  -z[u, v] + z[u, h + v] }
        h                        h                        h
```

$\lim\limits_{h\to 0}\left(\frac{\text{leg1}[u,v,h]}{h}\right)$ is given by:

```
first = {D[x[u, v], u], D[y[u, v], u], D[z[u, v], u]}
{x^(1,0)[u, v], y^(1,0)[u, v], z^(1,0)[u, v]}
```

$\lim\limits_{h\to 0}\left(\frac{\text{leg2}[u,v,h]}{h}\right)$ is given by:

```
second = {D[x[u, v], v], D[y[u, v], v], D[z[u, v], v]}
{x^(0,1)[u, v], y^(0,1)[u, v], z^(0,1)[u, v]}
```

The area conversion factor $SA_{xyz}[u, v]$ for the surface is the length of cross product:

```
Clear[cross, SAxyz]
cross[u_, v_] = first × second;
SAxyz[u_, v_] = √cross[u, v] . cross[u, v]
```

$$\sqrt{\big((y^{(0,1)}[u,v]\,x^{(1,0)}[u,v] - x^{(0,1)}[u,v]\,y^{(1,0)}[u,v]\big)^2 +}$$
$$\big(-z^{(0,1)}[u,v]\,x^{(1,0)}[u,v] + x^{(0,1)}[u,v]\,z^{(1,0)}[u,v]\big)^2 +$$
$$\big(z^{(0,1)}[u,v]\,y^{(1,0)}[u,v] - y^{(0,1)}[u,v]\,z^{(1,0)}[u,v]\big)^2 \big)$$

The final formula is a mess to remember, but you can get the same formula quickly via:

```
Clear[cross, SAxyz]
cross[u_, v_] =
 D[{x[u, v], y[u, v], z[u, v]}, u] × D[{x[u, v], y[u, v], z[u, v]}, v];
SAxyz[u_, v_] = √cross[u, v] . cross[u, v]
```

$$\sqrt{\big((y^{(0,1)}[u,v]\,x^{(1,0)}[u,v] - x^{(0,1)}[u,v]\,y^{(1,0)}[u,v]\big)^2 +}$$
$$\big(-z^{(0,1)}[u,v]\,x^{(1,0)}[u,v] + x^{(0,1)}[u,v]\,z^{(1,0)}[u,v]\big)^2 +$$
$$\big(z^{(0,1)}[u,v]\,y^{(1,0)}[u,v] - y^{(0,1)}[u,v]\,z^{(1,0)}[u,v]\big)^2 \big)$$

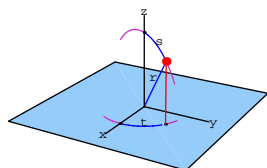And this is the explanation.

□**B.2.c.i)**

Measure the surface area of the surface of a sphere of radius r.

□**Answer:**

Go to spherical coordinates.

```
Show[all, DisplayFunction → $DisplayFunction];
```

The surface of the sphere of radius r is described by

$$\{x[s, t], y[s, t], z[s, t]\} =$$
$$\{r\,\mathrm{Sin}[s]\,\mathrm{Cos}[t], r\,\mathrm{Sin}[s]\,\mathrm{Sin}[t], r\,\mathrm{Cos}[s]\}$$

with s running from 0 to π and t running from 0 to 2π.

```
Clear[x, y, z, r, s, t]
{x[s_, t_], y[s_, t_], z[s_, t_]} =
 {r Sin[s] Cos[t], r Sin[s] Sin[t], r Cos[s]}
{r Cos[t] Sin[s], r Sin[s] Sin[t], r Cos[s]}
```

To measure the area of surface of the sphere of radius r, all you have to do is calculate

$$\int_0^\pi \int_0^{2\pi} SA_{xyz}[s, t]\, dt\, ds:$$

```
Clear[SAxyz, cross]
cross[s_, t_] =
 D[{x[s, t], y[s, t], z[s, t]}, s] × D[{x[s, t], y[s, t], z[s, t]}, t];
SAxyz[s_, t_] = √TrigExpand[cross[s, t] . cross[s, t]]
```

$$\sqrt{\frac{r^4}{2} - \frac{1}{2}r^4\,\mathrm{Cos}[s]^2 + \frac{1}{2}r^4\,\mathrm{Sin}[s]^2}$$

$$\int_0^\pi \int_0^{2\pi} SA_{xyz}[s, t]\, dt\, ds$$

$$4\pi\sqrt{r^4}$$

If your version of Mathematica gave you 0, then your version of Mathematica screwed up. Here is another way of getting the correct answer:

$$2\int_0^{\frac{\pi}{2}} \int_0^{2\pi} SA_{xyz}[s, t]\, dt\, ds$$

$$4\pi\sqrt{r^4}$$

$4\pi r^2$ is correct.

B.3) Surface integrals

Just as you can integrate functions on curves with respect to arc length, you can integrate functions on surfaces with respect to surface area.
And you don't have to go to a lot of trouble to do it.

□**B.3.a)**

Here is a parameterized surface C:

```
Clear[x, y, z, u, v]
x[u_, v_] = u Cos[v];
y[u_, v_] = -u Sin[v];
z[u_, v_] = v;
{{ulow, uhigh}, {vlow, vhigh}} = {{1, 3}, {0, 3π/2}};
surface = ParametricPlot3D[{x[u, v], y[u, v], z[u, v]},
  {u, ulow, uhigh}, {v, vlow, vhigh}, ViewPoint → CMView,
  Boxed → False, AxesLabel → {"x", "y", "z"}];
```
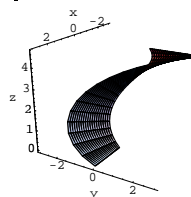
Calculate
$$\iint_R (x^2 + y^2)\, dA$$
where the integral is taken with respect to surface area.

□**Answer:**

$$\iint_R (x^2 + y^2)\, dA$$

is calculated via the formula
$$\int_{vlow}^{vhigh} \int_{ulow}^{uhigh} (x[u, v]^2 + y[u, v]^2)\, SA_{xyz}[u, v]\, du\, dv.$$

Calculating this is duck soup.

First you need the area conversion factor $SA_{xyz}[u, v]$:

```
Clear[SAxyz]
cross =
 D[{x[u, v], y[u, v], z[u, v]}, u] × D[{x[u, v], y[u, v], z[u, v]}, v];
SAxyz[u_, v_] = √TrigExpand[cross . cross]
```
$$\sqrt{1 + u^2}$$

Now turn the integral
$$\int_{vlow}^{vhigh} \int_{ulow}^{uhigh} (x[u, v]^2 + y[u, v]^2)\, SA_{xyz}[u, v]\, du\, dv.$$
over to the machine:

```
integral = NIntegrate[(x[u, v]^2 + y[u, v]^2) SAxyz[u, v],
  {u, ulow, uhigh}, {v, vlow, vhigh}]
103.125
```
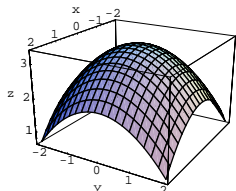
Not exciting, but not hard.

B.4) Surface integrals for measuring flow across surfaces

□**B.4.a)**

Here's a surface:

```
Clear[x, y, z, u, v]
x[u_, v_] = u;
y[u_, v_] = v;
z[u_, v_] = 1/3 (10 - u^2 - v^2);
{ulow, uhigh} = {-2, 2};
```

```
{vlow, vhigh} = {-2, 2};
surface = ParametricPlot3D[Evaluate[{x[u, v], y[u, v], z[u, v]}],
  {u, ulow, uhigh}, {v, vlow, vhigh}, AxesLabel → {"x", "y", "z"},
  ViewPoint → CMView];
```


Take {a, b} = {0.5, 1.0} and look at this display of the two curves
$$P_1[u] = \{x[u, b], y[u, b], z[u, b]\}$$
with ulow ≤ u ≤ uhigh
and
$$P_2[v] = \{x[a, v], y[a, v], z[a, v]\}$$
with vlow ≤ v ≤ vhigh
together with the surface plotted above:

```
{a, b} = {0.5, 1.0};
Clear[P1, P2]
P1[u_] = {x[u, b], y[u, b], z[u, b]};
P2[v_] = {x[a, v], y[a, v], z[a, v]};
curve1 = ParametricPlot3D[Evaluate[P1[u]],
  {u, ulow, uhigh}, DisplayFunction → Identity];
curve2 = ParametricPlot3D[Evaluate[P2[v]],
  {v, vlow, vhigh}, DisplayFunction → Identity];
surface = Insert[surface, EdgeForm[], {1, 1}];

curves =
 Show[surface, curve1, curve2, Boxed → False, ViewPoint → CMView,
  BoxRatios → Automatic, DisplayFunction → $DisplayFunction];
```
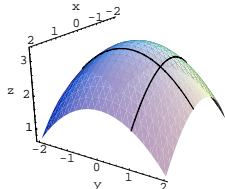

Two intersecting curves running on the surface.
Now calculate the area conversion factor $SA_{xyz}[u, v]$

```
Clear[SAxyz, cross]
cross[u_, v_] =
 D[{x[u, v], y[u, v], z[u, v]}, u] ×D[{x[u, v], y[u, v], z[u, v]}, v];
SAxyz[u_, v_] = √cross[u, v] . cross[u, v]
```

$$\sqrt{1 + \frac{4\,u^2}{9} + \frac{4\,v^2}{9}}$$

Keeping the same {a, b} as above, add to the last plot the three vectors
$$D[\{x[u, v], y[u, v], z[u, v]\}, u] \,/.\, \{u \to a, v \to b\},$$
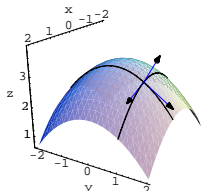$$D[\{x[u, v], y[u, v], z[u, v]\}, v]\} \,/.\, \{u \to a, v \to b\}, \text{and}$$
$$cross[a, b],$$
all with their tails at the point of intersection:

```
tail = {x[a, b], y[a, b], z[a, b]};
vector1 = D[{x[u, v], y[u, v], z[u, v]}, u] /. {u → a, v → b};
vector2 = D[{x[u, v], y[u, v], z[u, v]}, v] /. {u → a, v → b};
vector3 = cross[a, b];

Show[curves, Arrow[vector1, Tail → tail], Arrow[vector2, Tail → tail],
 Arrow[vector3, Tail → tail]];
```


Totally radical or what!
Explain why this happened and how it relates to the area conversion
factor $SA_{xyz}[u, v]$.

□ Answer:

Both
$$D[\{x[u, v], y[u, v], z[u, v]\}, u], \text{ and}$$
$$D[\{x[u, v], y[u, v], z[u, v]\}, v]$$
are tangent to the surface at {x[u, v], y[u, v], z[u, v]},

so their cross product, cross[u, v], is automatically normal
(perpendicular) to the surface at {x[u, v], y[u, v], z[u, v]}.
The upshot:
The area conversion factor
$$SA_{xyz}[u, v] = \sqrt{cross[u, v] . cross[u, v]}$$
is the same as the length of the normal vector
cross[u, v].

□ **B.4.b)**

Given a 3D vector field
$$Field[x, y, z] = \{m[x, y, z], n[x, y, z], p[x, y, z]\},$$
and given a surface parameterized by
$$\{x[u, v], y[u, v], z[u, v]\}$$
with ulow ≤ u ≤ uhigh and vlow ≤ v ≤ vhigh,
you can measure the flow of Field[x, y, z] across the surface by
calculating
$$\int_{vlow}^{vhigh} \int_{ulow}^{uhigh} Field[x[u, v], y[u, v]] . normal[u, v]\, d u\, d v$$
where normal[u, v] = cross[u, v] is given by:

```
Clear[x, y, z, u, v, normal]
normal[u_, v_] =
 D[{x[u, v], y[u, v], z[u, v]}, u] ×D[{x[u, v], y[u, v], z[u, v]}, v]
```
$$\{z^{(0,1)}[u, v]\, y^{(1,0)}[u, v] - y^{(0,1)}[u, v]\, z^{(1,0)}[u, v],$$
$$-z^{(0,1)}[u, v]\, x^{(1,0)}[u, v] + x^{(0,1)}[u, v]\, z^{(1,0)}[u, v],$$
$$y^{(0,1)}[u, v]\, x^{(1,0)}[u, v] - x^{(0,1)}[u, v]\, y^{(1,0)}[u, v]\}$$
                Don't try to memorize this formula.

Where does the flow-across formula
$$\int_{vlow}^{vhigh} \int_{ulow}^{uhigh} Field[x[u, v], y[u, v]] . normal[u, v]\, d u\, d v$$
come from?

□ **Answer:**

Put normcompField[x[u, v], y[u, v], z[u, v]] equal to the component of
Field[x[u, v], y[u, v], z[u, v]] in the direction perpendicular to the

surface at {x[u, v], y[u, v], z[u, v]}.
You measure the flow of Field[x, y, z] across the surface by calculating
$$flowacross = \iint_{surface} normcompField[x[u, v], y[u, v], z[u, v]]\, d A$$
$$= \int_{vlow}^{vhigh} \int_{ulow}^{uhigh} normcompField[x[u, v], y[u, v]]\, SA_{xyz}[u, v]\, d u\, d v.$$
Put this in your pocket for a minute.

Remember that normal[u, v] as calculated above is perpendicular to the
surface at the point {x[u, v], y[u, v], z[u, v]}. At this point, the
component of Field[x[u, v], y[u, v], z[u, v]] in the direction
perpendicular to the surface is
$$normcompField[x[u, v], y[u, v], z[u, v]]$$
$$= Field[x[u, v], y[u, v], z[u, v]] . \frac{normal[u,v]}{\sqrt{normal[u,v].normal[u,v]}}.$$

Because $SA_{xyz}[u, v] = \sqrt{normal[u, v] . normal[u, v]}$,
$$flowacross = \iint_{surface} normcompField[x[u, v], y[u, v], z[u, v]]\, d A$$

$$= \int_{vlow}^{vhigh} \int_{ulow}^{uhigh} normcompField[x[u, v], y[u, v]]\, SA_{xyz}[u, v]\, d u\, d v$$

$$= \int_{vlow}^{vhigh} \int_{ulow}^{uhigh} Field[x[u, v], y[u, v]] . \left(\frac{normal[u,v]}{SA_{xyz}[u,v]}\right) SA_{xyz}[u, v]\, d u\, d v$$

$$= \int_{vlow}^{vhigh} \int_{ulow}^{uhigh} Field[x[u, v], y[u, v]] . \left(\frac{normal[u,v]}{SA_{xyz}[u,v]}\right) d u\, d v$$

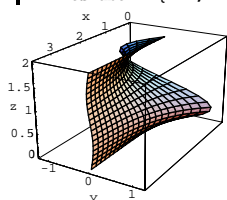$$= \int_{vlow}^{vhigh} \int_{ulow}^{uhigh} Field[x[u, v], y[u, v]] . normal[u, v]\, d u\, d v.$$
because the $SA_{xyz}[u, v]$ terms miraculously cancel out.
Explanation finished.

□B.4.c)

Here's a new surface:

```
Clear[x, y, z, u, v]
x[u_, v_] = v;
y[u_, v_] = 2 E^-v Sin[π u];
z[u_, v_] = u;
{ulow, uhigh} = {0, 2};
{vlow, vhigh} = {0.5, 3.5};
surface = ParametricPlot3D[Evaluate[{x[u, v], y[u, v], z[u, v]}],
  {u, ulow, uhigh}, {v, vlow, vhigh}, ViewPoint → CMView,
  AxesLabel → {"x", " y", "z"}];
```


Here is this surface together with some of the normal vectors whose lengths measure the area conversion factor

$SA_{xyz}[u, v]$,

which converts uv-paper area measurements into xyz-area measurements on the surface:

