

2D and 3D Measurements

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VC.05 Flow Measurements by Integrals Basics

B.1) Measuring flow across a curve with the integral

$$\int_{\text{flow}}^{\text{thigh}} (\text{Field}[x[t], y[t]] \cdot \{y'[t], -x'[t]\}) dt$$

Here's a vector field:

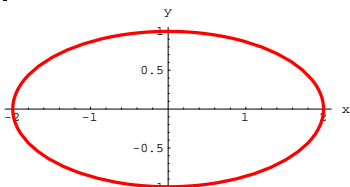
```
Clear[Field, m, n, x, y]
m[x_, y_] = 0.8 Sin[y];
n[x_, y_] = 0.4 x + 0.1 y;

Field[x_, y_] = {m[x, y], n[x, y]}
{0.8 Sin[y], 0.4 x + 0.1 y}
```

Here's a curve:

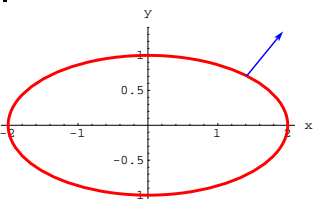
```
Clear[t]
{x[t_], y[t_]} = {2 Cos[t], Sin[t]};
tlow = 0;
thigh = 2 π;

curveplot = ParametricPlot[{x[t], y[t]}, {t, tlow, thigh},
  PlotStyle -> {{Thickness[0.01], Red}}, AxesLabel -> {"x", "y"}];
```



Pick a point on the curve, say $\{x[\frac{\pi}{4}], y[\frac{\pi}{4}]\}$, and look at the field vector at this point:

```
tt = π/4;
Clear[setup]
setup[t_] := Show[curveplot, Arrow[Field[x[t], y[t]],
  Tail -> {x[t], y[t]}], DisplayFunction -> Identity];
Show[setup[tt], DisplayFunction -> $DisplayFunction];
```



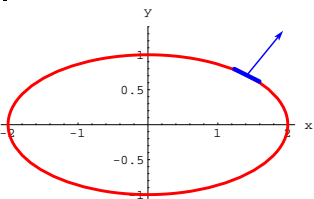
Now look at the curve with a little segment of fluid centered at the base of the field vector:

```
Clear[fluidbit, center, height, width, h, unitn, center]
unitn[t_] = {x'[t], y'[t]} /
  Sqrt[x'[t]^2 + y'[t]^2];

center[t_] = {x[t], y[t]};

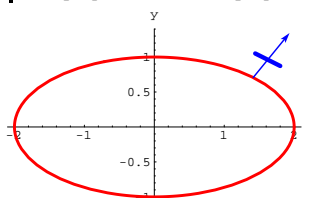
fluidbit[center_, t_, h_] := Graphics[{Thickness[0.015],
  Blue, Line[{center - h unitn[t], center + h unitn[t]}]}];

h = 0.2;
Show[setup[tt], fluidbit[center[tt], tt, h], PlotRange -> All,
  DisplayFunction -> $DisplayFunction];
```



When you check out how the fluid segment moves in a short time you'll see approximately this:

```
flowtime = 0.4;
Show[
  setup[tt], fluidbit[center[tt] + flowtime Field[x[tt], y[tt]], tt, h],
  DisplayFunction -> $DisplayFunction];
```

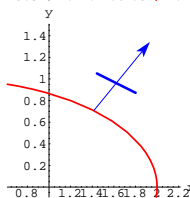
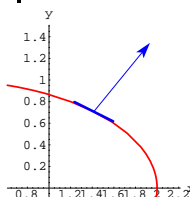


Grab the last two plots and animate.

To get an even better idea, put a microscope on it:

```
Show[setup[tt], fluidbit[center[tt], tt, h],
  PlotRange -> {{x[tt] - 4 h, x[tt] + 4 h}, {y[tt] - 4 h, y[tt] + 4 h}},
  DisplayFunction -> $DisplayFunction];

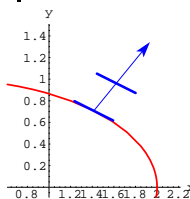
Show[
  setup[tt], fluidbit[center[tt] + flowtime Field[x[tt], y[tt]], tt, h],
  PlotRange -> {{x[tt] - 4 h, x[tt] + 4 h}, {y[tt] - 4 h, y[tt] + 4 h}},
  DisplayFunction -> $DisplayFunction];
```



Grab the last two plots and animate, running at a slow speed.

Now look at both together:

```
Show[setup[tt], fluidbit[center[tt], tt, h],
  fluidbit[center[tt] + flowtime Field[x[tt], y[tt]], tt, h],
  PlotRange -> {{x[tt] - 4 h, x[tt] + 4 h}, {y[tt] - 4 h, y[tt] + 4 h}},
  DisplayFunction -> $DisplayFunction];
```



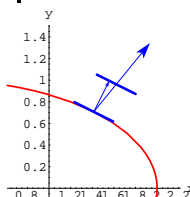
The approximate amount of fluid that flowed across the curve in this short time is measured by the area of the parallelogram defined by the two line segments shown above.

Put in the vector

$\text{flowtime} (\text{Field}[x[tt], y[tt]] \cdot \text{unitnormal}[tt]) \text{unitnormal}[tt]$
for $tt = \frac{\pi}{4}$, and $\text{flowtime} = 0.4$ as above:

```
Clear[unitnormal]
unitnormal[t_] := {y'[t], -x'[t]} /
  Sqrt[y'[t]^2 + x'[t]^2};

Show[setup[tt], fluidbit[center[tt], tt, h],
  fluidbit[center[tt] + flowtime Field[x[tt], y[tt]], tt, h],
  Arrow[flowtime Field[x[tt], y[tt]] . unitnormal[tt],
  unitnormal[tt], Tail -> {x[tt], y[tt]}],
  PlotRange -> {{x[tt] - 4 h, x[tt] + 4 h}, {y[tt] - 4 h, y[tt] + 4 h}},
  DisplayFunction -> $DisplayFunction];
```



Bingo!

The length of the new vector plotted above is
 $\text{flowtime}(\text{Field}[x[t], y[t]] \cdot \text{unitnormal}[t])$.

Putting $ds = \text{length of the fluid segment}$, you can say that in this
 flowtime, approximately

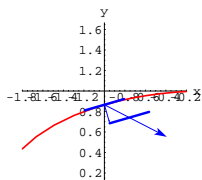
$\text{flowtime}(\text{Field}[x[t], y[t]] \cdot \text{unitnormal}[t]) ds$
 units of fluid flow across the segment of the curve in the above plot.

The flow goes with the normal $\{y'[t], -x'[t]\}$ at this point because
 $\text{Field}[x[t], y[t]] \cdot \text{unitnormal}[t] > 0$:

```
N[Field[x[t], y[t]] . unitnormal[t]]
0.801631
```

See what happens at another point on the curve:

```
tt = 2 π / 3;
Show[setup[tt], fluidbit[center[tt], tt, h],
fluidbit[center[tt] + flowtime Field[x[tt], y[tt]], tt, h],
Arrow[flowtime Field[x[tt], y[tt]] . unitnormal[tt],
unitnormal[tt], Tail -> {x[tt], y[tt]}],
PlotRange -> {{x[tt] - 4 h, x[tt] + 4 h}, {y[tt] - 4 h, y[tt] + 4 h}},
DisplayFunction -> $DisplayFunction];
```



Again, putting $ds = \text{length of the fluid segment}$, you can say that in this
 flowtime, approximately

$\text{flowtime}(\text{Field}[x[t], y[t]] \cdot \text{unitnormal}[t]) ds$
 units of fluid flow across the curve in the above plot.