```
Clear[normal]
normal[u_, v_] =
 D[{x[u, v], y[u, v], z[u, v]}, u] × D[{x[u, v], y[u, v], z[u, v]}, v];
normals = Table[Arrow[normal[u, v], Tail → {x[u, v], y[u, v], z[u, v]}],
  {u, 0.5, 1.5, 0.5}, {v, 1.5, 2.5}];
surface = Insert[surface, EdgeForm[], {1, 1}];

Show[surface, normals, ViewPoint → CMView, Boxed → False,
 AxesLabel → {"x", "y", "z"}];
```
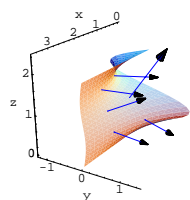

The normals are pointing out from the side of the surface you are sitting on.
Determine whether the net flow of the vector field

Field[x, y, z] = {y z, x z, x y}

across this surface is with the plotted normals or is against the plotted normals.

□Answer:

Enter the vector field:

```
Clear[x, y, z, m, n, p, Field]
{m[x_, y_, z_], n[x_, y_, z_], p[x_, y_, z_]} = {y z, x z, x y};
Field[x_, y_, z_] = {m[x, y, z], n[x, y, z], p[x, y, z]}
{y z, x z, x y}
```

Reenter the parameterization of the surface, and calculate normal[u, v]:

```
Clear[u, v]
x[u_, v_] = v;
y[u_, v_] = 2 E^-v Sin[π u];
z[u_, v_] = u;
{ulow, uhigh} = {0, 2};
{vlow, vhigh} = {0.5, 3.5};
Clear[normal]
normal[u_, v_] =
 D[{x[u, v], y[u, v], z[u, v]}, u] × D[{x[u, v], y[u, v], z[u, v]}, v]
{2 E^-v Sin[π u], 1, -2 E^-v π Cos[π u]}
```

Calculate

$\int_{vlow}^{vhigh} \int_{ulow}^{uhigh}$ Field[x[u, v], y[u, v]] . normal[u, v] du dv:

```
NIntegrate[Field[x[u, v], y[u, v], z[u, v]] . normal[u, v],
 {u, ulow, uhigh}, {v, vlow, vhigh}, AccuracyGoal → 2]
12.7343
```
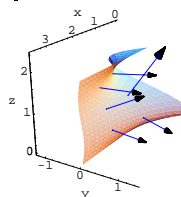
Positive.

This tells you that the net flow of Field[x, y, z] across this surface is in the direction of the plotted normals.

Take another look:

```
Show[surface, normals, ViewPoint → CMView, Boxed → False,
 AxesLabel → {"x", "y", "z"}];
```


In short, the net flow of

Field[x, y, z] = {y z, x z, x y}

across this surface is from behind the surface to the front of the surface.

B.5) Flux of the electric field

Way, way back, at the end of the 18th century, a fellow named Coulomb played with electrostatic problems. He rubbed cats with amber rods, and had an all-around good time with his science. More than that, he managed to do some calculations of forces that electrically charged objects exert on each other.

He found out that in free open space, the force that a particle located at {x, y, z} with charge q[x, y, z] exerts on a particle located at {xx, yy, zz} with charge q[xx, yy, zz] is given by:

```
Clear[K, q, x, y, z, xx, yy, zz]
R = {x, y, z} - {xx, yy, zz};
Force = K q[x, y, z] q[xx, yy, zz] R
        ───────────────────────────
               (R . R)^(3/2)
```

$$\{ \frac{K (x - xx) q[x, y, z] q[xx, yy, zz]}{((x - xx)^2 + (y - yy)^2 + (z - zz)^2)^{3/2}}, \frac{K (y - yy) q[x, y, z] q[xx, yy, zz]}{((x - xx)^2 + (y - yy)^2 + (z - zz)^2)^{3/2}},$$
$$\frac{K (z - zz) q[x, y, z] q[xx, yy, zz]}{((x - xx)^2 + (y - yy)^2 + (z - zz)^2)^{3/2}} \}$$

Note that R runs parallel to the vector that points from {xx, yy, zz} to {x, y, z}.

          Most folks call this formula Coulomb's law.

Look at the magnitude of this force:

```
√ Together[Force . Force]
```

$$\sqrt{ \frac{K^2 q[x, y, z]^2 q[xx, yy, zz]^2}{(x^2 - 2 x xx + xx^2 + y^2 - 2 y yy + yy^2 + z^2 - 2 z zz + zz^2)^2} }$$

This is the same as:

```
K q[x, y, z] q[xx, yy, zz]
──────────────────────────
       Expand[R . R]
```

$$\frac{K q[x, y, z] q[xx, yy, zz]}{x^2 - 2 x xx + xx^2 + y^2 - 2 y yy + yy^2 + z^2 - 2 z zz + zz^2}$$

This means that the force is directed on the line between the two points, and its magnitude is

$$\frac{K q[x,y,z] q[xx,yy,zz]}{(\text{distance between the two points})^2}$$

because the denominator

R . R = (distance between the two points)^2.

          This is what a lot of folks call an "inverse square law."

That constant K involves several other physical quantities and ultimately depends only on the system of units used in the calculation. As usual in mathematics, you specify standard math course units of measure by adjusting the units so that K = 1, and then you leave it to your physics friends to tell you about the K they actually use.
For visualizations of Coulomb's law, electrical folks like the idea of the electric field. The three dimensional electric field of assumes a charge

q[x, y, z] = 1 at {x, y, z},

and thus is the force per unit charge at {x, y, z}:

```
Clear[ElectField]
ElectField[x_, y_, z_] = Force /. {K → 1, q[x, y, z] → 1}
```

$$\left\{ \frac{(x - xx)\, q[xx, yy, zz]}{\left((x - xx)^2 + (y - yy)^2 + (z - zz)^2\right)^{3/2}}, \ \frac{(y - yy)\, q[xx, yy, zz]}{\left((x - xx)^2 + (y - yy)^2 + (z - zz)^2\right)^{3/2}}, \right.$$
$$\left. \frac{(z - zz)\, q[xx, yy, zz]}{\left((x - xx)^2 + (y - yy)^2 + (z - zz)^2\right)^{3/2}} \right\}$$

Here is a plot of the 3D electric field for {xx, yy, zz} = {0, 0, 0} and q[xx, yy, zz] = 6:

```
{xx, yy, zz} = {0, 0, 0};
q[xx, yy, zz] = 6;
Show[Graphics3D[{Red, PointSize[0.06], Point[{xx, yy, zz}]}],
 Table[Arrow[ElectField[x, y, z], Tail → {x, y, z}, VectorColor → Red],
  {x, -3, 3, 2}, {y, -3, 3, 2}, {z, -3, 3, 2}]];
```
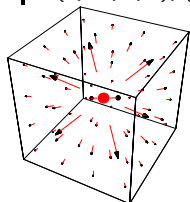

The field vectors point in the direction of greatest voltage drop because the electric field is the gradient of the negative voltage.

☐ **B.5.a)**

Measure the flux across the surface of the sphere
$$x^2 + y^2 + z^2 = 1$$
resulting from a single charge of strength 3 placed at {0, 0, 0}.
Then take any positive radius r and calculate the flux across the surface of the sphere
$$x^2 + y^2 + z^2 = r^2$$
of the electric field plotted above.
Discuss any noteworthy outcomes.

☐ **Answer:**

The flux of the electric field across a surface is just another name for the flow of the electric field across that surface.

Load in the electric field:

```
Clear[K, q, x, y, z, a];
R = {x, y, z} - {0, 0, 0};
Force = K q[x, y, z] q[0, 0, 0] R / (R . R)^(3/2)
{ K x q[0, 0, 0] q[x, y, z] / (x^2 + y^2 + z^2)^(3/2), K y q[0, 0, 0] q[x, y, z] / (x^2 + y^2 + z^2)^(3/2),
  K z q[0, 0, 0] q[x, y, z] / (x^2 + y^2 + z^2)^(3/2) }
q[0, 0, 0] = 3;
Clear[ElectField]
ElectField[x_, y_, z_] = Force /. {K → 1, q[x, y, z] → 1}
{ 3 x / (x^2 + y^2 + z^2)^(3/2), 3 y / (x^2 + y^2 + z^2)^(3/2), 3 z / (x^2 + y^2 + z^2)^(3/2) }
```

→ To measure the flux through $x^2 + y^2 + z^2 = 1$:

Use spherical coordinates for the sphere $x^2 + y^2 + z^2 = 1$:

```
Clear[x, y, z, r, s, t, normal]
r = 1;
{x[s_, t_], y[s_, t_], z[s_, t_]} =
 {r Sin[s] Cos[t], r Sin[s] Sin[t], r Cos[s]};
{{slow, shigh}, {tlow, thigh}} = {{0, π}, {0, 2 π}};
normal[s_, t_] = TrigExpand[
  D[{x[s, t], y[s, t], z[s, t]}, s] × D[{x[s, t], y[s, t], z[s, t]}, t]]
{Cos[t] Sin[s]^2, Sin[s]^2 Sin[t], Cos[s] Sin[s]}
```

You can see that these normals point out from the sphere because the third slot is positive for $0 < s < \frac{\pi}{2}$ and the third slot is negative for $\frac{\pi}{2} < s < \pi$.

Confirm with a plot:

```
Show[Table[Arrow[normal[s, t], Tail → {x[s, t], y[s, t], z[s, t]},
   VectorColor → Blue], {s, π/4, 3π/4, π/4}, {t, π/4, 7π/4, π/2}],
  ViewPoint → CMView, Axes → True, AxesLabel → {"x", "y", "z"}];
```
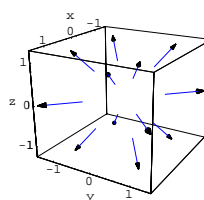

Good.

The flux of an electric field across a surface is just the flow of the same electric field across the surface.

See B.4) to learn why this is given by

$$\int\int_{\text{surface}} \text{ElectField . unitnormal}\, d\text{A}$$
$$= \int_{\text{slow}}^{\text{shigh}} \int_{\text{tlow}}^{\text{thigh}} \text{ElectField}[x[s, t], y[s, t], z[s, t]] \text{ . normal}[s, t]\, d\text{t}\, d\text{s}$$

where normal[s, t] is as calculated above.

```
You can't use Gauss's formula here because the electric field has a
singularity inside the sphere.
```

The flux is:

```
 ∫_slow^shigh ∫_tlow^thigh ElectField[x[s, t], y[s, t], z[s, t]] . normal[s, t] dt ds
12 π
```

The flux measurement is 12π flowing across the sphere from inside to outside.

→ To measure the flux through $x^2 + y^2 + z^2 = r^2$:

Use spherical coordinates for the sphere $x^2 + y^2 + z^2 = r^2$:

```
Clear[x, y, z, r, s, t, normal]
{x[s_, t_], y[s_, t_], z[s_, t_]} =
 {r Sin[s] Cos[t], r Sin[s] Sin[t], r Cos[s]};
{{slow, shigh}, {tlow, thigh}} = {{0, π}, {0, 2 π}};
normal[s_, t_] = TrigExpand[
  D[{x[s, t], y[s, t], z[s, t]}, s] × D[{x[s, t], y[s, t], z[s, t]}, t]];
flux =
 ∫_slow^shigh ∫_tlow^thigh ElectField[x[s, t], y[s, t], z[s, t]] . normal[s, t] dt ds
12 π √(r^2)
─────────
   r
```

This is the same as 12π because the r terms cancel out.

This reveals that the flux across the sphere is
$$12\pi$$
no matter what the radius is.

Think of it this way:

The charge at {0, 0, 0} continually sends out barrages of "bullets" in all directions. That the flux across any sphere
$$x^2 + y^2 + z^2 = r^2$$
is the same no mattter what r is means that at all times the same number of "bullets" passes through each of the spheres.

This also reflects the fact that this electric field has no sources or sinks other than the source at the singularity at {0, 0, 0}.

All this is a direct consequence of the inverse square law.

VC.10 3D Surface Measurements Tutorials

T.1) Measuring flow across surfaces:

Gauss's 3D formula versus calculation by surface integrals

□ **T.1.a.i) The situation suggests a calculation by Gauss's 3D formula**

Here's a surface:

```
Clear[x, y, z, r, s, t]
{x[r_, s_, t_], y[r_, s_, t_], z[r_, s_, t_]} =
 {0.6 r Sin[s] Cos[t], r Sin[s] Sin[t], 2 r Cos[s]};
{{slow, shigh}, {tlow, thigh}} = {{0, π}, {0, 2 π}};
Clear[surfaceplotter];
surfaceplotter[s_, t_] = {0, 1, 0} + {x[3, s, t], y[3, s, t], z[3, s, t]};
surface =
 ParametricPlot3D[Evaluate[surfaceplotter[s, t]], {s, slow, shigh},
  {t, tlow, thigh}, Boxed → False, BoxRatios → Automatic,
  ViewPoint → CMView, AxesLabel → {"x", "y", "z"}];
```
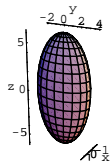

The surface is the skin of a solid egg.
Measure the flow of
$\quad$ Field[x, y, z] = $\{-4 x, y^2, 2 z\}$
across this surface.
Determine whether the net flow is from inside to outside or from outside to inside.

□ **Answer:**

This is a natural for Gauss's 3D formula because the surface is all of the outside skin of a solid region.

Enter the vector field:

```
Clear[x, y, z, m, n, p, Field]
{m[x_, y_, z_], n[x_, y_, z_], p[x_, y_, z_]} = {-4 x, y^2, 2 z};
Field[x_, y_, z_] = {m[x, y, z], n[x, y, z], p[x, y, z]}
{-4 x, y^2, 2 z}
```

Calculate divField[x, y, z]:

```
Clear[divField]
divField[x_, y_, z_] =
 D[m[x, y, z], x] + D[n[x, y, z], y] + D[p[x, y, z], z]
-2 + 2 y
```

Call the surface you see above C and call the solid consisting of all points inside and on the surface R. Gauss's formula tells you that flow of Field[x, y, z] across C is

$$\iiint_R \text{divField[x, y, z]} \, dx \, dy \, dz.$$

To calculate this integral, take advantage of the parameterization of the surface. Checking the plotting instructions above you see that R is described by

```
Clear[x, y, z, r, s, t]
{x[r_, s_, t_], y[r_, s_, t_], z[r_, s_, t_]} =
 {0.6 r Sin[s] Cos[t], r Sin[s] Sin[t], 2 r Cos[s]}
{0.6 r Cos[t] Sin[s], r Sin[s] Sin[t], 2 r Cos[s]}
```

With:

```
{{rlow, rhigh}, {slow, shigh}, {tlow, thigh}} =
 {{0, 3}, {0, π}, {0, 2 π}}
{{0, 3}, {0, π}, {0, 2 π}}
```

Yyou know that the net flow of Field[x, y, z] across C is measured by

$$\iiint_R \text{divField[x, y, z]} \, dx \, dy \, dz$$

$$= \int_{rlow}^{rhigh} \int_{slow}^{shigh} \int_{tlow}^{thigh} \text{divField[x[r, s, t], y[r, s, t], z[r, s, t]]} \, V_{xyz}[r, s, t] \, dt \, d$$

Unleash Mathematica:

```
Clear[gradx, grady, gradz, Vxyz]
gradx[r_, s_, t_] =
 {D[x[r, s, t], r], D[x[r, s, t], s], D[x[r, s, t], t]};
grady[r_, s_, t_] =
 {D[y[r, s, t], r], D[y[r, s, t], s], D[y[r, s, t], t]};
gradz[r_, s_, t_] =
 {D[z[r, s, t], r], D[z[r, s, t], s], D[z[r, s, t], t]};
Vxyz[r_, s_, t_] =
 TrigExpand[Det[{gradx[r, s, t], grady[r, s, t], gradz[r, s, t]}]]
```

$1.2 r^2 \text{Sin[s]} + 0. r^2 \text{Cos[s]}^2 \text{Sin[s]} +$
$0. r^2 \text{Cos[t]}^2 \text{Sin[s]} + 0. r^2 \text{Cos[s]}^2 \text{Cos[t]}^2 \text{Sin[s]} +$
$0. r^2 \text{Sin[s]}^3 + 0. r^2 \text{Cos[t]}^2 \text{Sin[s]}^3 + 0. r^2 \text{Cos[s] Cos[t] Sin[t]} +$
$0. r^2 \text{Cos[s]}^3 \text{Cos[t] Sin[t]} + 0. r^2 \text{Cos[s] Cos[t] Sin[s]}^2 \text{Sin[t]} +$
$0. r^2 \text{Sin[s] Sin[t]}^2 + 0. r^2 \text{Cos[s]}^2 \text{Sin[s] Sin[t]}^2 + 0. r^2 \text{Sin[s]}^3 \text{Sin[t]}^2$

Good, this never goes negative for the r, s, and t used here.

Here comes the measurement of the net flow of Field[x, y, z] across C

$$\iiint_R \text{divField[x, y, z]} \, dx \, dy \, dz$$

$$= \int_{rlow}^{rhigh} \int_{slow}^{shigh} \int_{tlow}^{thigh} \text{divField[x[r, s, t], y[r, s, t], z[r, s, t]]} \, V_{xyz}[r, s, t] \, dt \, d$$

```
∫_rlow^rhigh ∫_slow^shigh ∫_tlow^thigh  divField[
    x[r, s, t], y[r, s, t], z[r, s, t]] Vxyz[r, s, t] dt ds
  dr
-271.434
```

Negative.

Because Gauss's 3D formula says

$$\iiint_R \text{divField[x, y, z]} \, dx \, dy \, dz$$

$$= \iint_R \text{Field . outerunitnormal} \, dA$$

$$= \text{net flow of Field[x, y, z] across C},$$

this negative result tells you that the net flow of this vector field is against the outernormals.

The bottom line:
The net flow of this vector field across this surface is from outside to inside. There must be lots of sinks inside the surface.

□ **T.1.a.ii)**

Go with the same surface as in part i) above, but this time measure the flow of the 3D vector field
$\quad$ Field[x, y, z] = $\{x + \text{Sin[y]}, y + 4 e^{-x}, -2 z + x^3\}$
across this surface.
Determine whether the net flow across the surface is from inside to outside or from outside to inside.

□ **Answer:**

Enter the vector field:

```
Clear[x, y, z, m, n, p, Field]
{m[x_, y_, z_], n[x_, y_, z_], p[x_, y_, z_]} =
 {x + Sin[y], y + 4 E^-z, -2 z + x^3};
Field[x_, y_, z_] = {m[x, y, z], n[x, y, z], p[x, y, z]}
{x + Sin[y], 4 E^-z + y, x^3 - 2 z}
```

Calculate divField[x, y, z]:

```
Clear[divField]
divField[x_, y_, z_] =
 D[m[x, y, z], x] + D[n[x, y, z], y] + D[p[x, y, z], z]
0
```

No sources or sinks.

The net flow of this vector field across this surface is 0. In fact the net flow of this vector field across any surface that is the whole skin of a solid region is 0. The outside-to-inside flow is exactly balanced by the inside-to-outside flow.
You could've done this one by hand.

□T.1.a.iii)

Go with the same surface as in part i) above, but this time measure the flow of the 3D vector field

Field[x, y, z] = {x + Sin[y], y + 4 e⁻ˣ, −5 z + x³}

across this surface.

Determine whether the net flow is from inside to outside or from outside to inside.

□Answer:

Enter the vector field:

```
Clear[x, y, z, m, n, p, Field]
{m[x_, y_, z_], n[x_, y_, z_], p[x_, y_, z_]} =
 {x + Sin[y], y + 4 E^-z, -5 z + x^3};
Field[x_, y_, z_] = {m[x, y, z], n[x, y, z], p[x, y, z]}
```
$\{x + \text{Sin}[y], 4 E^{-z} + y, x^3 - 5 z\}$

Calculate divField[x, y, z]:

```
Clear[divField]
divField[x_, y_, z_] =
 D[m[x, y, z], x] + D[n[x, y, z], y] + D[p[x, y, z], z]
```
-3

Negative; every point is a sink.

The net flow of this vector field across this surface is from outside to inside. In fact, the net flow of this vector field across any surface that is the whole skin of a solid region is from outside to inside.

You could've done this one by hand.

□T.1.b) The situation dictates a direct calculation by a surface integral

Here's a surface:

```
Clear[x, y, z, u, v]
x[u_, v_] = 3 v;
y[u_, v_] = -2 u v;
z[u_, v_] = 3 u - v;
{ulow, uhigh} = {0, 2};
{vlow, vhigh} = {0, 1.3};
```

```
surface = ParametricPlot3D[Evaluate[{x[u, v], y[u, v], z[u, v]}],
  {u, ulow, uhigh}, {v, vlow, vhigh}, ViewPoint → CMView,
  AxesLabel → {"x", " y", "z"}];
```
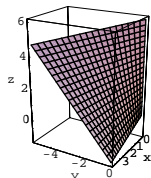

Go with

f[x, y, z] = ((x + 4) (y + 3) z)²,

and measure the flow of the gradient field of f[x, y, z] across this surface.

□Answer:

You can't use Gauss's formula because this surface is not the skin of a solid region, so you've got to go with a direct calculation via the surface integral.

Enter the function and its gradient field:

```
Clear[f, x, y, z, gradf]
f[x_, y_, z_] = ((x + 4) (y + 3) z)^2;
gradf[x_, y_, z_] = {D[f[x, y, z], x], D[f[x, y, z], y], D[f[x, y, z], z]}
```
$\{2 (4 + x) (3 + y)^2 z^2, 2 (4 + x)^2 (3 + y) z^2, 2 (4 + x)^2 (3 + y)^2 z\}$

Enter the vector field:

```
Clear[x, y, z, m, n, p, Field]
{m[x_, y_, z_], n[x_, y_, z_], p[x_, y_, z_]} = gradf[x, y, z];
Field[x_, y_, z_] = {m[x, y, z], n[x, y, z], p[x, y, z]}
```
$\{2 (4 + x) (3 + y)^2 z^2, 2 (4 + x)^2 (3 + y) z^2, 2 (4 + x)^2 (3 + y)^2 z\}$

Reenter the parameterization of the surface and calculate normal[u, v]:

```
Clear[x, y, z, u, v]
x[u_, v_] = 3 v;
y[u_, v_] = -2 u v;
z[u_, v_] = 3 u - v;
{ulow, uhigh} = {0, 2};
{vlow, vhigh} = {0, 1.3};
```

```
Clear[normal]
normal[u_, v_] =
 D[{x[u, v], y[u, v], z[u, v]}, u] × D[{x[u, v], y[u, v], z[u, v]}, v]
```
$\{6 u + 2 v, 9, 6 v\}$

Calculate

$\int_{ulow}^{uhigh} \int_{vlow}^{vhigh} \text{Field}[x[u, v], y[u, v], z[u, v]] \, . \, \text{normal}[u, v] \, dv \, du$:

```
flowacross =
 ∫_ulow^uhigh ∫_vlow^vhigh Field[x[u, v], y[u, v], z[u, v]] . normal[u, v] dv du;
N[flowacross]
```
19086.8

Strongly positive.

To interpret this, look at:

```
{u, ulow, uhigh}
```
$\{u, 0, 2\}$
```
{v, vlow, vhigh}
```
$\{v, 0, 1.3\}$
```
scalefactor = 0.3;
normals = Table[Arrow[normal[u, v], Tail → {x[u, v], y[u, v], z[u, v]},
  ScaleFactor → scalefactor], {u, 0.5, 1.5, 1}, {v, 0.2, 1.0, 0.8}];
```

```
Show[surface, normals, ViewPoint → CMView, Boxed → False,
 PlotRange → All, AxesLabel → {"x", "y", "z"}];
```
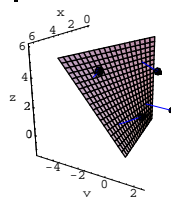

The normals are pointing out from the side of the surface you are sitting on.

Because the flow measurement is hugely positive, this tells you that the net flow of gradf[x, y, z] across this surface is strongly from the side you can't see to the side you can see.

T.2) Using Gauss's formula to avoid a calculational nightmare:

**Calculating flow across an oddball surface by calculating the flow across a substitute surface**

C_1 is the top half of a sphere of radius 2 centered at the origin:

```
Clear[x1, y1, z1, s, t]
x1[s_, t_] = 2 Sin[s] Cos[t];
y1[s_, t_] = 2 Sin[s] Sin[t];
z1[s_, t_] = 2 Cos[s];
{{slow, shigh}, {tlow, thigh}} = {{0, π/2}, {0, 2 π}};
C1plot = ParametricPlot3D[{x1[s, t], y1[s, t], z1[s, t]},
  {s, slow, shigh}, {t, tlow, thigh}, Boxed → False,
  ViewPoint → CMView, AxesLabel → {"x", "y", "z"}, PlotRange → All];
```
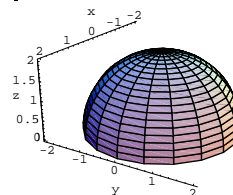

C_2 is an oddball surface that fits under C_1 but agrees with C_1 on their common boundary curve.

```
Clear[x2, y2, z2, r, t]
x2[r_, t_] = r Cos[t];
y2[r_, t_] = r Sin[t];
z2[r_, t_] = 0.2 r Sin[t/2]^2 (E^√(4-r²) - 1);
{{rlow, rhigh}, {tlow, thigh}} = {{0, 2}, {0, 2 π}};
C2plot = ParametricPlot3D[{x2[r, t], y2[r, t], z2[r, t]},
  {r, rlow, rhigh}, {t, tlow, thigh}, Boxed → False,
  ViewPoint → CMView, AxesLabel → {"x", "y", "z"}, PlotRange → All];
```
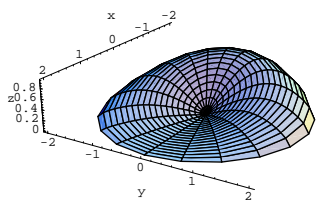
Take a look at the surfaces together from underneath:

```
Show[Clplot, C2plot, ViewPoint → {1, 1, -1}];
```
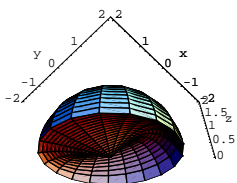

Together C_1 and C_2 make the skin of a solid region.

□**T.2.a.i)**

How do you know that when you go with a 3D vector field
Field[x, y, z] with divField[x, y, z] = 0 throughout the solid region
whose top skin is C_1 and whose bottom skin is C_2, then
$$\iint_{C_1} \text{Field . topunitnormal } d\,A$$
$$= \iint_{C_2} \text{Field . topunitnormal } d\,A,$$
so that the flow of Field[x, y, z] across both surfaces is the same?

□**Answer:**

Remember that C_1 and C_2 share a common boundary curve.

Make a solid region R whose top skin is C_1 and whose bottom skin is
C_2. Agree that C stands for the skin of R.

You are armed with the fact that divField[x, y, z] = 0 throughout R.

Use this fact and Gauss's formula to see that

$$0 = \iiint_R 0 \, d\,x \, d\,y \, d\,z$$
$$= \iiint_R \text{divField[x, y, z] } d\,x \, d\,y \, d\,z$$
$$= \iint_C \text{Field . outerunitnormal } d\,A$$
$$= \iint_{C_1} \text{Field . topunitnormal } d\,A - \iint_{C_2} \text{Field . topunitnormal } d\,A,$$

because along C_1, the outer unit normal of C agrees with the top unit
normal of C_1, but along C_2 the outer unit normal of C agrees with the
negative of the top unit normal of C_2.

The upshot:
$$\iint_{C_1} \text{Field . topunitnormal } d\,A = \iint_{C_2} \text{Field . topunitnormal } d\,A.$$
Accordingly,

flow of Field[x, y, z] across C_1
$$= \iint_{C_1} \text{Field . topunitnormal } d\,A$$
$$= \iint_{C_2} \text{Field . topunitnormal } d\,A$$
$$= \text{flow of Field[x, y, z] across } C_2.$$

□**T.2.a.ii)**

What calculational nightmare does this help you to avoid?

□**Answer:**

Because the parameterization of C involves screwy functions like
$$z2[r, t] = 0.2 \, r \, \text{Sin}[\tfrac{t}{2}] \left(e^{\sqrt{4-r^2}} - 1 \right),$$
calculating
$$\iint_{C_1} \text{Field . topunitnormal } d\,A$$
will probably be a nightmare. Its normals make you want to vomit.

Take a look:

```
Clear[normal2]
normal2[r_, t_] = D[{x2[r, t], y2[r, t], z2[r, t]}, r] ×
   D[{x2[r, t], y2[r, t], z2[r, t]}, t]
```

$$\{0.2 \, r \, \text{Cos[t] Sin}\left[\tfrac{t}{2}\right]^2 - 0.2 \, E^{\sqrt{4-r^2}} \, r \, \text{Cos[t] Sin}\left[\tfrac{t}{2}\right]^2 +$$
$$\frac{0.2 \, E^{\sqrt{4-r^2}} \, r^3 \, \text{Cos[t] Sin}\left[\tfrac{t}{2}\right]^2}{\sqrt{4-r^2}} - 0.2 \, r \, \text{Cos}\left[\tfrac{t}{2}\right] \text{Sin}\left[\tfrac{t}{2}\right] \text{Sin[t]} +$$
$$0.2 \, E^{\sqrt{4-r^2}} \, r \, \text{Cos}\left[\tfrac{t}{2}\right] \text{Sin}\left[\tfrac{t}{2}\right] \text{Sin[t]}, \, 0.2 \, r \, \text{Cos}\left[\tfrac{t}{2}\right] \text{Cos[t] Sin}\left[\tfrac{t}{2}\right] -$$
$$0.2 \, E^{\sqrt{4-r^2}} \, r \, \text{Cos}\left[\tfrac{t}{2}\right] \text{Cos[t] Sin}\left[\tfrac{t}{2}\right] + 0.2 \, r \, \text{Sin}\left[\tfrac{t}{2}\right]^2 \text{Sin[t]} -$$
$$0.2 \, E^{\sqrt{4-r^2}} \, r \, \text{Sin}\left[\tfrac{t}{2}\right]^2 \text{Sin[t]} + \frac{0.2 \, E^{\sqrt{4-r^2}} \, r^3 \, \text{Sin}\left[\tfrac{t}{2}\right]^2 \text{Sin[t]}}{\sqrt{4-r^2}},$$
$$r \, \text{Cos[t]}^2 + r \, \text{Sin[t]}^2\}$$

These normals will make you choke and they will probably make
Mathematica choke too.

But you know that

flow of Field[x, y, z] across C_1
$$= \iint_{C_1} \text{Field . topunitnormal } d\,A$$
$$= \iint_{C_2} \text{Field . topunitnormal } d\,A$$
$$= \text{flow of Field[x, y, z] across } C_2,$$

so instead of confronting the nightmarish calculation of
$$\iint_{C_2} \text{Field . topunitnormal } d\,A,$$
you can opt for the simpler calculation of
$$\iint_{C_1} \text{Field . topunitnormal } d\,A.$$

□**T.2.a.iii)**

Illustrate this idea by calculating the flow of the 3D vector field
$$\text{Field[x, y, z]} = \{x - y, -2y - x, z + 3\}$$
across C_2.

□**Answer:**

Enter the vector field:

```
Clear[x, y, z, m, n, p, Field]
{m[x_, y_, z_], n[x_, y_, z_], p[x_, y_, z_]} = {x - y, -2 y - x, z + 3};
Field[x_, y_, z_] = {m[x, y, z], n[x, y, z], p[x, y, z]}
```

$$\{x - y, -x - 2y, 3 + z\}$$

Check divField[x, y, z]:

```
Clear[divField]
divField[x_, y_, z_] =
  D[m[x, y, z], x] + D[n[x, y, z], y] + D[p[x, y, z], z]
```
$$0$$

Good.

Now you know you can measure the flow of this vector field across the
squirrely surface C_2 by measuring the flow of this vector field across
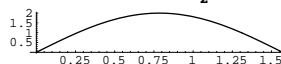the clean surface C_1.

Here you go:
$$\iint_{C_1} \text{Field . topunitnormal } d\,A$$
$$= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \text{Field[x1[s, t], y1[s, t], z1[s, t]] . normal1[s, t] } d\,s \, d\,t$$
where normal1[s, t] is:

```
Clear[normal1]
normal1[s_, t_] = TrigExpand[D[{x1[s, t], y1[s, t], z1[s, t]}, s] ×
   D[{x1[s, t], y1[s, t], z1[s, t]}, t]]
```
$$\{4 \, \text{Cos[t] Sin[s]}^2, \, 4 \, \text{Sin[s]}^2 \text{Sin[t]}, \, 4 \, \text{Cos[s] Sin[s]}\}$$

You can see that normal1[s, t] is an top normal by looking at the third
slot, which is 2 Sin[2 s]. In the parameterization of C_1, s runs from 0 to
$\frac{\pi}{2}$ and 2 Sin[2 s] remains nonnegative for $0 \le s \le \frac{\pi}{2}$:

```
Plot[2 Sin[2 s], {s, 0, π/2}, AspectRatio → 1/(2 π)];
```


Now you're ready to calculate:

flow of Field[x, y, z] across C_2
$$= \text{flow of Field[x, y, z] across } C_1$$

$$= \int\int_{C_1} \text{Field . topunitnormal } dA$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \text{Field[x1[s, t], y1[s, t], z1[s, t]] . normal1[s, t] } ds\, dt$$

```
∫₀²π ∫₀^(π/2) Field[x1[s, t], y1[s, t], z1[s, t]] . normal1[s, t] ds dt
```
12 π

Strong flow from low to high across C_2 (as well as C_1).

The direct calculation of the flow of this field across C_2 is not practical.

Gauss was a master at avoiding calculational nightmares.

For a story about Gauss as young kid, click the box.
Once when Gauss was in grade school, he misbehaved in class. As
punishment his teacher told him to add the consecutive numbers

$$1 + 2 + 3 + 4 + 5 + \ldots + 999 + 1000.$$

Young Gauss immediately came back with the answer:

```
Clear[k]
   1000
    ∑   k
   k=1
```
500500

The way he did it was to write:

$$\text{sum} = \quad 1 + \quad 2 + \quad 3 + \ldots + 999 + 1000$$
$$\text{sum} = 1000 + 999 + 998 + \ldots + \quad 2 + \quad 1.$$

He added vertically to get

$$2\,\text{sum} = 1001 + 1001 + 1001 + \ldots + 1001 + 1001$$

$$= 1000\,(1001)$$

because there are 1000 terms.

Consequently
$$\text{sum} = \tfrac{1000\,(1001)}{2} = 500\,(1001) = 500500.$$

Gauss was a genius.

T.3) Using Gauss's formula to take advantage of singularities: Calculating flow across the skin of a solid region by calculating the flow across a substitute sphere

Here's an elliptical cylinder with top and bottom:

```
radial1 = {1, -2, -1};
radial2 = {1, 1, -1};
core = {0.5, 2, 2};
Clear[sideplotter, topplotter, bottomplottter, r, u, t]
sideplotter[u_, t_] = Cos[t] radial1 + Sin[t] radial2 + u core;
{{ulow, uhigh}, {tlow, thigh}} = {{-1, 2}, {0, 2 π}};
topplotter[r_, t_] = r Cos[t] radial1 + r Sin[t] radial2 + uhigh core;
bottomplotter[r_, t_] = r Cos[t] radial1 + r Sin[t] radial2 + ulow core;
{rlow, rhigh} = {0, 1};
sides = ParametricPlot3D[
  Evaluate[sideplotter[u, t]], {u, ulow, uhigh}, {t, tlow, thigh},
  PlotPoints → {2, Automatic}, DisplayFunction → Identity];
top = ParametricPlot3D[
  Evaluate[topplotter[r, t]], {r, rlow, rhigh}, {t, tlow, thigh},
  PlotPoints → {2, Automatic}, DisplayFunction → Identity];
bottom = ParametricPlot3D[
  Evaluate[bottomplotter[r, t]], {r, rlow, rhigh}, {t, tlow, thigh},
  PlotPoints → {2, Automatic}, DisplayFunction → Identity];

Show[sides, top, bottom, Boxed → False,
 ViewPoint → CMView, AxesLabel → {"x", "y", "z"}, PlotRange → All,
 DisplayFunction → $DisplayFunction];
```
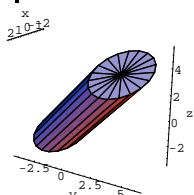

Go with the electric field
$$\text{Field[x, y, z]} = \left\{ \tfrac{2x}{(x^2+y^2+z^2)^{3/2}}, \tfrac{2y}{(x^2+y^2+z^2)^{3/2}}, \tfrac{2z}{(x^2+y^2+z^2)^{3/2}} \right\}$$
resulting from a charge of strength 2 placed at the point $\{0, 0, 0\}$ which is inside this skin.

□**T.3.a)**

Try to use Gauss's 3D formula to calculate the flow of this vector field across the skin of the surface plotted above.

□**Answer:**

Enter the vector field:

```
Clear[x, y, z, m, n, p, Field]
{m[x_, y_, z_], n[x_, y_, z_], p[x_, y_, z_]} =
  {   2 x            2 y            2 z       };
   ─────────── , ─────────── , ───────────
   (x²+y²+z²)^(3/2) (x²+y²+z²)^(3/2) (x²+y²+z²)^(3/2)
Field[x_, y_, z_] = {m[x, y, z], n[x, y, z], p[x, y, z]}
{   2 x            2 y            2 z       }
 ─────────── , ─────────── , ───────────
 (x²+y²+z²)^(3/2) (x²+y²+z²)^(3/2) (x²+y²+z²)^(3/2)
```

Calculate divField[x, y, z]:

```
Clear[divField]
divField[x_, y_, z_] =
 Together[D[m[x, y, z], x] + D[n[x, y, z], y] + D[p[x, y, z], z]]
0
```

Good; Gauss's 3D formula tells you if you let R stand for the solid of which the plotted surface above is the outside skin, then you are guaranteed that the flow of this vector field across this surface is given by

$$\int\int\int_R \text{divField[x, y, z] } dx\, dy\, dz$$
$$= \int\int\int_R 0 \, dx\, dy\, dz = 0.$$

You happily report that the flow of Field[x, y, z] across the skin plotted above is 0 and go on to the next problem.

□**T.3.b)**

Was the answer given in part a) correct?

□**Answer:**

No way.

□**T.3.c.i)**

What went wrong?

□**Answer:**

Look at:

```
Field[0, 0, 0]
{Indeterminate, Indeterminate, Indeterminate}
```

Field[x, y, z] goes nuts at $\{0, 0, 0\}$.

In short, $\{0, 0, 0\}$ is a singularity of Field[x, y, z], and the damage comes from the fact that $\{0, 0, 0\}$ is inside the plotted surface above.

Anytime you have a singularity on or in a solid region R, Gauss's formula

$$\int\int\int_R \text{divField[x, y, z] } dx\, dy\, dz$$
$$= \text{flow of Field[x, y, z] across C}$$

has the possibility of failing.

And it did fail in the "answer" given in part a).

In the absence of singularities of Field[x, y, z]
or (of divField[x, y, z]) inside
or on the surface of a solid region R,
Gauss's formula cannot fail.

□**T.3.c.ii)**

Are there other singularities?

□**Answer:**

Look at:

```
Field[x, y, z]
{   2 x            2 y            2 z       }
 ─────────── , ─────────── , ───────────
 (x²+y²+z²)^(3/2) (x²+y²+z²)^(3/2) (x²+y²+z²)^(3/2)
```

A short examination of this formula for Field[x, y, z] shows that the only point at which the denominators are 0 is {0, 0, 0}. There are no other singularities.

□**T.3.d)**

One way to try to do the calculation of the flow of Field[x, y, z] across this surface is to let C stand for the whole surface and to try to do a direct calculation of

$$\int\int_C \text{Field[x, y, z]} . \text{outerunitnormal} \, d\,A$$

by breaking it up in the form:

$$\int\int_{C_{sides}} \text{Field[x, y, z]} . \text{outerunitnormal} \, d\,A$$
$$+ \int\int_{C_{top}} \text{Field[x, y, z]} . \text{outerunitnormal} \, d\,A$$
$$+ \int\int_{C_{bottom}} \text{Field[x, y, z]} . \text{outerunitnormal} \, d\,A.$$

This is a calculational nightmare because it involves setting up three different normal vectors and then hoping that Mathematica can do the resulting integrals.

Part of the art of mathematics is knowing how to avoid calculational nightmares.

How can you avoid this nightmare?

□**Answer:**

You center a little sphere called R_{little} at the lone singularity:

```
singularity = {0, 0, 0};
Clear[xlittle, ylittle, zlittle, s, t, littlesphereplotter]
littleradius = 0.1;
{xlittle[s_, t_], ylittle[s_, t_], zlittle[s_, t_]} =
 singularity + {littleradius Sin[s] Cos[t],
   littleradius Sin[s] Sin[t], littleradius Cos[s]};
littlesphereplot = ParametricPlot3D[
  Evaluate[{xlittle[s, t], ylittle[s, t], zlittle[s, t]}], {s, 0, π},
  {t, 0, 2 π}, Boxed → False, ViewPoint → CMView, PlotRange → All];
```
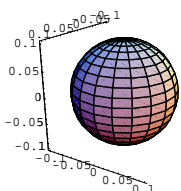

The singularity at {0, 0, 0} is encapsulated inside the little sphere, and the little sphere is inside the original surface.

Now you make a new hollow solid, taking everything inside the original skin C, but rejecting everything inside the skin, C_{little}, of the little sphere. Let R_{new} stand for the new hollow solid and let C_{new} stand for its skin. Note that there are no singularities inside R_{new} because the lone singularity lies inside the hollow part. Apply Gauss's formula to the new hollow solid R_{new} to see that

$$\int\int_{C_{new}} \text{Field[x, y, z]} . \text{outerunitnormal} \, d\,A$$
$$= \int\int\int_{R_{new}} \text{divField[x, y, z]} \, d\,x \, d\,y \, d\,z$$
$$= 0$$

because divField[x, y, z] = 0 throughout R_{new}.

This tells you that

$$\int\int_{C_{new}} \text{Field[x, y, z]} . \text{outerunitnormal} \, d\,A = 0.$$

But if you let C_{little} stand for the skin of the little sphere centered at the singularity, and you take the outer unit normal from the little sphere, then you get

$$0 = \int\int_{C_{new}} \text{Field[x, y, z]} . \text{outerunitnormal} \, d\,A$$

$$= \int\int_C \text{Field[x, y, z]} . \text{outerunitnormal} \, d\,A$$
$$- \int\int_{C_{little}} \text{Field[x, y, z]} . \text{outerunitnormal} \, d\,A$$

because the outer unit normal on the new solid points inside the little sphere.

This is big news because it tells you that

$$\int\int_C \text{Field[x, y, z]} . \text{outerunitnormal} \, d\,A$$
$$= \int\int_{C_{little}} \text{Field[x, y, z]} . \text{outerunitnormal} \, d\,A$$

As a result, you can avoid the calculation of the gruesome integral

$$\int\int_C \text{Field[x, y, z]} . \text{outerunitnormal} \, d\,A$$

by calculating one single, easy integral:

$$\int\int_{C_{little}} \text{Field[x, y, z]} . \text{outerunitnormal} \, d\,A$$
$$= \int_0^{2\pi} \int_0^\pi \text{Field[x}_{little}, \text{y}_{little}, \text{z}_{little}] . \text{normal[s, t]} \, d\,s \, d\,t$$

where normal[s, t] is given by:

```
Clear[normal]
normal[s_, t_] =
 TrigExpand[D[{xlittle[s, t], ylittle[s, t], zlittle[s, t]}, s] ×
   D[{xlittle[s, t], ylittle[s, t], zlittle[s, t]}, t]]
{0.01 Cos[t] Sin[s]², 0.01 Sin[s]² Sin[t], 0.01 Cos[s] Sin[s] +
  0. Cos[s] Cos[t]² Sin[s] + 0. Cos[t] Sin[t] + 0. Cos[s]² Cos[t] Sin[t] +
  0. Cos[t] Sin[s]² Sin[t] + 0. Cos[s] Sin[s] Sin[t]²}
```

Make sure that these normals are outer normals:

```
scalefactor = 5;
Show[littlesphereplot, Table[Arrow[normal[s, t],
  Tail → {xlittle[s, t], ylittle[s, t], zlittle[s, t]},
  ScaleFactor → scalefactor],
  {s, π/4, 3π/4, π/4}, {t, 0, 2 π, π/2}], Boxed → False,
 ViewPoint → CMView, PlotRange → All];
```
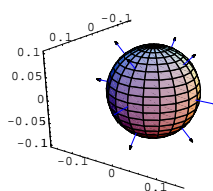

They are outer normals.

Now you can correctly calculate the flow of Field[x, y, z] across the original cylindrical surface in one short, sweet calculation:

```
∫₀^π ∫₀^{2π} Field[xlittle[s, t], ylittle[s, t], zlittle[s, t]] . normal[s, t]
  dt ds
25.1327
```

And you're out of here.

T.4) Combined electric fields and multiple singularities

If you understand B.5) and T.3), then you understand why it is that when you take a charge of strength q and place it at a point {a, b, c} inside the whole skin of a solid region R, then the flow of the resulting electric field across the whole skin is the same as its flow across any small sphere centered at {a, b, c}.

This calculates out to:

```
Clear[x, y, z, a, b, c, Field]
Field[x_, y_, z_] = {q (x - a), q (y - b), q (z - c)} / ((x - a)² + (y - b)² + (z - c)²)^(3/2);
Clear[x, y, z, r, s, t, q, normal]
center = {a, b, c};
{x[s_, t_], y[s_, t_], z[s_, t_]} =
 center + {r Sin[s] Cos[t], r Sin[s] Sin[t], r Cos[s]};
{{slow, shigh}, {tlow, thigh}} = {{0, π}, {0, 2π}};
normal[s_, t_] = TrigExpand[
  D[{x[s, t], y[s, t], z[s, t]}, s] ×D[{x[s, t], y[s, t], z[s, t]}, t]];
∫slow^shigh ∫tlow^thigh Field[x[s, t], y[s, t], z[s, t]] . normal[s, t] dt ds
```

$$\frac{4 \pi q \sqrt{r^2}}{r}$$

This is the same as $4\pi q$.

The upshot:

If you place a charge of strength q inside the skin of a solid region, then flow of the electric field across this skin is $4\pi q$.

If q > 0, the flow is from inside to outside, and if q < 0, the flow is from outside to inside.

□T.4.a)

Take any skin of any solid region you like.

Here is the electric field resulting from a charge of strength q_1 placed at a point {a, b, c} inside this skin.