The flow is opposite the direction of the normal $\{y'[t], -x'[t]\}$ at this
 point because

$$\text{Field}[x[t], y[t]] \cdot \text{unitnormal}[t] < 0:$$

```
N[Field[x[tt], y[tt]] . unitnormal[tt]]
-0.470122
```

This tells you that, in a short flowtime, approximately

$$\text{flowtime}(\text{Field}[x[t], y[t]] \cdot \{y'[t], -x'[t]\}) dt$$

units of fluid flow across the small segment of the curve.

Now, cover the whole curve with non-overlapping little segments of
 length ds as above. Adding the individual measurements together, you
 get the approximate flow-across measurement

$$\text{flowtime} \text{Sum}[(\text{Field}[x[t], y[t]] \cdot \{y'[t], -x'[t]\}) dt], \\ (t, \text{tlow}, \text{thigh} - dt, dt)]$$

As dt closes in on 0, these approximate measurements close in on the
 exact measurement

$$\text{flowtime} \int_{\text{tlow}}^{\text{thigh}} (\text{Field}[x[t], y[t]] \cdot \{y'[t], -x'[t]\}) dt$$

of the net flow across the curve in the given flowtime.

To arrive at the measurement of the net flow across the whole curve
 per time unit, divide by flowtime to learn that

$$\int_{\text{tlow}}^{\text{thigh}} (\text{Field}[x[t], y[t]] \cdot \{y'[t], -x'[t]\}) dt$$

measures the net flow over the curve per time unit.

If measurements are in gallons and seconds the calculation:

```
N[Integrate[Field[x[t], y[t]] . {y'[t], -x'[t]}, {t, 0, 2 π}]
0.628319
```

tells you that the net flow of this vector field across the curve above is
 0.628319 gallons per second.

□B.1.a.i)

Explain why this tells you that the integral

$$\int_{\text{tlow}}^{\text{thigh}} \text{Field}[x[t], y[t]] \cdot \{y'[t], -x'[t]\} dt$$

```
tlow = 0;
thigh = 2 π;
N[Integrate[Field[x[t], y[t]] . {y'[t], -x'[t]}, {t, tlow, thigh}]
0.628319
```

measures the net amount of fluid flowing across this curve per unit of
 flow time.

□Answer:

Fixing a particular t , you know that in a short flowtime approximately

$$\text{flowtime}(\text{Field}[x[t], y[t]] \cdot \text{unitnormal}[t]) ds$$

units of fluid flow across the small segment of the curve of length ds
 centered at $\{x[t], y[t]\}$.

Because you integrate $\sqrt{x'[t]^2 + y'[t]^2}$ to measure length on the
 curve, you know that

$$ds = \sqrt{x'[t]^2 + y'[t]^2} dt.$$

This tells you that, in a short flowtime, approximately

$$\text{flowtime} \text{Field}[x[t], y[t]] \cdot \text{unitnormal}[t] \sqrt{x'[t]^2 + y'[t]^2} dt$$

units of fluid flow across the a small segment of the curve of length ds
 centered at $\{x[t], y[t]\}$.

But

$$\text{unitnormal}[t] = \frac{\{y'[t], -x'[t]\}}{\sqrt{x'[t]^2 + y'[t]^2}},$$

so

$$(\text{Field}[x[t], y[t]] \cdot \text{unitnormal}[t]) \sqrt{x'[t]^2 + y'[t]^2} \\ = \text{Field}[x[t], y[t]] \cdot \{y'[t], -x'[t]\}.$$

□B.1.a.ii)

Take another look at the integral

$$\int_{\text{tlow}}^{\text{thigh}} (\text{Field}[x[t], y[t]] \cdot \{y'[t], -x'[t]\}) dt$$

for the vector field and the curve specified above:

```
N[Integrate[Field[x[t], y[t]] . {y'[t], -x'[t]}, {t, tlow, thigh}]
0.628319
```

This tells you that the net flow of this vector field across this curve is
 about 0.63 fluid units per time unit.

Why does this also tell you that the net flow is from inside to outside?

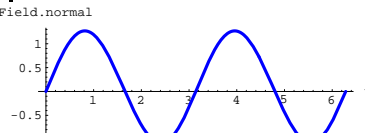
□Answer:

Look at this plot of

$$\text{Field}[x[t], y[t]] \cdot \{y'[t], -x'[t]\}$$

for $\text{tlow} \leq t \leq \text{thigh}$:

```
Plot[Field[x[t], y[t]] . {y'[t], -x'[t]},
{t, 0, 2 π}, PlotStyle -> {{Thickness[0.01], Blue}},
AxesLabel -> {"t", "Field.normal"}];
```



$$\int_{\text{tlow}}^{\text{thigh}} (\text{Field}[x[t], y[t]] \cdot \{y'[t], -x'[t]\}) dt \\ = \text{area above} - \text{area below}.$$

→ area above the t -axis = total flow of this vector field across this
 curve in the direction of the normals

$$\{y'[t], -x'[t]\}.$$

→ area below the t -axis = total flow of this vector field across this
 curve opposite the direction of the normals

$$\{y'[t], -x'[t]\}.$$

In the case above,

```


$$\int_{\text{tlow}}^{\text{thigh}} (\text{Field}[\text{x}[t], \text{y}[t]] \cdot \{y'[t], -x'[t]\}) dt > 0 :$$

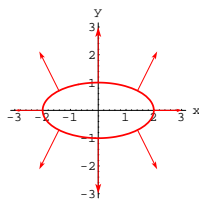
NIntegrate[Field[x[t], y[t]] \cdot {y'[t], -x'[t]}, {t, tlow, thigh}]
0.628319
    
```

This tells you that the flow in the direction of the normals is greater than the flow against the normals.

Take a look at the normals:

```

Show[curveplot,
Table[Arrow[{y'[t], -x'[t]}, Tail -> {x[t], y[t]}, VectorColor -> Red],
{t, tlow, thigh, (thigh - tlow) / 8}]];
    
```



The normals point out way from the curve.

The upshot: The net flow of the given vector field across this curve is from inside to outside because

```


$$\int_{\text{tlow}}^{\text{thigh}} (\text{Field}[\text{x}[t], \text{y}[t]] \cdot \{y'[t], -x'[t]\}) dt > 0 :$$

NIntegrate[Field[x[t], y[t]] \cdot {y'[t], -x'[t]}, {t, tlow, thigh}]
0.628319
    
```

□B.1.b.i)

Here's a new vector field:

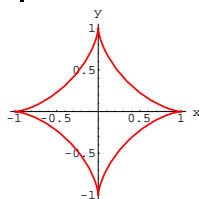
```

Clear[Field, m, n, x, y]
m[x_, y_] = 2 x y;
n[x_, y_] = x^2 - y^2;
Field[x_, y_] = {m[x, y], n[x, y]}
{2 x y, x^2 - y^2}
    
```

Here's a new curve:

```

Clear[t]
{x[t_], y[t_]} = {Cos[t]^3, Sin[t]^3};
tlow = 0;
thigh = 2 \pi;
curveplot = ParametricPlot[{x[t], y[t]}, {t, tlow, thigh},
PlotStyle -> {{Thickness[0.01], Red}}, AxesLabel -> {"x", "y"}];
    
```



Use the integral

$\int_{\text{tlow}}^{\text{thigh}} (\text{Field}[\text{x}[t], \text{y}[t]] \cdot \{y'[t], -x'[t]\}) dt$
to analyze the net flow of this vector field across this curve.