```
Clear[a, b, c, q1, x, y, z, Field1]
point1 = {a, b, c};
Field1[x_, y_, z_] = q1 ({x, y, z} - point1) / ((x - a)² + (y - b)² + (z - c)²)^(3/2)
```

$$\left\{ \frac{q1 \, (-a + x)}{((-a+x)^2 + (-b+y)^2 + (-c+z)^2)^{3/2}}, \frac{q1 \, (-b + y)}{((-a+x)^2 + (-b+y)^2 + (-c+z)^2)^{3/2}}, \frac{q1 \, (-c + z)}{((-a+x)^2 + (-b+y)^2 + (-c+z)^2)^{3/2}} \right\}$$

Here is the electric field resulting from a charge of strength q_2 placed at a different point {aa, bb, cc} inside the same skin.

```
Clear[aa, bb, cc, Field2]
point2 = {aa, bb, cc};
Field2[x_, y_, z_] = q2 ({x, y, z} - point2) / ((x - aa)² + (y - bb)² + (z - cc)²)^(3/2)
```

$$\left\{ \frac{q2 \, (-aa + x)}{((-aa+x)^2 + (-bb+y)^2 + (-cc+z)^2)^{3/2}}, \frac{q2 \, (-bb + y)}{((-aa+x)^2 + (-bb+y)^2 + (-cc+z)^2)^{3/2}}, \frac{q2 \, (-cc + z)}{((-aa+x)^2 + (-bb+y)^2 + (-cc+z)^2)^{3/2}} \right\}$$

Measure the flow of the combined electric field
    Field1[x, y, z] + Field2[x, y, z]
across this skin.

□Answer:

Again you try to practice the art of mathematics in an effort to avoid a calculational nightmare.

Check the divergence of Field1[x, y, z], Field2[x, y, z], and

Field1[x, y, z] + Field2[x, y, z]:

```
Clear[m1, n1, p1, divF1]
{m1[x_, y_, z_], n1[x_, y_, z_], p1[x_, y_, z_]} = Field1[x, y, z];
divFie1d1[x_, y_, z_] =
 Together[D[m1[x, y, z], x] + D[n1[x, y, z], y] + D[p1[x, y, z], z]]
0
Clear[m2, n2, p2, divField2]
{m2[x_, y_, z_], n2[x_, y_, z_], p2[x_, y_, z_]} = Field2[x, y, z];
divField2[x_, y_, z_] =
 Together[D[m2[x, y, z], x] + D[n2[x, y, z], y] + D[p2[x, y, z], z]]
0
Clear[m, n, p, divField1plusField2]
{m[x_, y_, z_], n[x_, y_, z_], p[x_, y_, z_]} =
 Field1[x, y, z] + Field2[x, y, z];
divField1plusField2[x_, y_, z_] =
 Together[D[m[x, y, z], x] + D[n[x, y, z], y] + D[p[x, y, z], z]]
0
```

Good!

→ divField1[x, y, z] = 0 at all points inside the skin except at its

singularity at {a, b, c};

→ divField2[x, y, z] = 0 at all points inside the skin except at its

singularity at {aa, bb, cc};

→ The divergence of the combined electric field

    Field1[x, y, z] + Field2[x, y, z]

is 0 at all points inside the skin except at the two singularities at

{a, b, c} and {aa, bb, cc}.

Now on to the calculation:

Center a little sphere C_1 at the singularity at {a, b, c} and center another

little sphere C_2 at the singularity at {aa, bb, cc}. If you understand

what happened in T.3), then you'll understand that the flow of

    Field1[x, y, z] + Field2[x, y, z]

across the skin is the same as the sum

$$\iint_{C_1} (\text{Field1} + \text{Field2}) \, . \, \text{outerunitnormal} \, dA$$
$$+ \iint_{C_2} (\text{Field1} + \text{Field2}) \, . \, \text{outerunitnormal} \, dA$$
$$= \iint_{C_1} \text{Field1} \, . \, \text{outerunitnormal} \, dA$$
$$+ \iint_{C_1} \text{Field2} \, . \, \text{outerunitnormal} \, dA$$
$$+ \iint_{C_2} \text{Field1} \, . \, \text{outerunitnormal} \, dA$$
$$+ \iint_{C_2} \text{Field2} \, . \, \text{outerunitnormal} \, dA$$

Now, because divField1[x, y, z] = 0 inside C_2, and because

Field1[x, y, z] has no singularity inside C_2, you know that

$$\iint_{C_2} \text{Field1} \, . \, \text{outerunitnormal} \, dA = 0.$$

Similarly,

$$\iint_{C_1} \text{Field2} \, . \, \text{outerunitnormal} \, dA = 0.$$

So the flow of the combined electric field

Field1[x, y, z] + Field2[x, y, z] across the original skin reduces to

$$\iint_{C_1} \text{Field1} \, . \, \text{outerunitnormal} \, dA$$
$$+ \iint_{C_2} \text{Field2} \, . \, \text{outerunitnormal} \, dA$$

The calculation done at the very beginning tells you that this is just

$$4 q_1 \pi + 4 q_2 \pi = 4 \pi (q1 + q2).$$

By using your head instead of the machine, you side-stepped another

gruesome calculation.

By the way, electrical folks and physicists really get off on this stuff.

You should like it too because it is clean, useful mathematics.

T.5) Surface packaging: Parameterized, explicit and implicit

If you have a surface to work with, the best way it can come into your
hands is in parameterized form
    {x[u, v], y[u, v], z[u, v]},
where {u, v} varies in a region R_{uv}. In this form:
→ You can plot it with ParametricPlot3D.
→ You can measure its surface area with the integral
    $$\iint_{R_{uv}} SA_{xyz}[u, v] \, du \, dv$$
where $SA_{xyz}[u, v]$ is given by:

```
Clear[x, y, z, u, v, cross, SAxyz]
cross[u_, v_] =
 D[{x[u, v], y[u, v], z[u, v]}, u] ×D[{x[u, v], y[u, v], z[u, v]}, v];
SAxyz[u_, v_] = √cross[u, v] . cross[u, v]
```

$$\sqrt{ \left((y^{(0,1)}[u, v] \, x^{(1,0)}[u, v] - x^{(0,1)}[u, v] \, y^{(1,0)}[u, v])^2 + (-z^{(0,1)}[u, v] \, x^{(1,0)}[u, v] + x^{(0,1)}[u, v] \, z^{(1,0)}[u, v])^2 + (z^{(0,1)}[u, v] \, y^{(1,0)}[u, v] - y^{(0,1)}[u, v] \, z^{(1,0)}[u, v])^2 \right) }$$

→ You can integrate a function f[x, y, z] with respect to surface area
on this surface by means of the integral
    $$\iint_{\text{surface}} f[x, y, z] \, dA$$
    $$= \iint_{R_{uv}} f[x[u, v], y[u, v], z[u, v]] \, SA_{xyz}[u, v] \, du \, dv.$$
→ You can measure the flow of a given vector field Field[x, y, z]

across the surface by means of the integral

$\int\int_{\text{surface}}$ Field . unitnormal dA

$= \int\int_{R_{uv}}$ Field[x[u, v], y[u, v], z[u, v]] . normal du dv.

where normal[u, v] is given by:

```
Clear[normal]
normal[u_, v_] =
D[{x[u, v], y[u, v], z[u, v]}, u] × D[{x[u, v], y[u, v], z[u, v]}, v]
```
$\{z^{(0,1)}[u, v] y^{(1,0)}[u, v] - y^{(0,1)}[u, v] z^{(1,0)}[u, v],$
$-z^{(0,1)}[u, v] x^{(1,0)}[u, v] + x^{(0,1)}[u, v] z^{(1,0)}[u, v],$
$y^{(0,1)}[u, v] x^{(1,0)}[u, v] - x^{(0,1)}[u, v] y^{(1,0)}[u, v]\}$

If you have the surface already parameterized, you can go right to work using these formulas to make measurements like a bandit. But often, the surface is not given to you in parametric form. When this happens, you can be in the lucky or the (often) unlucky case.

> The rain, it rains on the just,
> and the unjust fella,
> but chiefly on the just,
> because the unjust steals the just's umbrella.
> -Author unknown

□**T.5.a) The lucky case: An explicit formula**

You are lucky if you get a surface in one of the forms
z = f[x, y], y = g[x, z], or x = h[y, z].
The fancy folks call each of these an explicit formula of a surface.
Why are you lucky when you get an explicit equation for a surface?

□**Answer:**

You are lucky because the explicit form carries with it a built-in

parameterization.

If the surface comes to you with the explicit formula

z = f[x, y],

then you can parameterize it with the formulas

x[u, v] = u,

y[u, v] = v, and

z[u, v] = f[u, v].

If the surface comes to you with the explicit formula

y = g[x, z],

then you can parameterize it with the formulas

x[u, v] = u,

y[u, v] = g[u, v], and

z[u, v] = v.

If the surface comes to you with the explicit formula

x = h[y, z],

then you can parameterize it with the formulas

x[u, v] = h[u, v],

y[u, v] = u, and

z[u, v] = v.

□**T.5.b) The (often) unlucky case: An implicit equation**

You can be in deep doo-doo if you get a surface in the form
f[x, y, z] = constant.
    The fancy folks call this an implicit formula of a surface.
Why can you be unlucky when you get an implicit equation for a surface?
What is the first thing you should try to do in order to try to work with surface specified with an implicit equation?

□**Answer:**

You can't do diddly-squat with an implicit equation until you come up

with a parameterization or an explicit equation.

So the first thing you should try to do is to try get a parameterization.

This is not always a pleasant task. Sometimes it's impossible.

Experience with the starred problems will be useful for understanding developments later in the course.

G.1) How do you know?*

□**G.1.a)**

Here's a 3D vector field:

```
Clear[x, y, z, m, n, p, Field]
{m[x_, y_, z_], n[x_, y_, z_], p[x_, y_, z_]} =
 {-6 x + 3 x Cos[z], 4 y + 5 E^-0.2 (x+z), 5 z - 3 Sin[z]};
Field[x_, y_, z_] = {m[x, y, z], n[x, y, z], p[x, y, z]}
```
$\{-6 x + 3 x \cos[z], 5 E^{-0.2 (x+z)} + 4 y, 5 z - 3 \sin[z]\}$

How do you know that the net flow of this vector field across any surface that is the whole skin of a solid region is from inside to outside?

□**G.1.b)**

Here's another 3D vector field:

```
Clear[x, y, z, m, n, p, Field]
{m[x_, y_, z_], n[x_, y_, z_], p[x_, y_, z_]} =
 {-6 x + 3 x Cos[z], 2 y + x/(1 + x²), z - 3 Sin[z]};
Field[x_, y_, z_] = {m[x, y, z], n[x, y, z], p[x, y, z]}
```
$\{-6 x + 3 x \cos[z], \frac{x}{1 + x^2} + 2 y, z - 3 \sin[z]\}$

How do you know that the net flow of this vector field across any surface that is the whole skin of a solid region is from outside to inside?

□**G.1.c)**

Here's another 3D vector field involving a parameter b:

```
Clear[b, x, y, z, m, n, p, Field]
{m[x_, y_, z_], n[x_, y_, z_], p[x_, y_, z_]} =
 {-6 x + 3 x E^z, b y + 12 z, 8 x + 4 z - 3 E^z};
Field[x_, y_, z_] = {m[x, y, z], n[x, y, z], p[x, y, z]}
```

$\{-6 x + 3 E^z x, b y + 12 z, -3 E^z + 8 x + 4 z\}$

How do you know that if b > 2, then the net flow of this vector field across any surface that is the whole skin of a solid region is from inside to outside?
How do you know that if b < 2, then the net flow of this vector field across any surface that is the whole skin of a solid region is from outside to inside?
How do you know that if b = 2, then the net flow of this vector field across any surface that is the whole skin of a solid region is 0?

□**G.1.d)**

Here's another 3D vector field:

```
Clear[b, x, y, z, m, n, p, Field]
{m[x_, y_, z_], n[x_, y_, z_], p[x_, y_, z_]} =
 {6 x² + 3 x E^z, 9 y + 12 z, 8 x + 3 z - 3 E^z};
Field[x_, y_, z_] = {m[x, y, z], n[x, y, z], p[x, y, z]}
```
$\{3 E^z x + 6 x^2, 9 y + 12 z, -3 E^z + 8 x + 3 z\}$

How do you know that any point
{x, y, z} with x > −1
is a source for this vector field?
How do you know that any point
{x, y, z} with x < −1
is a sink for this vector field?
How do you know that the net flow of this vector field across the sphere of radius = 1 centered at {1, 0, 0} is from inside to outside, but the net flow of this vector field across the sphere of radius = 1 centered at {−3, 0, 0} is from outside to inside?

□**G.1.e)**

Here's another 3D vector field involving cleared functions f[x, y], g[y, z], and h[x, z]:

```
Clear[f, g, h, x, y, z, m, n, p, Field]
{m[x_, y_, z_], n[x_, y_, z_], p[x_, y_, z_]} =
 {g[y, z], h[x, z], f[x, y]};
Field[x_, y_, z_] = {m[x, y, z], n[x, y, z], p[x, y, z]}
```
$\{g[y, z], h[x, z], f[x, y]\}$

This tells you that
→ m[x, y, z] = g[y, z] does not change as x changes;
→ n[x, y, z] = h[x, z] does not change as y changes and
→ p[x, y, z] = f[x, y] does not change as z changes.
Give a concrete example of this kind of vector field.
How do you know in advance that this kind of vector field has no sources or sinks?

□G.1.f)

Look at this plot:

```
Clear[x, y, z, s, t]
x[s_, t_] = (5 Sin[s] Cos[t]) / (1 + (s - π/2)^2);
y[s_, t_] = (4 - Sin[t]) Sin[s] Sin[t];
z[s_, t_] = 4 Cos[s];
{{slow, shigh}, {tlow, thigh}} = {{0, π/2}, {0, π}};
surface = ParametricPlot3D[Evaluate[{x[s, t], y[s, t], z[s, t]}],
   {s, slow, shigh}, {t, tlow, thigh}, DisplayFunction → Identity];
Clear[normal]
normal[s_, t_] =
 D[{x[s, t], y[s, t], z[s, t]}, s] × D[{x[s, t], y[s, t], z[s, t]}, t];
jump = π/8;
scalefactor = 0.2;
normals = Table[Arrow[normal[s, t], Tail → {x[s, t], y[s, t], z[s, t]},
    ScaleFactor → scalefactor], {s, slow + jump, shigh - jump, jump},
   {t, tlow + jump, thigh - jump, jump}];

Show[surface, normals, ViewPoint → CMView,
 Boxed → False, AxesLabel → {"x", "y", "z"}, BoxRatios → Automatic,
 DisplayFunction → $DisplayFunction];
```
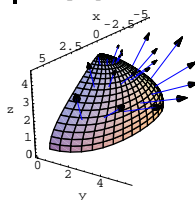

This plot shows a surface together with some of the normal vectors whose lengths measure the area conversion factor
    $SA_{xyz}[s, t]$,
which converts st-paper area measurements into xyz-area measurements on the surface.
Notice this:
The longer normal vectors have their tails on the larger plates on the Mathematica plot, and the shorter normal vectors have their tails on the smaller plates on the Mathematica plot.
How do you know that this is likely to happen no matter what surface you go with?

G.2) Meat-and-potatoes measurements*

□G.2.a.i)

Here's a vector field:

```
Clear[x, y, z, m, n, p, Field]
{m[x_, y_, z_], n[x_, y_, z_], p[x_, y_, z_]} = {y^2, z^2, x^2};
Field[x_, y_, z_] = {m[x, y, z], n[x, y, z], p[x, y, z]}
{y^2, z^2, x^2}
```

Measure the net flow of this vector field across the surface of the three dimensional box
    $-1 \le x \le 2, -3 \le y \le 3, 0 \le z \le 4$.
Is the net flow of this vector field across this skin from inside to outside, from outside to inside, or 0?

□ **Tip:**

If you are not using Gauss's 3D formula, then you are working too hard.

□G.2.a.ii)

Here's a new vector field:

```
Clear[x, y, z, m, n, p, Field]
{m[x_, y_, z_], n[x_, y_, z_], p[x_, y_, z_]} = {x^2, y^2, z^2};
Field[x_, y_, z_] = {m[x, y, z], n[x, y, z], p[x, y, z]}
{x^2, y^2, z^2}
```

Measure the net flow of this vector field across the surface of the three dimensional box
    $-1 \le x \le 2, -3 \le y \le 3, 0 \le z \le 4$.
Is the net flow across this skin from inside to outside, from outside to inside, or 0?

□G.2.b)

Measure the net flow of the vector field
    Field[x, y, z] = {x^2, y^2, z^2}
across the surface of the sphere of radius 3 centered at {0, 0, 2}.
Is the net flow across this skin from inside to outside, from outside to inside, or 0?

□G.2.c)

Here's a new vector field:

```
Clear[x, y, z, m, n, p, Field]
{m[x_, y_, z_], n[x_, y_, z_], p[x_, y_, z_]} = {x^2, y^3, z^2};
Field[x_, y_, z_] = {m[x, y, z], n[x, y, z], p[x, y, z]}
{x^2, y^3, z^2}
```

Use Gauss's formula to measure the net flow of this vector field across the surface of the 3D solid region between the sphere of radius 1 centered at {0, 0, 0} and the sphere of radius 4 centered at {0, 0, 1}.

□ **Tip:**

Measure the flow of Field[x, y, z] across the sphere of radius 1 centered at {0, 0, 0}. Then measure the flow of Field[x, y, z] across the sphere of radius 4 centered at {0, 0, 1}.
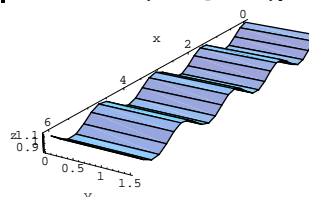
Put the two measurements together in the right way to get the answer you are after.

□G.2.d.i)

Here's a magic carpet:

```
Clear[x, y, z, s, t]
{x[s_, t_], y[s_, t_], z[s_, t_]} = {t, 0, 1 + 0.15 Cos[4 t]} + s {0, 1, 0};
{{slow, shigh}, {tlow, thigh}} = {{0, 1.5}, {0, 6}};
```

```
carpet = ParametricPlot3D[Evaluate[{x[s, t], y[s, t], z[s, t]}],
   {s, slow, shigh}, {t, tlow, thigh}, PlotPoints → {2, 30},
   ViewPoint → CMView, BoxRatios → Automatic, Boxed → False,
   AxesLabel → {"x", "y", "z"}];
```


Look at the numbers displayed on the axes, and use them to say why it is clear that the surface area measurement of this magic carpet is a bit more than 9 square units.
Then move in with a surface integral to give a more accurate measurement.

□ **Tip:**

Use NIntegrate to calculate the surface integral.

□G.2.d.ii)

While you're at it, take the same magic carpet as in part i) above and measure the flow of the 3D vector field
    Field[x, y, z] = {y − z, x + z, x − y}
across (through) this carpet. Determine whether the net flow of this vector field across (through) this carpet is from under to over or from over to under.

□G.2.d.iii) Calculus Cal screws up

When that lab pest, Calculus Cal, looked over your shoulder and saw your answer to part ii) above, he said: "Your answer is wrong, and I'll tell you why. Look at the vector field and its divergence:"

```
Clear[x, y, z, m, n, p, Field, divField]
{m[x_, y_, z_], n[x_, y_, z_], p[x_, y_, z_]} = {y - z, x + z, x - y};
Field[x_, y_, z_] = {m[x, y, z], n[x, y, z], p[x, y, z]};
divField[x_, y_, z_] =
 D[m[x, y, z], x] + D[n[x, y, z], y] + D[p[x, y, z], z]
```

0

Calculus Cal went on to say: "It says right in B.1) that any vector field Field[x, y, z] with divField[x, y, z] = 0 for all points {x, y, z} has no sources or sinks. Therefore, its flow across any old surface is 0."
Say why Calculus Cal is wrong and then tell Cal where to go.

□**G.2.e)**

Here's a plot of the uv-paper ellipse
$$(\tfrac{u}{2})^2 + (\tfrac{v}{4})^2 = 1$$
on the plane
$$x + 2\,y + 4\,z = 0$$
under the parameterization
$$x[u, v] = u,$$
$$y[u, v] = v, \text{ and}$$
$$z[u, v] = \tfrac{1}{4}\,(-u - 2\,v).$$

```
Clear[x, y, z, u, v, t]
x[u_, v_] = u;
y[u_, v_] = v;
z[u_, v_] = 1/4 (-u - 2 v);
u[t_] = 2 Cos[t];
v[t_] = 4 Sin[t];
curve = ParametricPlot3D[{x[u[t], v[t]], y[u[t], v[t]], z[u[t], v[t]]},
   {t, 0, 2 π}, DisplayFunction → Identity];
{a, b} = {-5, -5};
h = 10;
plane = Graphics3D[Polygon[
   {{x[a, b], y[a, b], z[a, b]}, {x[a + h, b], y[a + h, b], z[a + h, b]},
    {x[a + h, b + h], y[a + h, b + h], z[a + h, b + h]},
    {x[a, b + h], y[a, b + h], z[a, b + h]}}]];

Show[curve, plane, AxesLabel → {"x", "y", "z"}, ViewPoint → CMView,
 DisplayFunction → $DisplayFunction];
```
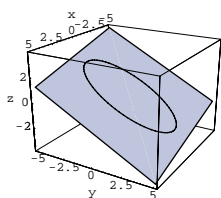

Come up with a measurement of the area enclosed by this curve as measured on the plotted plane.

G.3) Substitute surfaces for avoiding calculational nightmares*

□**G.3.a)**

Here is a surface called C_1:

```
Clear[x1, y1, z1, s, t]
x1[s_, t_] = 2 + 2 Sin[s] Cos[t];
y1[s_, t_] = 1 + 2 Sin[s] Sin[t];
z1[s_, t_] = 1 + E^Sin[y1[s,t]-1] Cos[s];
{{s1low, s1high}, {t1low, t1high}} = {{0, π/2}, {0, 2 π}};
C1plot = ParametricPlot3D[Evaluate[{x1[s, t], y1[s, t], z1[s, t]}],
   {s, s1low, s1high}, {t, t1low, t1high}, Boxed → False,
   ViewPoint → CMView, PlotRange → All, AxesLabel → {"x", "y", "z"}];
```
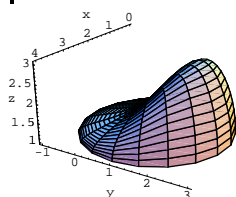

C_2 is a circular disk in the plane z = 1 centered at the point {2, 1, 1}:

```
Clear[x2, y2, z2, s, t]
x2[s_, t_] = 2 + s Cos[t];
y2[s_, t_] = 1 + s Sin[t];
z2[s_, t_] = 1;
```

```
{{s2low, s2high}, {t2low, t2high}} = {{0, 2}, {0, 2 π}};
C2plot = ParametricPlot3D[
   Evaluate[{x2[s, t], y2[s, t], z2[s, t]}], {s, s2low, s2high},
   {t, t2low, t2high}, PlotPoints → {2, Automatic}, Boxed → False,
   ViewPoint → CMView, PlotRange → All, AxesLabel → {"x", "y", "z"}];
```
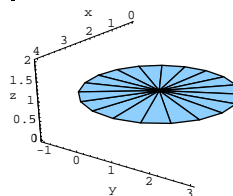

C_1 and C_2 share the same boundary curve, as you can see by studying the code or by looking from underneath:

```
Show[C1plot, C2plot, ViewPoint → {5, 5, -5}];
```
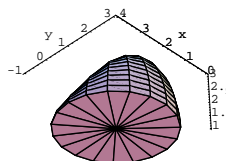

Here's a vector field:

```
Clear[x, y, z, Field]
Field[x_, y_, z_] = {z - y, x + z, y - x}
```
$\{-y + z, \ x + z, \ -x + y\}$

And its divergence:

```
Clear[m, n, p, divField]
{m[x_, y_, z_], n[x_, y_, z_], p[x_, y_, z_]} = Field[x, y, z];
divField[x_, y_, z_] =
 D[m[x, y, z], x] + D[n[x, y, z], y] + D[p[x, y, z], z]
```
0

Calculate the net flow of this vector field across C_1,
$$\iint_{C_1} \text{Field . topunitnormal}\, d\,A,$$
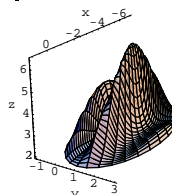by calculating the net flow of this vector field across C_2,

$$\iint_{C_2} \text{Field . topunitnormal}\, d\,A,$$
and explain why this gives the correct answer.

□**G.3.b)**

Here is a new surface called C_1:

```
Clear[x1, y1, z1, r, s]
x1[r_, s_] = -3 + 2 r Cos[s];
y1[r_, s_] = 1 + r Sin[s];
z1[r_, s_] = 2 + E^Sin[4 (y1[r,s]-1)] (2 - r);
{{r1low, r1high}, {s1low, s1high}} = {{0, 2}, {0, 2 π}};
C1plot = ParametricPlot3D[Evaluate[{x1[r, s], y1[r, s], z1[r, s]}],
   {r, r1low, r1high}, {s, s1low, s1high}, Boxed → False,
   ViewPoint → CMView, PlotRange → All, PlotPoints → {30, 30},
   AxesLabel → {"x", "y", "z"}];
```


Here is a vector field:

```
Clear[x, y, z, Field]
Field[x_, y_, z_] = {3 z - y, 3 x + z, 6 y - x}
```
$\{-y + 3 z, \ 3 x + z, \ -x + 6 y\}$

And its divergence:

```
Clear[m, n, p, divField]
{m[x_, y_, z_], n[x_, y_, z_], p[x_, y_, z_]} = Field[x, y, z];
divField[x_, y_, z_] =
 D[m[x, y, z], x] + D[n[x, y, z], y] + D[p[x, y, z], z]
```
0

Calculate the net flow of this vector field across C_1,
by calculating
$$\iint_{C_2} \text{Field . topunitnormal}\, d\,A$$
where C_2 is a substitute surface of your own choice.
Explain why you believe in your answer.

G.4) Avoiding another calculational nightmare*

Here's a vector field:

```
Clear[Field, x, y, z, m, n, p, divField1]
Field[x_, y_, z_] = {z, 4 x, -3 y} + (3 ({x, y, z} - {0, 0.5, 0}))/((x + 0)^2 + (y - 0.5)^2 + (z + 0)^2)^(3/2);
{m[x_, y_, z_], n[x_, y_, z_], p[x_, y_, z_]} = Field[x, y, z]
```

$$\left\{z + \frac{3x}{(x^2 + (-0.5+y)^2 + z^2)^{3/2}}, 4x + \frac{3(-0.5+y)}{(x^2 + (-0.5+y)^2 + z^2)^{3/2}}, -3y + \frac{3z}{(x^2 + (-0.5+y)^2 + z^2)^{3/2}}\right\}$$

Note the big singularity at {0, 0.5, 0}.
Here is the divergence of Field[x, y, z]:

```
divField[x_, y_, z_] =
 Together[D[m[x, y, z], x] + D[n[x, y, z], y] + D[p[x, y, z], z]]
```

$$\frac{0. y}{\sqrt{x^2 + (-0.5+y)^2 + z^2}\,(0.25 + x^2 - 1. y + y^2 + z^2)^2}$$

Here comes a surface:

```
Clear[xskin, yskin, zskin, radius, s, t]
radius[s_, t_] = 5 (Cos[t]^2 + 1) (1.5 - Sin[s]);
{xskin[s_, t_], yskin[s_, t_], zskin[s_, t_]} =
 radius[s, t] {Sin[s] Cos[t], Cos[s], Sin[s] Sin[t]};
{{slow, shigh}, {tlow, thigh}} = {{0, π}, {0, 2 π}};
surfaceplot =
 ParametricPlot3D[Evaluate[{xskin[s, t], yskin[s, t], zskin[s, t]}],
  {s, slow, shigh}, {t, tlow, thigh}, Boxed → False,
  ViewPoint → CMView, PlotRange → All, AxesLabel → {"x", "y", "z"}];
```
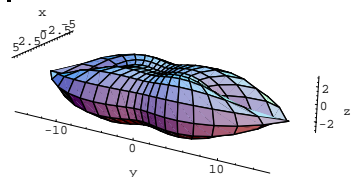

□G4.a.i)

Is the singularity of Field[x, y, z] inside the surface plotted above?

□G.4.a.ii)

Why is it a bad idea to assume in advance that since divField[x, y, z] calculated out to 0, then the net flow of Field[x, y, z] across the surface plotted above is 0?

□G.4.a.iii)

Measure the flow of Field[x, y, z] across this surface.
Is the flow from outside to inside or inside to outside?
□ **Big Tip:**

Look at:

```
Clear[Field, x, y, z]
Field[x_, y_, z_] = {z, 4 x, -3 y} + (3 ({x, y, z} - {0, 0.5, 0}))/((x + 0)^2 + (y - 0.5)^2 + (z + 0)^2)^(3/2)
```

$$\left\{z + \frac{3x}{(x^2 + (-0.5+y)^2 + z^2)^{3/2}}, 4x + \frac{3(-0.5+y)}{(x^2 + (-0.5+y)^2 + z^2)^{3/2}}, -3y + \frac{3z}{(x^2 + (-0.5+y)^2 + z^2)^{3/2}}\right\}$$

Notice that Field[x, y, z] is the sum of two vector fields

Field1[x, y, z] and Field2[x, y, z] where

```
Clear[Field1, Field2]
Field1[x_, y_, z_] = {z, 4 x, -3 y}
```

$\{z, 4x, -3y\}$

and

```
Field2[x_, y_, z_] = (3 ({x, y, z} - {0, 0.5, 0}))/((x + 0)^2 + (y - 0.5)^2 + (z + 0)^2)^(3/2)
```

$$\left\{\frac{3x}{(x^2 + (-0.5+y)^2 + z^2)^{3/2}}, \frac{3(-0.5+y)}{(x^2 + (-0.5+y)^2 + z^2)^{3/2}}, \frac{3z}{(x^2 + (-0.5+y)^2 + z^2)^{3/2}}\right\}$$

You might want to make the calculations for Field1[x, y, z] and

Field2[x, y, z] separately, remembering that Field[x, y, z] is an electric

field and one of the tutorials told you all you need to know about

electric fields.
If you are grinding a lot of Mathematica code on this, you are not
practicing the art of avoiding calculation; Gauss would not be proud of
you.
Go ahead and make Gauss proud of you.

□G.4.b.i)

Here is a new vector field and its divergence:

```
Clear[Field, x, y, z, m, n, p, divField]
Field[x_, y_, z_] = {z, 4 x, -3 y} + (3 ({x, y, z} - {10, 0, 0}))/((x - 10)^2 + (y + 0)^2 + (z + 0)^2)^(3/2);
{m[x_, y_, z_], n[x_, y_, z_], p[x_, y_, z_]} = Field[x, y, z]
```

$$\left\{z + \frac{3(-10+x)}{((-10+x)^2 + y^2 + z^2)^{3/2}}, 4x + \frac{3y}{((-10+x)^2 + y^2 + z^2)^{3/2}}, -3y + \frac{3z}{((-10+x)^2 + y^2 + z^2)^{3/2}}\right\}$$