□Answer:

```


$$\int_0^{\text{thigh}} \text{Field}[\text{x}[t], \text{y}[t]] \cdot \{y'[t], -x'[t]\} dt$$

0
    
```

This tells you that the inside-to-outside flow across this curve exactly balances the outside-to-inside flow across the curve. The net flow of this vector field across this curve is 0.

□B.1.b.ii)

Here's another new vector field:

```

Clear[Field, m, n, x, y]
m[x_, y_] = x + y;
n[x_, y_] = x^2 + y^2;
Field[x_, y_] = {m[x, y], n[x, y]}
{x + y, x^2 + y^2}
    
```

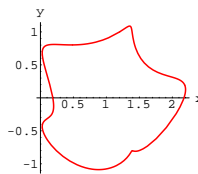
Here's another new curve:

```

Clear[t]
{x[t_], y[t_]} =
    
```

```

{1, 0} + {Sin[\pi t] + \frac{1}{5} Cos[4 \pi t], Cos[\pi t] + \frac{1}{8} Sin[5 \pi t]};
tlow = 0;
thigh = 2;
curveplot = ParametricPlot[{x[t], y[t]}, {t, tlow, thigh},
PlotStyle -> {{Thickness[0.01], Red}}, AxesLabel -> {"x", "y"}];
    
```



Use the integral

$\int_{\text{tlow}}^{\text{thigh}} (\text{Field}[\text{x}[t], \text{y}[t]] \cdot \{y'[t], -x'[t]\}) dt$
to help analyze the net flow of this vector field across this curve.

□Answer:

The parameterization of the curve involves squirrely functions, so try

NIntegrate:

```

NIntegrate[Field[x[t], y[t]] \cdot {y'[t], -x'[t]}, {t, tlow, thigh},
AccuracyGoal -> 2]
-2.82743
    
```

This tells you

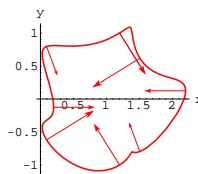
$$\int_{\text{tlow}}^{\text{thigh}} (\text{Field}[\text{x}[t], \text{y}[t]] \cdot \{y'[t], -x'[t]\}) dt < 0.$$

The net flow across this curve is in the opposite direction of the normals.

See which way the normals point:

```

scalefactor = 0.2;
Show[curveplot, Table[Arrow[{y'[t], -x'[t]}, Tail -> {x[t], y[t]}, ScaleFactor -> scalefactor, VectorColor -> Red],
{t, tlow, thigh, (thigh - tlow) / 8}]];
    
```



The normals point to the inside of the curve and the net flow is opposite to the direction of these inward-pointing normals. The result: The net flow of this vector field across this curve is from inside to outside. Math happens again.

B.2) Measuring flow along a curve with the integral

$$\int_{\text{tlow}}^{\text{thigh}} (\text{Field}[\text{x}[t], \text{y}[t]] \cdot \{x'[t], y'[t]\}) dt$$

□B.2.a)

Given a vector field $\text{Field}[x, y]$ and a curve specified through parametric equations $\{x[t], y[t]\}$ with $\text{tlow} \leq t \leq \text{thigh}$, you use the integral

$$\int_{\text{tlow}}^{\text{thigh}} \text{Field}[\text{x}[t], \text{y}[t]] \cdot \{y'[t], -x'[t]\} dt$$

to measure the flow of the field across the curve.

What do you measure when you calculate

$$\int_{\text{tlow}}^{\text{thigh}} \text{Field}[\text{x}[t], \text{y}[t]] \cdot \{x'[t], y'[t]\} dt?$$

□Answer:

$\{y'[t], -x'[t]\}$ is a normal vector, and when you calculate

$$\int_{\text{tlow}}^{\text{thigh}} \text{Field}[\text{x}[t], \text{y}[t]] \cdot \{y'[t], -x'[t]\} dt,$$

you measure net flow across the curve.

Analogously, $\{x'[t], y'[t]\}$ is a tangent vector, and when you calculate

$\int_{t_{low}}^{t_{high}} \text{Field}[x[t], y[t]] \cdot \{x'[t], y'[t]\} dt$,
 you measure net flow ALONG the curve.

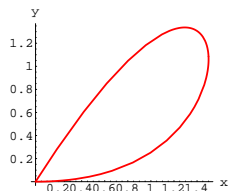
□B.2.b)

Here's a vector field:

```
Clear[Field, m, n, x, y, t]
m[x_, y_] = y Sin[x];
n[x_, y_] = -x Cos[y];
Field[x_, y_] = {m[x, y], n[x, y]}
{y Sin[x], -x Cos[y]}
```

And here is a curve C:

```
tlow = 0;
thigh = 1;
x[t_] = 6 t (1 - t);
y[t_] = 6 t^2 Cos[ $\frac{\pi t}{2}$ ];
curveplot = ParametricPlot[{x[t], y[t]}, {t, 0, 1},
PlotStyle -> {{Red, Thickness[0.01]}}, AxesLabel -> {"x", "y"},
PlotRange -> All];
```



Use the integral

$\int_{t_{low}}^{t_{high}} \text{Field}[x[t], y[t]] \cdot \{x'[t], y'[t]\} dt$
 to determine whether the net flow of Field[x, y] along C is clockwise or is counterclockwise.

□Answer:

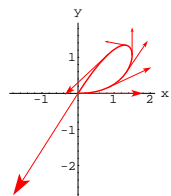
Make the flow-along-the-curve measurement

```
 $\int_{t_{low}}^{t_{high}} \text{Field}[x[t], y[t]] \cdot \{x'[t], y'[t]\} dt$ 
NIntegrate[Field[x[t], y[t]] . {x'[t], y'[t]}, {t, tlow, thigh}]
-1.46585
```

Negative. This means that the flow of Field[x, y] along C is against the direction of the tangent vectors {x'[t], y'[t]} of this parameterization of C.

Take a look at some of these tangent vectors:

```
scalefactor = 0.3;
tanvectors =
Table[Arrow[{x'[t], y'[t]}, Tail -> {x[t], y[t]}, VectorColor -> Red,
ScaleFactor -> scalefactor], {t, tlow, thigh,  $\frac{\text{thigh} - \text{tlow}}{6}$ };
Show[curveplot, tanvectors];
```



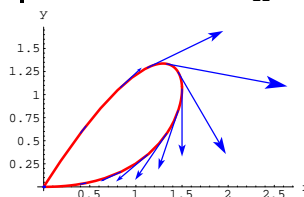
This curve is parameterized in the counterclockwise way. Because

$\int_{t_{low}}^{t_{high}} (\text{Field}[x[t], y[t]] \cdot \{x'[t], y'[t]\}) dt$
 turned out negative, this tells you that the flow of this field along this curve is clockwise.

Confirm with a plot of some of the tangential components of the field vectors on the curve:

```
Clear[tan, tancomp]
tan[t_] = {x'[t], y'[t]};
```

```
tancomp[t_] =  $\frac{\text{Field}[x[t], y[t]] \cdot \text{tan}[t]}{\text{tan}[t] \cdot \text{tan}[t]} \text{tan}[t];$ 
Show[curveplot,
Table[Arrow[tancomp[t], Tail -> {x[t], y[t]}, VectorColor -> Blue],
{t, tlow, thigh,  $\frac{\text{thigh} - \text{tlow}}{12}$ }}];
```



Yessiree-Bob.

Net clockwise flow, just as the measurement predicted.

Math continues to work.

B.3) Measurements by path integrals

$$\int_C m[x, y] dx + n[x, y] dy$$

□B.3.a.i)

Here's some crazy looking notation:

$$\int_C m[x, y] dx + n[x, y] dy$$

Folks call this a path integral.

Just what is a path integral?

□Answer:

A path integral needs the following ingredients:

- It needs two functions m[x, y] and n[x, y].
- It needs a curve C with a specified direction.

The resulting path integral written as by

$$\int_C m[x, y] dx + n[x, y] dy$$

The path integral

$$\int_C m[x, y] dx + n[x, y] dy$$

is calculated by evaluating the old-fashioned integral

$$\int_{t_{low}}^{t_{high}} (m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t]) dt$$

where {x[t], y[t]}, tlow ≤ t ≤ thigh is any parameterization of C that gives the curve the same direction as the specified direction of C.

Look familiar?

When you work with closed curves (like deformed circles with no loops), folks all across our planet have agreed to specify the counterclockwise direction. There is a nifty piece of notation to do this. For closed curves C without loops,

$$\oint_C m[x, y] dx + n[x, y] dy$$

means that the parameterization you use to evaluate the path integral is counterclockwise.

□B.3.a.ii) Calculating a path integral

Calculate

$$\oint_C m[x, y] dx + n[x, y] dy$$

for the case in which

$$m[x, y] = y - x,$$

$$n[x, y] = 2xy$$

and C is the ellipse

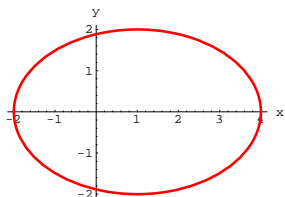
$$\left(\frac{x-1}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1.$$

□Answer:

Enter the vector field and go with a counterclockwise parameterization of the ellipse:

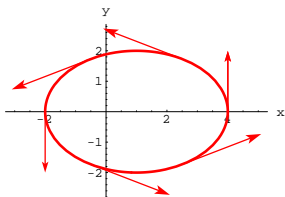
```
Clear[x, y, m, n, t]
m[x_, y_] = y - x;
n[x_, y_] = 2 x y;
{x[t_], y[t_]} = {1, 0} + {3 Cos[t], 2 Sin[t]};
tlow = 0;
thigh = 2 π;

curveplot = ParametricPlot[{x[t], y[t]}, {t, tlow, thigh},
  PlotStyle -> {Thickness[0.01], Red}], AxesLabel -> {"x", "y"}];
```



Check to be sure that the parameterization is counterclockwise:

```
Show[curveplot,
  Table[Arrow[{x'[t], y'[t]}, Tail -> {x[t], y[t]}, VectorColor -> Red],
    {t, tlow, thigh, (thigh - tlow) / 6}]]];
```



Good.

Here is the calculation of

$$\oint_C m[x, y] dx + n[x, y] dy:$$

```
NIntegrate[m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t], {t, tlow, thigh}]
```

-18.8496

Done.

□B.3.b.i) Path integrals as a flow-along-the-curve measurement

Most folks say the path integral $\oint_C m[x, y] dx + n[x, y] dy$ measures the flow of the vector field $\text{Field}[x, y] = \{m[x, y], n[x, y]\}$ along a closed curve C.

They go on to say that if $\oint_C m[x, y] dx + n[x, y] dy > 0$,

then the net flow of $\text{Field}[x, y] = \{m[x, y], n[x, y]\}$ along C is counterclockwise, but if

$\oint_C m[x, y] dx + n[x, y] dy < 0$, then the net flow of $\text{Field}[x, y] = \{m[x, y], n[x, y]\}$ along C is clockwise.

Where do they get this idea?

□Answer:

Remember:

$$\oint_C m[x, y] dx + n[x, y] dy$$

demands a counterclockwise parameterization.

Also remember:

When a closed curve C is parameterized in the counterclockwise way with a parameterization $\{x[t], y[t]\}$ with $tlow \leq t \leq thigh$, then the tangent vectors $\{x'[t], y'[t]\}$ automatically point in the counterclockwise direction.

When you measure the flow of a vector field

$$\text{Field}[x, y] = \{m[x, y], n[x, y]\}$$

along such a curve, you calculate

$$\begin{aligned} & \int_{tlow}^{thigh} (\text{Field}[x[t], y[t]] \cdot \{x'[t], y'[t]\}) dt \\ &= \int_{tlow}^{thigh} (m[x[t], y[t]] n[x[t], y[t]] \cdot \{x'[t], y'[t]\}) dt \\ &= \int_{tlow}^{thigh} (m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t]) dt \\ &= \oint_C m[x, y] dx + n[x, y] dy. \end{aligned}$$

The result:

$$\oint_C m[x, y] dx + n[x, y] dy$$

measures the flow of the vector field $\text{Field}[x, y] = \{m[x, y], n[x, y]\}$ around a closed curve C.

Consequently, if

$$\oint_C m[x, y] dx + n[x, y] dy > 0,$$

then the net flow of $\text{Field}[x, y] = \{m[x, y], n[x, y]\}$ around C is counterclockwise, but if

$$\oint_C m[x, y] dx + n[x, y] dy < 0,$$

then the net flow of $\text{Field}[x, y] = \{m[x, y], n[x, y]\}$ around C is clockwise.

□B.3.b.ii)

Calculate

$$\oint_C 5xy dx + (x^3 + y^2) dy$$

where C is the circle

$$x^2 + (y - 1)^2 = 4$$

and interpret the result.

□Answer:

$$\oint_C 5xy dx + (x^3 + y^2) dy$$

measures the net flow of the vector field

$$\text{Field}[x, y] = \{5xy, x^3 + y^2\}$$

along the circle.

Here are

→ a counterclockwise parameterization of C, and

→ the calculation of

$$\oint_C 5xy dx + (x^3 + y^2) dy:$$

```
Clear[m, n, x, y, t]
tlow = 0;
thigh = 2 π;

{m[x_, y_], n[x_, y_]} = {5 x y, x^3 + y^2};
{x[t_], y[t_]} = {0, 1} + 2 {Cos[t], Sin[t]};

NIntegrate[m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t], {t, tlow, thigh}]
```

37.6991

Big time positive.

This tells you that the net flow of the vector field

$$\text{Field}[x, y] = \{5xy, x^3 + y^2\}$$

along the circle is strongly counterclockwise.