```
divField[x_, y_, z_] =
 Together[D[m[x, y, z], x] + D[n[x, y, z], y] + D[p[x, y, z], z]]
```

0

Is the singularity of Field[x, y, z] inside the surface plotted above?

□G.4.b.ii)

Why is it a good idea to assume in advance that since divField[x, y, z] calculated out to 0, the net flow of Field[x, y, z] across the surface plotted above is 0?

G.5) Flux of the electric field and Gauss's law
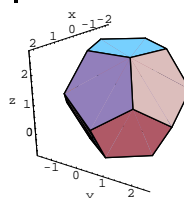
□G.5.a.i)

Before you start, excecute the cell below:

```
Needs["Graphics`Polyhedra`"]
```

Here is a surface that is the whole skin of a solid region:

```
surfaceplot =
 Show[Polyhedron[Dodecahedron, {0, 0.5, 1}, 2], Axes → True,
  Boxed → False, ViewPoint → CMView, AxesLabel → {"x", "y", "z"}];
```


The point {1, 0, 1} is inside this skin as you can see by including a vector whose tip is at {1, 0, 1} and whose tail is outside the skin:

```
point = {1, 0, 1};
tail = {2, 3, 0};
pointer = Arrow[point - tail, Tail → tail, VectorColor → Blue];
Show[surfaceplot, pointer, PlotRange → All];
```
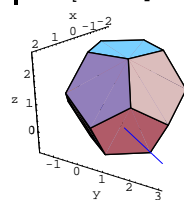

An electrical charge of strength 3 is placed at {1, 0, 1}, resulting in the electric field

Field[x, y, z]
$$= \left\{\frac{3(x-1)}{((x-1)^2 + y^2 + (z-1)^2)^{3/2}}, \frac{3y}{((x-1)^2 + y^2 + (z-1)^2)^{3/2}}, \frac{3(z-1)}{((x-1)^2 + y^2 + (z-1)^2)^{3/2}}\right\}$$

Measure the flux of this electric field across this skin.
□ **Tip:**

The flux of this electric field across this surface is the same as the flow of this field across this surface.

If you are going to heavy calculation here, then you are doing too much work.

G.5.a.ii)

Continue to go with the same surface as in part i) above.
This time place an electrical charge of strength 4 at {0, 0, 0} and another electical charge of strength −2 at {1, 0, 0}.
Measure the flux of the resulting combined electric field across this skin.

G.5.a.iii)

Continue to go with the same surface as in part i) above.
This time place an electrical charge of strength 4 at {2, 3, 2} and another electrical charge of strength −2 at {5, 0, 0}.
Measure the flux of the resulting combined electric field across this skin.

□**Tip:**

Check whether these points are inside the skin.

□**G.5.b.i) Gauss's 3D law**

Gauss's law in physics says that if you have a surface C that is the boundary surface of a solid in three dimensions, and if you take n different points
$\{a_1, b_1, c_1\}, \{a_2, b_2, c_2\}, \ldots, \{a_n, b_n, c_n\}$
inside C, and you place electrical charges of strength
q_1 at $\{a_1, b_1, c_1\}$,
q_2 at $\{a_2, b_2, c_2\}$,
…
q_n at $\{a_n, b_n, c_n\}$,
then the flux (= flow) of the resulting electric field across C is simply
$4\pi(q_1 + q_2 + q_3 + \ldots + q_n)$.
Explain where Gauss's law comes from.

□**G.5.b.ii) Gauss's law for dipoles**

You put two electric charges of the same strength but opposite signs at two points that are very close to each other. The resulting electrical field is called a dipole field.
Take a surface C that is the boundary surface of a solid in three dimensions with both points inside C. What is the flux (= flow) of the resulting dipole field across C?

G.6) Sources and sinks in the 3D gradient field, the Laplacian

$$\frac{\partial^2 f[x,y,z]}{\partial x^2} + \frac{\partial^2 f[x,y,z]}{\partial y^2} + \frac{\partial^2 f[x,y,z]}{\partial z^2}$$

and steady state heat*

One way to come up with a 3D vector field is to take a function f[x, y, z] and put
Field[x, y, z] = gradf[x, y, z].
Try it out:

```
Clear[f, gradf, Field, x, y]

f[x_, y_, z_] = 1 - 1/2 (x² + y² + z²);
```

```
gradf[x_, y_, z_] =
  {D[f[x, y, z], x], D[f[x, y, z], y], D[f[x, y, z], z]};
Field[x_, y_, z_] = gradf[x, y, z]
```
{-x, -y, -z}

This is the gradient field of
$f[x, y, z] = 1 - \frac{1}{2}(x^2 + y^2 + z^2)$.
Note that {x, y, z} = {0, 0, 0} maximizes f[x, y, z].
Here's a plot of the scaled gradient field of f[x, y, z] shown with the maximizer at {0, 0, 0}:

```
maximizerplot = Graphics3D[{Red, PointSize[0.06], Point[{0, 0, 0}]}];
scalefactor = 0.4;
gradfieldplot = Table[Arrow[gradf[x, y, z],
   Tail → {x, y, z}, VectorColor → Blue, ScaleFactor → scalefactor],
   {x, -1, 1, 2}, {y, -1, 1, 1}, {z, -1, 1, 1}];
```

```
Show[maximizerplot, gradfieldplot, ViewPoint → CMView,
 Axes → Automatic, AxesLabel → {"x", "y", "z"}];
```
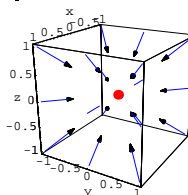

□**G.6.a.i)**

Why did this happen?
Where are the trajectories in this gradient field headed?
What do you try to learn about a function f[x, y, z] by looking at a plot of its gradient field?

□**G.6.a.ii)**

You are given a certain function f[x, y, z], and the information that
f[a, b, c] > f[x, y, z]
for all {x, y, z} close to but not equal to {a, b, c}.
You center a small sphere on {a, b, c} and plot the gradient field of f[x, y, z] on this sphere.
Is the net flow of the gradient field of f[x, y, z] across this sphere from inside to outside or from outside to inside?

□**G.6.b.i) Minimizers**

This time go with
$f[x, y] = e^{(x^2 + y^2 + z^2)/16}$.

```
Clear[f, gradf, Field, x, y, z]

f[x_, y_, z_] = E^(1/16 (x²+y²+z²));
gradf[x_, y_, z_] =
  {D[f[x, y, z], x], D[f[x, y, z], y], D[f[x, y, z], z]};
Field[x_, y_, z_] = gradf[x, y, z]
```
$\{\frac{1}{8} E^{\frac{1}{16}(x^2+y^2+z^2)} x, \frac{1}{8} E^{\frac{1}{16}(x^2+y^2+z^2)} y, \frac{1}{8} E^{\frac{1}{16}(x^2+y^2+z^2)} z\}$

Note that {x, y, z} = {0, 0, 0} minimizes f[x, y, z].
Here's a plot of the scaled gradient field of f[x, y, z] shown with the minimizer at {0, 0, 0}:

```
minimizerplot = Graphics3D[{Red, PointSize[0.06], Point[{0, 0, 0}]}];
scalefactor = 2.5;
gradfieldplot = Table[Arrow[gradf[x, y, z],
   Tail → {x, y, z}, VectorColor → Blue, ScaleFactor → scalefactor],
   {x, -1, 1, 2}, {y, -1, 1, 1}, {z, -1, 1, 1}];
```

```
Show[minimizerplot, gradfieldplot, Axes → Automatic,
 AxesLabel → {"x", "y", "z"}];
```
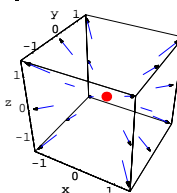
Why did this happen?
What are the trajectories in this gradient field trying to get away from?

☐G.6.b.ii)

You are given a certain function f[x, y, z], and the information that
$$f[a, b, c] < f[x, y, z]$$
for all {x, y, z} close to but not equal to {a, b, c}.
You center a small sphere at {a, b, c} and plot the gradient field of
f[x, y, z] on this sphere.
Is the net flow of the gradient field of f[x, y, z] across this sphere from
inside to outside or from outside to inside?

☐G.6.c.i) The Laplacian $\frac{\partial^2 f[x,y,z]}{\partial x^2} + \frac{\partial^2 f[x,y,z]}{\partial y^2} + \frac{\partial^2 f[x,y,z]}{\partial z^2}$

Here is a cleared function and its gradient field:

```
Clear[x, y, z, f, gradf, m, n, p, Field, divField]
gradf[x_, y_, z_] =
 {D[f[x, y, z], x], D[f[x, y, z], y], D[f[x, y, z], z]};
{m[x_, y_, z_], n[x_, y_, z_], p[x_, y_, z_]} = gradf[x, y, z];
Field[x_, y_, z_] = {m[x, y, z], n[x, y, z], p[x, y, z]}
```
$\{f^{(1,0,0)}[x, y, z], f^{(0,1,0)}[x, y, z], f^{(0,0,1)}[x, y, z]\}$

Here is the divergence of this gradient field:

```
divField[x_, y_, z_] =
 D[m[x, y, z], x] + D[n[x, y, z], y] + D[p[x, y, z], z]
```
$f^{(0,0,2)}[x, y, z] + f^{(0,2,0)}[x, y, z] + f^{(2,0,0)}[x, y, z]$

The Laplacian of f[x, y, z] is given by
$\frac{\partial^2 f[x,y,z]}{\partial x^2} + \frac{\partial^2 f[x,y,z]}{\partial y^2} + \frac{\partial^2 f[x,y,z]}{\partial z^2}$:

```
Clear[Laplacianf]
Laplacianf[x_, y_, z_] =
 D[f[x, y, z], {x, 2}] + D[f[x, y, z], {y, 2}] + D[f[x, y, z], {z, 2}]
```
$f^{(0,0,2)}[x, y, z] + f^{(0,2,0)}[x, y, z] + f^{(2,0,0)}[x, y, z]$

How does the divergence of the gradient field compare with the
Laplacian?

☐G.6.c.ii)

How do you check the Laplacian
$\frac{\partial^2 f[x,y,z]}{\partial x^2} + \frac{\partial^2 f[x,y,z]}{\partial y^2} + \frac{\partial^2 f[x,y,z]}{\partial z^2}$
to look for sources and sinks in the gradient field of a given function
f[x, y, z]?
How do you check the Laplacian
$\frac{\partial^2 f[x,y,z]}{\partial x^2} + \frac{\partial^2 f[x,y,z]}{\partial y^2} + \frac{\partial^2 f[x,y,z]}{\partial z^2}$
to check whether the gradient field of a given function f[x, y, z] is free
of sources or sinks at points other than singularities?

☐G.6.c.iii)

Explain this:
If f[x, y, z] is a function with no singularities, and with
$\frac{\partial^2 f[x,y,z]}{\partial x^2} + \frac{\partial^2 f[x,y,z]}{\partial y^2} + \frac{\partial^2 f[x,y,z]}{\partial z^2} = 0$
for all points {x, y, z}, then there can be no point {a, b, c} with
$$f[a, b, c] > f[x, y, z]$$
for all {x, y, z} close to but not equal to {a, b, c}.
And explain this:
If f[x, y, z] is a function with no singularities, and with
$\frac{\partial^2 f[x,y,z]}{\partial x^2} + \frac{\partial^2 f[x,y,z]}{\partial y^2} + \frac{\partial^2 f[x,y,z]}{\partial z^2} = 0$
for all points {x, y, z}, then there can be no point {a, b, c} with
$$f[a, b, c] < f[x, y, z]$$
for all {x, y, z} close to but not equal to {a, b, c}.

☐Tip:

Review what you said in parts a) and b) above.

☐G.6.d.i)

A large solid region in three dimensions represents a big rock. Part of
the surface of the rock is kept at a prescribed temperature - maybe
hotter at one point than at another. The remainder of the surface is
perfectly insulated.
You wait until the temperature inside the rock settles into its steady
state condition.
Say temp[x, y, z] represents the steady state temperature at a position
{x, y, z} inside the rock.
In the steady state, no point inside the rock and not on the surface can
be a source of new heat flow or a sink for old heat.
Why does this tell you that if
$$Field[x, y, z] = gradtemp[x, y, z],$$
then
divField[x, y, z]
$= \frac{\partial^2 temp[x,y,z]}{\partial x^2} + \frac{\partial^2 temp[x,y,z]}{\partial y^2} + \frac{\partial^2 temp[x,y,z]}{\partial z^2}$
$= 0$
at each point {x, y, z} inside but not on the surface of the rock?

☐G.6.d.ii)

Explain why your answer to part i), immediately above, tells you that
the hottest and the coldest locations of the rock must be on the outside
skin of the rock and not inside the rock.

G.7) Morphing and Moebius strips

Here's a flat ribbon:

```
Clear[s, t, flatribbon]
flatribbon[s_, t_] = {-2, t, s};
flatribbonplot = ParametricPlot3D[flatribbon[s, t], {t, -2 π, 2 π},
 {s, -0.5, 0.5}, PlotPoints → {2, 2}, DisplayFunction → Identity];
threedims = Axes3D[1.];

Show[flatribbonplot, threedims, ViewPoint → CMView,
 PlotRange → {{-2.5, 2.5}, {-2 π, 2 π}, {-1.5, 1.5}}, Boxed → False,
 DisplayFunction → $DisplayFunction];
```
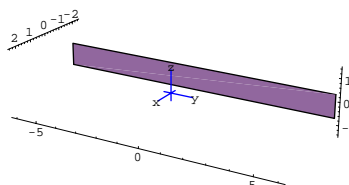

You can pull the two ends around the z-axis, and join them together to
make a nice cylindrical ribbon:

```
Clear[P, x, y, z, t, unittan, mainunitnormal, binormal, roundribbon]
P[t_] = 2 {-Cos[t], Sin[t], 0}
```
$\{-2 Cos[t], 2 Sin[t], 0\}$

```
unittan[t_] = TrigExpand[ P'[t] / √(P'[t] . P'[t]) ]
```
$\{Sin[t], Cos[t], 0\}$

```
mainunitnormal[t_] = unittan'[t] / √(TrigExpand[unittan'[t] . unittan'[t]])
```
$\{Cos[t], -Sin[t], 0\}$

```
binormal[t_] = TrigExpand[unittan[t] × mainunitnormal[t]]
```
$\{0, 0, -1\}$

```
roundribbon[s_, t_] = P[t] - s binormal[t];
roundribbonplot = ParametricPlot3D[roundribbon[s, t], {s, -0.5, 0.5},
 {t, -π, π}, PlotPoints → {2, Automatic}, DisplayFunction → Identity];

Show[roundribbonplot, threedims, ViewPoint → CMView,
 PlotRange → {{-2.5, 2.5}, {-2 π, 2 π}, {-1.5, 1.5}}, Boxed → False,
 DisplayFunction → $DisplayFunction];
```
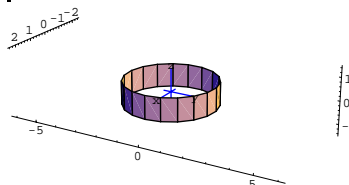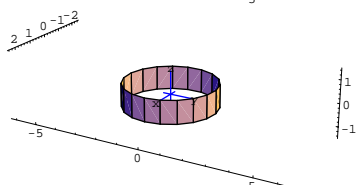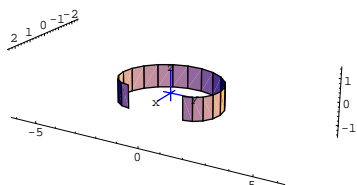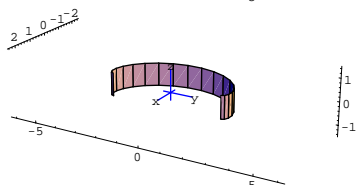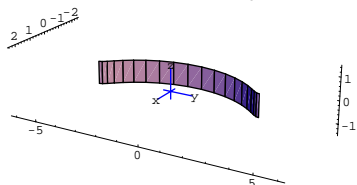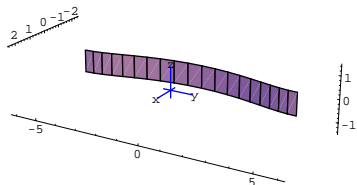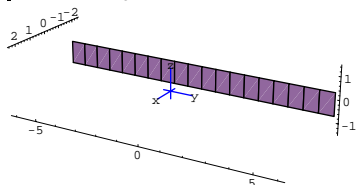

Here's an animation about how you go from the flat ribbon to the
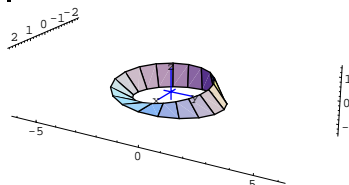round ribbon:

```
Clear[morph, h]
morph[h_] := ParametricPlot3D[
```

```
    (1 - h) flatribbon[s, 2 t] + h roundribbon[s, t], {s, -0.5, 0.5},
    {t, -π, π}, PlotPoints → {2, Automatic}, DisplayFunction → Identity];
Table[Show[morph[h], threedims, ViewPoint → CMView,
    PlotRange → {{-2.5, 2.5}, {-2 π, 2 π}, {-1.5, 1.5}},
    Boxed → False, DisplayFunction → $DisplayFunction],
    {h, 0, 1, 1/5}];
```
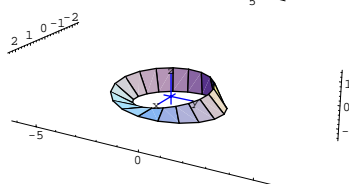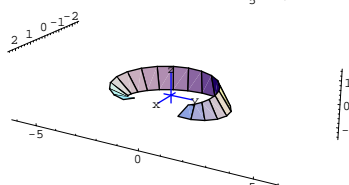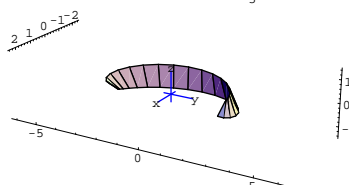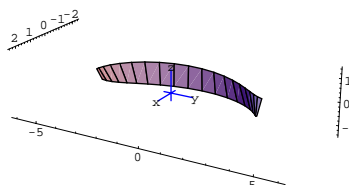

Animate these.

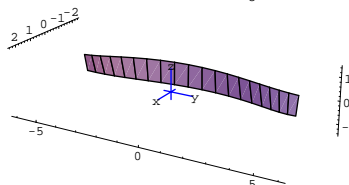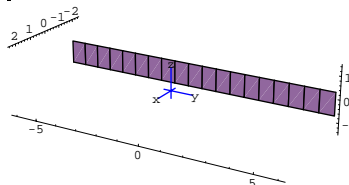### □G.7.a)

You can start with the same flat ribbon:

```
Show[flatribbonplot, threedims, ViewPoint → CMView,
    PlotRange → {{-2.5, 2.5}, {-2 π, 2 π}, {-1.5, 1.5}}, Boxed → False,
    DisplayFunction → $DisplayFunction];
```


You can join the ends giving it a twist along the way:

```
Clear[twistedribbon]
twistedribbon[s_, t_] =
  P[t] + s (Sin[t/2] mainunitnormal[t] - Cos[t/2] binormal[t]);
twistedribbonplot =
  ParametricPlot3D[twistedribbon[s, t], {s, -0.5, 0.5}, {t, -π, π},
    PlotPoints → {2, Automatic}, DisplayFunction → Identity];
```

```
Show[twistedribbonplot, threedims, ViewPoint → CMView,
    PlotRange → {{-2.5, 2.5}, {-2 π, 2 π}, {-1.5, 1.5}}, Boxed → False,
    DisplayFunction → $DisplayFunction];
```


The fancy folks like to call this a "Moebius strip."
Here's an animation about how you go from the flat ribbon to the Moebius strip:

```
Clear[morph, h]
morph[h_] := ParametricPlot3D[
    (1 - h) flatribbon[s, 2 t] + h twistedribbon[s, t], {s, -0.5, 0.5},
    {t, -π, π}, PlotPoints → {2, Automatic}, DisplayFunction → Identity];
Table[Show[morph[h], threedims, ViewPoint → CMView,
    PlotRange → {{-2.5, 2.5}, {-2 π, 2 π}, {-1.5, 1.5}},
    Boxed → False, DisplayFunction → $DisplayFunction],
    {h, 0, 1, 1/5}];
```
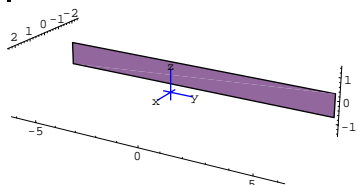

Animate these

Rip a strip of paper out of an old traditional calculus book and try it for yourself.
The round ribbon has two definite sides - the inside and the outside. If you stick your finger on the inside of the round ribbon and move your finger around without lifting it, then your finger never leaves the inside.

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If you stick your finger on the outside of the round ribbon and move your finger around without lifting it, then your finger never leaves the outside.
Can you say anything like this for the Moebius strip?
How many sides does the Moebius strip have?
Does it make any sense to talk about the flow of a vector field across a Moebius strip?

□**G.7.b)**

What happens when you twist the ribbon twice?

G.8) Bidding on rocket parts

You are an engineer working for the United Engineering Company, makers of specialty metal products for the industrial market. A call for bids comes in for some front sections of a rocket upon which ceramic nose cones will be mounted.
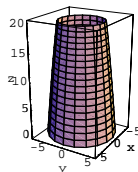The specifications say that the front section is to be made as follows:
You start with the inner parabolic surface with all measurements in meters:

```
Clear[innerx, innery, innerz, r, s]
innerx[r_, s_] = r Cos[s];
innery[r_, s_] = r Sin[s];
innerz[r_, s_] = 9 - r^2;
{innerrlow, innerrhigh} = {2, 3};
{innerslow, innershigh} = {0, 2 π};
innersurface =
 ParametricPlot3D[{innerx[r, s], innery[r, s], innerz[r, s]},
  {r, innerrlow, innerrhigh}, {s, innerslow, innershigh},
  ViewPoint → CMView, PlotRange → All, AxesLabel → {"x", "y", "z"}];
```
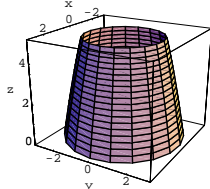
Then you have the outer parabolic surface with all measurements in meters:

```
Clear[outerx, outery, outerz, r, s]
outerx[r_, s_] = 2 r Cos[s];
outery[r_, s_] = 2 r Sin[s];
outerz[r_, s_] = 36 - (2 r)^2;
{outerrlow, outerrhigh} = {2, 3};
{outerslow, outershigh} = {0, 2 π};
outersurface =
 ParametricPlot3D[{outerx[r, s], outery[r, s], outerz[r, s]},
  {r, outerrlow, outerrhigh}, {s, outerslow, outershigh},
  ViewPoint → CMView, PlotRange → All, AxesLabel → {"x", "y", "z"}];
```

The specifications say that you get the top of the front section by running a straight line from each point on the top of the inner surface
    {innerx[2, s], innery[2, s], innerz[2, s]}
to the corresponding point
    {outerx[2, s], outery[2, s], outerz[2, s]}
on the top of the outer surface.
You get the base, on which the section rests, by running a straight line from each point on the bottom of the inner surface
    {innerx[3, s], innery[3, s], innerz[3, s]}
to the corresponding point
    {outerx[3, s], outery[3, s], outerz[3, s]}
on the bottom of the outer surface.

□**G.8.a)**

Measure:
1) The surface area of the inner skin.
2) The surface area of the outer skin.
3) The surface area of the flat ring at the bottom.
4) The surface area of the conical ring at the top.
5) The volume enclosed by the outer skins composed of the outer shell, the flat ring at the bottom, the conical ring at the top and the inner shell.
Report these measurements, and throw in a plot of the finished product including the flat ring at the bottom and the conical ring at the top.
Show it from several advantageous viewpoints.

□**Tip:**

This function might be of use:

```
Clear[section, r, s, t]
section[r_, s_, t_] = {innerx[r, s], innery[r, s], innerz[r, s]} +
  t ({outerx[r, s], outery[r, s], outerz[r, s]} -
    {innerx[r, s], innery[r, s], innerz[r, s]})
```
$$\{r \, \text{Cos}[s] + r\, t\, \text{Cos}[s], \, r\, \text{Sin}[s] + r\, t\, \text{Sin}[s], \, 9 - r^2 + (27 - 3\, r^2)\, t\}$$

G.9) Volumes, flow and bouyancy

□**G.9.a)**

Use Gauss's formula to explain why the flow measurement of the vector field
    Field[x, y, z] = $\{\frac{x}{3}, \frac{y}{3}, \frac{z}{3}\}$
across the surface of a solid region is the volume of the solid region.
How is the measurement of the flow of
    Field[x, y, z] = {x, 0, 0}
across the surface of a solid region related to the volume of the solid region?

□**G.9.b)**

One day a long, long time ago, Archimedes jumped out of his bathtub and ran around the streets of Syracuse yelling "Eureka!" Why? It's a pretty safe bet that he wasn't looking for a vacuum cleaner.
The plain fact is that he had just discovered the idea behind Ivory soap, and it was floating in his bathwater. More than that, he had figured out the reason it floated.
The bar of soap was able to displace only an amount of water of weight equal to that of the bar of soap.

□**G.9.b.i) Rub-a-dub-dub**

Three dirty old men in a tub set out floating around a rather large swimming pool in a definitely large wash tub. In the tub with them were three large pieces of lead, a case of something to drink and some fishing tackle. Archimedes's law says that as they launched their craft, water in an amount with mass equal to the total mass of tub, men, lead, beverage, and tackle was displaced. The only place for the water to go was up, so the level of the water in the pool raised a bit. As the day wore on, the lead weights began to get into their way, so they threw them out of the tub.
Use Archimedes's law to explain what happened to the level of the water in the pool when they got the lead out.

□**G.9.b.ii)**

Look at:

```
water = Plot[0.1 Sin[10 t] + 2, {t, -3, 3}, AspectRatio → Automatic,
  PlotStyle → {{Thickness[0.01], Blue}}, DisplayFunction → Identity];

arrows = Table[Arrow[0.4 {Cos[ (2 k π)/10 ], Sin[ (2 k π)/10 ]},
  Tail → {Cos[ (2 k π)/10 ], Sin[ (2 k π)/10 ]}, VectorColor → Red], {k, 0, 10}];

labels = Graphics[{Text["Ball", {0, 0}],
  Text["Waves", {1.5, 1.8}], Text["Force vectors", {0, 1.5}]}];

Show[{Graphics[{Thickness[0.01], Circle[{0, 0}, 1]}],
  water, arrows, labels}, Axes → None, PlotRange → All,
 DisplayFunction → $DisplayFunction];
```

Waves
Force vectors
Ball
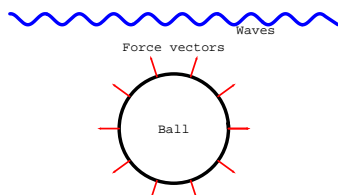
What you have here is a ball submerged in water. At each point (little patch) of the surface of the ball, a force is acting according to the rule

P[x, y, z] = PressureDensity

= FluidDensity g (−z)

where z is depth measured starting from z = 0 at the surface of the pool. The direction of the pressure is inward in the direction of the normal to the surface of the ball at each point. The total (buoyancy) force on the surface, C, of the ball is nothing more than

Force = $\iint_C$ P . n dA.

Your job is to explain Archimedes's law, namely,

Force = W {0, 0, 1},

where W is the weight of the water displaced.


G.10) Rotating and measuring

☐G.10.a)

Here is part of the curve z = Sin[2 x] plotted in the xz-plane:

```
Clear[f, x, y, z]
xlow = 0.5;
xhigh = 3.1;
f[x_] = Sin[2 x];
xzplane = Graphics3D[
  Polygon[{{0, 0, -1.5}, {0, 0, 1.5}, {4, 0, 1.5}, {4, 0, -1.5}}]];
threedims = Axes3D[2, 0.2];
curve = ParametricPlot3D[
  {x, 0, f[x]}, {x, xlow, xhigh}, DisplayFunction → Identity];

Show[xzplane, curve, threedims, ViewPoint → CMView, PlotRange → All,
 DisplayFunction → $DisplayFunction];
```
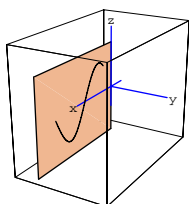

Here's a plot of the surface you get when you rotate this curve around the z-axis:

```
Clear[r, t]
rotated = ParametricPlot3D[{r Cos[t], r Sin[t], f[r]},
  {r, xlow, xhigh}, {t, 0, 2 π}, DisplayFunction → Identity];

Show[rotated, threedims, Axes → Automatic, ViewPoint → CMView,
 PlotRange → All, DisplayFunction → $DisplayFunction];
```
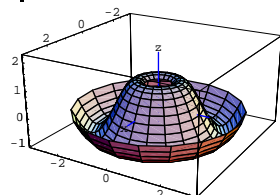

Measure the area of this surface.

☐G.10.b)

Take a function z = f[x]. If you plot it in the xz-plane as above for a ≤ x ≤ b with 0 ≤ a < b, then say how you can arrive at the formula

$$\int_a^b 2 \pi x \sqrt{1 + f'[x]^2} \, dx$$

to measure the area of the surface you get by rotating this curve about the z-axis.

Use this formula to confirm your answer above in part i).

Students in old-fashioned calculus courses spend a lot of time plugging and chugging with this formula.

Do what you did above but run with:

```
Clear[x, y, z, f, r, t]
x[r_, t_] = r Cos[t];
y[r_, t_] = r Sin[t];
z[r_, t_] = f[r];
```

☐G.10.c)

Here is part of the parametric curve

P[s] = {0, 3 s (3 − s), 2 + s} as it sits in the yz-plane:

```
Clear[x, y, z, s]
x[s_] = 0;
y[s_] = 8 - 3 s (3 - s);
z[s_] = 2 + s;
slow = 0;
shigh = 2;
yzplane =
 Graphics3D[Polygon[{{0, 0, 0}, {0, 0, 5}, {0, 8, 5}, {0, 8, 0}}]];
threedims = Axes3D[2, 0.2];
curve = ParametricPlot3D[{x[s], y[s], z[s]},
  {s, slow, shigh}, DisplayFunction → Identity];

Show[curve, yzplane, curve, threedims,
 Axes → Automatic, ViewPoint → CMView, PlotRange → All,
 DisplayFunction → $DisplayFunction];
```
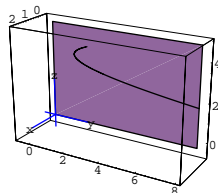

Rotate this curve about the y-axis and plot what you get.
Then measure the area of the surface you get.

☐G.10.d)

Calculus Cal, that calculus lab pest who hangs out in the calculus lab acting as if he knows everything, but in fact knows very little, taps you on the shoulder and tells you that a certain old-fashioned printed calculus text says:

If you want to measure the surface area of the surface explicitly given by

z = f[x, y]

over the rectangle a ≤ x ≤ b and c ≤ y ≤ d, then you calculate

$$\int_c^d \int_a^b \sqrt{1 + \left(\frac{\partial f[x,y]}{\partial x}\right)^2 + \left(\frac{\partial f[x,y]}{\partial y}\right)^2} \, dx \, dy$$
$$= \int_c^d \int_a^b \sqrt{1 + \left(\frac{\partial f[u,v]}{\partial u}\right)^2 + \left(\frac{\partial f[u,v]}{\partial v}\right)^2} \, du \, dv.$$

But Cal can't seem to figure out where the formula comes from and the calculus book really doesn't say where it comes from.

Tell Cal where the formula comes from and then tell Cal where to go.