Check it out with a plot if you don't believe it.

□B.3.c.i) Path integrals as a flow-across-the-curve measurement

Most folks say the path integral

$$\oint_C -n[x, y] dx + m[x, y] dy$$

measures the flow of the vector field

$$\text{Field}[x, y] = \{m[x, y], n[x, y]\}$$

across a closed curve C.

They go on to say that if

$$\oint_C -n[x, y] dx + m[x, y] dy > 0,$$

then the net flow of $\text{Field}[x, y] = \{m[x, y], n[x, y]\}$ across C is from inside to outside, but if

$$\oint_C -n[x, y] dx + m[x, y] dy < 0,$$

then the net flow of $\text{Field}[x, y] = \{m[x, y], n[x, y]\}$ across C is from outside to inside.

Where do they get this idea?

□Answer:

Remember:

$$\oint_C -n[x, y] dx + m[x, y] dy$$

demands a counterclockwise parameterization.

Also remember:

When a closed curve C is parameterized in the counterclockwise way with a parameterization

$$\{x[t], y[t]\} \text{ with } t_{\text{low}} \leq t \leq t_{\text{high}},$$

then the normal vectors $\{y'[t], -x'[t]\}$ automatically point out away from the inside to the outside of the curve.

When you measure the flow of a vector field

$$\text{Field}[x, y] = \{m[x, y], n[x, y]\}$$

across such a curve, you calculate

$$\int_{t_{\text{low}}}^{t_{\text{high}}} (\text{Field}[x[t], y[t]] \cdot \{y'[t], -x'[t]\}) dt$$

$$= \int_{t_{\text{low}}}^{t_{\text{high}}} \{m[x[t], y[t]], n[x[t], y[t]]\} \cdot \{y'[t], -x'[t]\} dt$$

$$= \int_{t_{\text{low}}}^{t_{\text{high}}} (m[x[t], y[t]] y'[t] - n[x[t], y[t]] x'[t]) dt$$

$$= \oint_C -n[x, y] dx + m[x, y] dy.$$

The upshot:

$$\oint_C -n[x, y] dx + m[x, y] dy$$

measures the flow of the vector field

$$\text{Field}[x, y] = \{m[x, y], n[x, y]\}$$

across C.

To wrap it up: If

$$\oint_C -n[x, y] dx + m[x, y] dy > 0,$$

then the net flow of $\text{Field}[x, y] = \{m[x, y], n[x, y]\}$ across C is from inside to outside, but if

$$\oint_C -n[x, y] dx + m[x, y] dy < 0,$$

then the net flow of $\text{Field}[x, y] = \{m[x, y], n[x, y]\}$ across C is from outside to inside.

□B.3.c.ii)

Calculate

$$\oint_C E^y dx - E^x dy$$

where C is the circle

$$x^2 + (y - 0.5)^2 = 0.7$$

and give two interpretations of the measurement.

□Answer:

Here are

→ a counterclockwise parameterization of C, and

→ the calculation of

$$\oint_C E^y dx - E^x dy :$$

```
Clear[m, n, x, y, t]
tlow = 0;
thigh = 2 π;

{m[x_, y_], n[x_, y_]} = {E^y, -E^x};
{x[t_], y[t_]} = {0, 0.5} + √0.7 {Cos[t], Sin[t]};

NIntegrate[m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t], {t, tlow, thigh}]
-6.3496
```

Negative.

→ Flow-along measurement interpretation:

$$\oint_C m[x, y] dx + n[x, y] dy$$

measures the net flow of

$$\text{Field}[x, y] = \{m[x, y], n[x, y]\} \text{ along C.}$$

The path integral calculated here was

$$\oint_C E^y dx - E^x dy < 0$$

So:

The net flow of the vector field

$$\text{Field}[x, y] = \{E^y, -E^x\}$$

along the circle is clockwise.

→ Flow-across measurement interpretation:

$$\oint_C -n[x, y] dx + m[x, y] dy$$

measures the net flow of

$$\text{Field}[x, y] = \{m[x, y], n[x, y]\} \text{ across C.}$$

The path integral calculated here was

$$\oint_C E^y dx - E^x dy < 0$$

So:

The net flow of the vector field

$$\text{Field}[x, y] = \{-E^x, -E^y\}$$

across the circle is from outside to inside.

Note carefully that the two interpretations involve different vector fields.

B.4) Directed curves; path integrals

$$\int_C m[x, y] dx + n[x, y] dy,$$

path independence, and gradient fields

□B.4.a)

Lots of folks say that a parameterization gives a curve a direction. What do they mean by this?

□Answer:

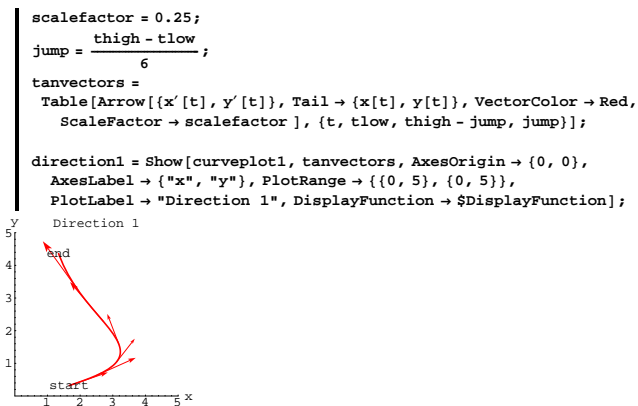
The direction your parameterization goes specifies a direction for the curve.

Here's a curve $\{x[t], y[t]\}$ with a few scaled tangent vectors $\{x'[t], y'[t]\}$:

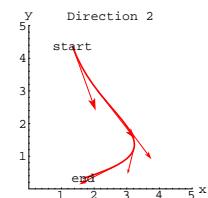
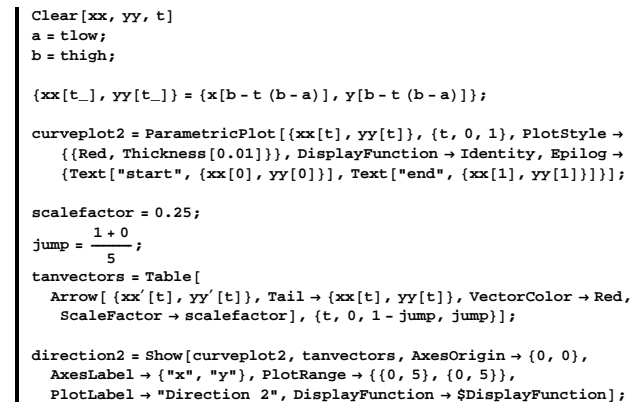
```
Clear[x, y, t, direction]
tlow = 0.25;
thigh = 1.25;

{x[t_], y[t_]} = {1 + 3 Sin[2 t]^2 Cos[t], t E^t};

curveplot1 = ParametricPlot[{x[t], y[t]}, {t, tlow, thigh},
PlotStyle -> {{Red, Thickness[0.01]}}, DisplayFunction -> Identity,
Epilog -> {Text["start", {x[tlow], y[tlow]}],
Text["end", {x[thigh], y[thigh]}]}];
```

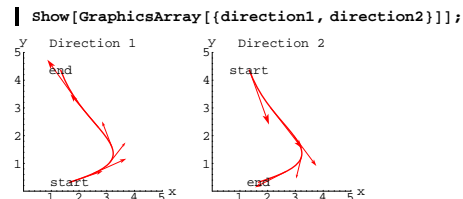


You can plot the same physical curve the reverse direction by changing the parameterization:

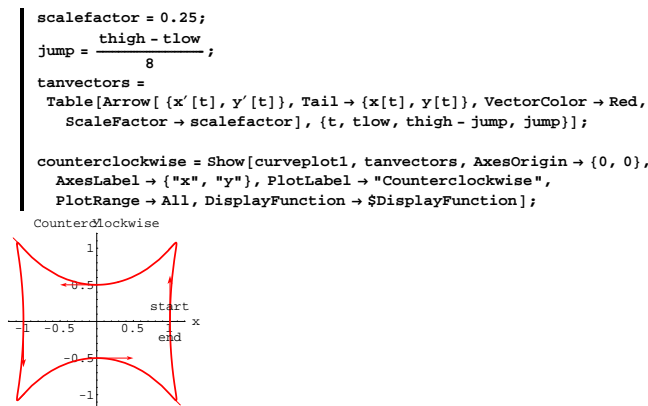
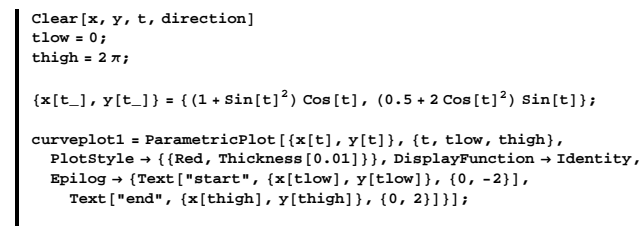


The curve is physically the same curve, but this new parameterization directs it to run from high to low.

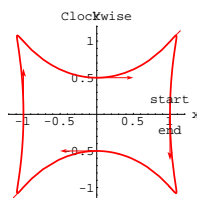
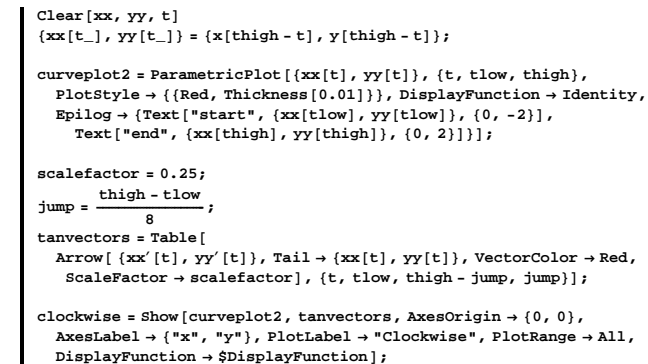
Compare:



If the curve C is closed (like a deformed circle) and has no loops, then there is no natural start or end. Your parametrization gives a start, an end, and a clockwise or counterclockwise direction. It's all up to you. Here is a closed curve parameterized in the counterclockwise direction with the start point the same as the end point:



Here is the same curve with a clockwise parameterization with the start point the same as the end point:



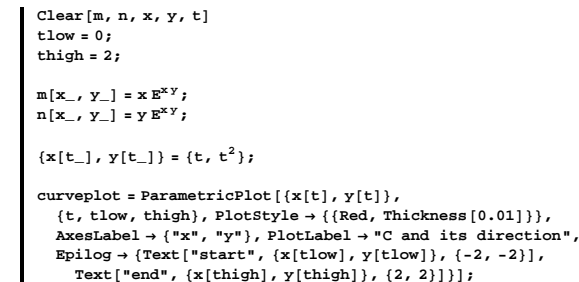
For closed curves without loops, the clockwise and counterclockwise directions are the only choices you have.

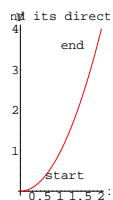
□B.4.b)

Calculate $\int_C x E^{xy} dx + y E^{xy} dy$ where C is the part of the parabola $y = x^2$ starting at {0, 0} and ending at {2, 4}. Interpret the meaning of the result.

□Answer:

Here's everything you need:





Yep, the curve C runs on the parabola $y = x^2$ and starts at $(0, 0)$ and ends at $(2, 4)$.

Here comes the calculation of $\int_C x E^{xy} dx + y E^{xy} dy$:

```
INIntegrate[m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t], {t, tlow, thigh}]
4312.75
```

Humongously positive.

→ Flow-along interpretation:

$$\int_C x E^{xy} dx + y E^{xy} dy$$

measures the net flow of the vector field

$$\text{Field}[x, y] = \{x E^{xy}, y E^{xy}\}$$

along C. The net flow of this vector field is strongly in the direction of the parameterization of the curve (from low to high).

→ Flow-across interpretation:

$$\int_C x E^{xy} dx + y E^{xy} dy$$

measures the net flow of the vector field

$$\text{Field}[x, y] = \{y E^{xy}, -x E^{xy}\}$$

across C. The net flow of this vector field across C is strongly in the direction of the normals $\{y'[t], -x'[t]\}$. These normals point to the right as you advance along the curve in the direction of the parameterization, so the net flow of the vector field

$$\text{Field}[x, y] = \{y E^{xy}, -x E^{xy}\}$$

across C is from above C to below C.

□B.4.c.i) Fundamental formula for path integrals of gradient fields

Here's a cleared gradient field:

```
Clear[f, x, y, m, n, gradf, Field]
gradf[x_, y_] := {D[f[x, y], x], D[f[x, y], y]};
{m[x_, y_], n[x_, y_]} = gradf[x, y];
Field[x_, y_] := {m[x, y], n[x, y]}
{f^(2,0)[x, y], f^(0,1)[x, y]}
```

Here's Mathematica's calculation of

$$\int_C m[x, y] dx + n[x, y] dy$$

for a cleared parameterization of a curve C that starts at $\{x[tlow], y[tlow]\}$ and ends at $\{x[thigh], y[thigh]\}$:

```
Clear[t]
Integrate[m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t] dt,
{t, tlow, thigh}]
-f[x[0], y[0]] + f[x[2], y[2]]
```

Explain where the answer comes from.

□Answer:

Put:

```
Clear[g]
g[t_] := f[x[t], y[t]]
f[x[t], y[t]]
```

Compare:

```
D[g[t], t] == m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t]
True
```

This tells you

$$g'[t] = m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t].$$

The fundamental formula of calculus tells you

$$f[x[thigh], y[thigh]] - f[x[tlow], y[tlow]]$$

$$= g[thigh] - g[tlow]$$

$$= \int_{tlow}^{thigh} g'[t] dt$$

$$= \int_{tlow}^{thigh} (m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t]) dt$$

$$= \int_C m[x, y] dx + n[x, y] dy.$$

So

$$\int_C m[x, y] dx + n[x, y] dy$$

$$= f[x[thigh], y[thigh]] - f[x[tlow], y[tlow]]$$

in the case that $\{m[x, y], n[x, y]\} = \nabla f[x, y]$.

The explanation is over.

□B.4.c.ii) Path independence for gradient fields

Now you know why you are guaranteed that

$$f[x_1, y_1] - f[x_0, y_0] = \int_C m[x, y] dx + n[x, y] dy$$

for any curve C starting at $\{x_0, y_0\}$ and ending at $\{x_1, y_1\}$ provided

$$\{m[x, y], n[x, y]\} = \nabla f[x, y]$$

for a function $f[x, y]$.

What calculational advantage do you get from this?

What theoretical advantage do you get from this?

□Answer:

→ Your calculational advantage:

Here's a gradient field:

```
Clear[f, x, y, gradf, m, n]
f[x_, y_] := Sin[π x y];
gradf[x_, y_] := {D[f[x, y], x], D[f[x, y], y]};
{m[x_, y_], n[x_, y_]} = gradf[x, y]
{π y Cos[π x y], π x Cos[π x y]}
```

Because $\{m[x, y], n[x, y]\} = \nabla f[x, y]$, you are guaranteed that if C is any curve running from $\{0, 0\}$ to $\{1, \frac{5}{2}\}$, then

$$\int_C m[x, y] dx + n[x, y] dy$$

is given by:

$$\left[f\left[1, \frac{5}{2}\right] - f\left[0, 0\right] \right]$$

Your calculational advantage is that you don't have to set up any parameterizations to calculate the path integral

$$\int_C m[x, y] dx + n[x, y] dy.$$

→ Your theoretical advantage:

When you know that

$$\{m[x, y], n[x, y]\} = \nabla f[x, y],$$

then you know that

$$\int_C m[x, y] dx + n[x, y] dy$$

DOES NOT DEPEND ON THE ROUTE OF THE PATH C takes from its start to its end. In fact, the value of

$$\int_C m[x, y] dx + n[x, y] dy$$

depends ONLY on the starting point and the ending point of C.

□B.4.c.iii) The net flow of a gradient field along any closed curve is 0

If $\text{Field}[x, y] = \nabla f[x, y]$ for a function $f[x, y]$, then how do you know that the flow of $\text{Field}[x, y]$ along any closed curve is 0?

□Answer:

Put

$$\text{Field}[x, y] = \{m[x, y], n[x, y]\} = \nabla f[x, y].$$

You are guaranteed that

$$f[x_1, y_1] - f[x_0, y_0] = \int_C m[x, y] dx + n[x, y] dy$$

for any curve C starting at $\{x_0, y_0\}$ and ending at $\{x_1, y_1\}$.

But for a closed curve C (like a deformed circle), you know that

$$\{x_0, y_0\} = \{x_1, y_1\}.$$

So

$$\oint_C m[x, y] dx + n[x, y] dy$$

$$= f[x_1, y_1] - f[x_0, y_0]$$

$$= f[x_0, y_0] - f[x_0, y_0] = 0$$

in the case that C is a closed curve.

That's all there is to it.

Try it out for $\{m[x, y], n[x, y]\} = \nabla f[x, y]$ with

$$f[x, y] = x^4 y^2,$$

and with C the circle of radius 0.5 centered at $\{0, 0\}$:

```
Clear[f, x, y, m, n, gradf, Field, t]
f[x_, y_] = x^4 y^2;

gradf[x_, y_] = {D[f[x, y], x], D[f[x, y], y]};
{m[x_, y_], n[x_, y_]} = gradf[x, y];

Field[x_, y_] = {m[x, y], n[x, y]};
tlow = 0;
thigh = 2 π;

{x[t_], y[t_]} = 0.5 {Cos[t], Sin[t]};

∫tlowthigh (m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t]) dt
0.
```

Just as theory predicted.

VC.05 Flow Measurements by Integrals Tutorials

T.1) Backward and forward

□T.1.a.i)

Explain why:

If C_1 and C_2 are the same physical curve, but
→ the starting point of C_1 = the ending point of C_2
and

→ the ending point of C_1 = the starting point of C_2 ,
then for any $m[x, y]$ and $n[x, y]$ that come down the pike, you will
always get

$$\int_{C_1} m[x, y] dx + n[x, y] dy$$

$$= - \int_{C_2} m[x, y] dx + n[x, y] dy.$$

In other words, if you reverse the direction, then you reverse the sign
of the path integral.

□Answer:

Both

$$\int_{C_1} m[x, y] dx + n[x, y] dy$$

and

$$\int_{C_2} m[x, y] dx + n[x, y] dy$$

measure the flow of the vector field

$$\text{Field}[x, y] = \{m[x, y], n[x, y]\}$$

along the same curve. But the interpretation is different in each case,
because the unit tangent vectors on C_1 point in the direction exactly
opposite of those on C_2 .

The opposite direction of the tangent vectors accounts for the minus
sign.

□T.1.a.ii)

Illustrate by calculating the path integral

$$\int_{C_1} y^3 dx + x^2 y dy,$$

where C_1 is the segment of the parabola

$$y = x^2$$

starting at $\{0, 0\}$ and ending at $\{2, 4\}$, and then calculating the path
integral

$$\int_{C_2} y^3 dx + x^2 y dy$$

where C_2 is the segment of the same parabola, but starting at $\{2, 4\}$
and ending at $\{0, 0\}$.

□Answer:

```
Clear[m, n, x, y]
m[x_, y_] = y^3;
n[x_, y_] = x^2 y;
```

To calculate the path integral

$$\int_{C_1} y^3 dx + x^2 y dy,$$

where C_1 is the segment of the parabola

$$y = x^2$$

starting at $\{0, 0\}$ and ending at $\{2, 4\}$, parameterize C_1 so that it runs
from $\{0, 0\}$ to $\{2, 4\}$ and integrate:

```
Clear[t]
x[t_] = t;
y[t_] = t^2;
tlow = 0;
thigh = 2;
start1 = {x[tlow], y[tlow]}
{0, 0}
end1 = {x[thigh], y[thigh]}
{2, 4}
C1pathintegral =
NIntegrate[m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t], {t, tlow, thigh}]
39.619
```

To calculate the path integral

$$\int_{C_2} y^3 dx + x^2 y dy$$

where C_2 is the segment of the parabola $y = x^2$ starting at $\{2, 4\}$ and
ending at $\{0, 0\}$, parameterize C_2 so that it starts at $\{2, 4\}$ and runs to
 $\{0, 0\}$ and integrate:

```
Clear[x, y]
x[t_] = 2 - t;
y[t_] = (2 - t)^2;
tlow = 0;
thigh = 2;
start2 = {x[tlow], y[tlow]}
{2, 4}
end2 = {x[thigh], y[thigh]}
{0, 0}
C2pathintegral =
NIntegrate[m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t], {t, tlow, thigh}]
-39.619
```

Compare:

```
{C1pathintegral, C2pathintegral}
{39.619, -39.619}
```

Just as you knew in advance; they are negatives of each other.

□T.1.b.i)

Calculate the path integral

$$\int_C -x^2 y dx + 2xy dy,$$

where C starts at $(-1, 2)$, runs on a straight line to $(8, -1)$, and then runs on the straight line from $(8, -1)$ to $(4, 7)$, where C ends.

□Answer:

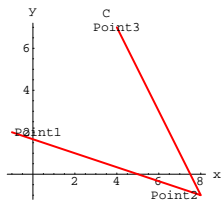
Enter the integrands:

```
Clear[m, n, x, y]
{m[x_, y_], n[x_, y_]} = {-x^2 y, 2 x y}
{-x^2 y, 2 x y}
```

Here's a look at C:

```
point1 = {-1, 2};
point2 = {8, -1};
point3 = {4, 7};
points = {point1, point2, point3};

Show[Graphics[{Red, Thickness[0.01], Line[points]}],
Axes -> True, AxesLabel -> {"x", "y"},
PlotLabel -> "C", Epilog -> {Text["Point1", point1, {-1, 0}],
Text["Point2", point2, {1, 0}], Text["Point3", point3]}];
```



Here comes the parameterization that makes C run from point1 to point2 and then to point3 and the calculation of

$$\int_C -x^2 y dx + 2xy dy:$$

```
Clear[x1, y1, x2, y2, t]
{x1[t_], y1[t_]} = point1 + t (point2 - point1);
```

```
{x2[t_], y2[t_]} = point2 + t (point3 - point2);
tlow = 0;
thigh = 1;

Cpathintegral = NIntegrate[
m[x1[t], y1[t]] x1'[t] + n[x1[t], y1[t]] y1'[t], {t, tlow, thigh}] +
NIntegrate[m[x2[t], y2[t]] x2'[t] + n[x2[t], y2[t]] y2'[t],
{t, tlow, thigh}]
```

624.583

And you're out of here.

□T.1.b.ii)

Calculate the path integral

$$\int_C -x^2 y dx + 2xy dy$$

where C starts at $(4, 7)$, runs on a straight line to $(8, -1)$ and then runs on the straight line from $(8, -1)$ to $(-1, 2)$:

□Answer:

This is almost the same as in part i) above.

The curve C is physically the same here as it was in part i).

The only difference is that this time the direction of C is the reverse of the direction in part i).

To do the calculation, take the answer to part i) above and multiply it by -1 .

□T.1.c.i) Clockwise versus counterclockwise

It's late and you're calculating

$$\int_C y^2 dx + (2x^2 + y) dy$$

where C is the ellipse $(\frac{x+1}{4})^2 + (\frac{y}{2})^2 = 1$.

In your haste to meet your date at the local hangout, you type:

```
Clear[m, n, x, y, t]
m[x_, y_] = y^2;
n[x_, y_] = 2 x^2 + y;
{x[t_], y[t_]} = {-1, 0} + {4 Sin[t], 2 Cos[t]};
answer =
NIntegrate[m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t], {t, 0, 2 pi}]
```

100.531

A friend looking over your shoulder says: "Good work, but your answer is wrong because your parameterization is clockwise and not counterclockwise."

After looking at a plot of some tangent vectors, you see that your parameterization is clockwise.

And then you say, "The correct answer is:"

```
correctanswer = -answer
-100.531
```

Are you right?

□Answer:

Yes.

T.2) Screwing up

□T.2.a)

What are the best ways of screwing up the calculation of a path integral

$$\oint_C m[x, y] dx + n[x, y] dy?$$

□Answer:

The best way to screw up is to give a clockwise parameterization instead of a counterclockwise parameterization of C.

The second best way to screw up is to give a counterclockwise parameterization that covers C more than once.

Case in point:

Calculate

$$\oint_C -y dx + x dy,$$

given that C is the circle $x^2 + y^2 = 1$.

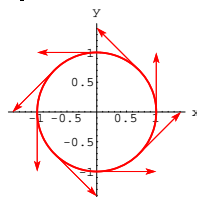
```
Clear[m, n, x, y]
m[x_, y_] = -y;
n[x_, y_] = x;
```

```
{x[t_], y[t_]} = {Cos[t], Sin[t]};
tlow = 0;
thigh = 4 pi;

pathintegral =
NIntegrate[m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t], {t, tlow, thigh}]
12.5664
```

Check the parameterization:

```
curveplot = ParametricPlot[{x[t], y[t]},
{t, tlow, thigh}, PlotStyle -> {{Red, Thickness[0.01]}},
AxesLabel -> {"x", "y"}, DisplayFunction -> Identity];
jump = pi/4;
tangentvectors =
Table[Arrow[{x'[t], y'[t]}, Tail -> {x[t], y[t]}, VectorColor -> Red],
{t, tlow, thigh - jump, jump}];
Show[curveplot, tangentvectors, DisplayFunction -> $DisplayFunction];
```



So far, so good. Everything looks fine.

Now look at:

```
{x[tlow], y[tlow]} == {x[thigh], y[thigh]}
True
```

The starting point and the ending point are the same.

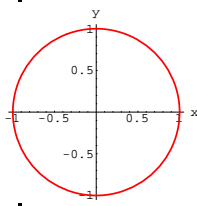
So this parameterization passes the usual tests, but the calculated value of

$$\oint_C -y dx + x dy$$

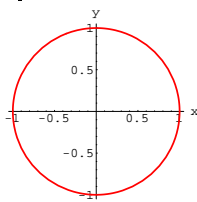
above is DEAD WRONG.

To see why, look at:

```
ParametricPlot[{x[t], y[t]}, {t, tlow, 2 π},
  PlotStyle → {{Red, Thickness[0.01]}}, AxesLabel → {"x", "y"}];
```



```
ParametricPlot[{x[t], y[t]}, {t, 2 π, thigh},
  PlotStyle → {{Red, Thickness[0.01]}}, AxesLabel → {"x", "y"}];
```



As t advances from $t_{low} = 0$ to $t_{high} = 4\pi$ as originally specified, the parameterization goes around the curve twice, and this is not what you want.

The right value for

$$\oint_C -y dx + x dy$$

is:

```
tlow = 0;
correctthigh = 2 π;
correctpathintegral = NIntegrate[
  m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t], {t, tlow, correctthigh}]
6.28319
```

T.3) Recognizing gradient fields: The gradient test

When you have to calculate a path integral

$$\int_C m[x, y] dx + n[x, y] dy$$

and you recognize that the vector field

$$\text{Field}[x, y] = \{m[x, y], n[x, y]\}$$

is the gradient field of a function $f[x, y]$, then a warm comfortable feeling radiates through your body.

If you want to exploit the advantages you get from a gradient field, you'll have to be in a position to recognize when a given vector field is a gradient field.

To this end, go with a cleared gradient field:

```
Clear[f, gradf, x, y, m, n]
gradf[x_, y_] = {D[f[x, y], x], D[f[x, y], y]};
{m[x_, y_], n[x_, y_]} = gradf[x, y]
{f^{(1,0)}[x, y], f^{(0,1)}[x, y]}
```

Look at this:

```
gradtest = D[m[x, y], y] - D[n[x, y], x]
0
```

This tells you that if you have a gradient field

$$\text{Field}[x, y] = \{m[x, y], n[x, y]\},$$

then

$$D[m[x, y], y] - D[n[x, y], x] = 0.$$

T.3.a.i)

If you are given a vector field

$$\text{Field}[x, y] = \{m[x, y], n[x, y]\}$$

and you learn that

$$D[m[x, y], y] - D[n[x, y], x] = 0,$$

then are you automatically guaranteed that $\text{Field}[x, y]$ is a gradient field?

Answer:

Yes, provided that neither $m[x, y]$ nor $n[x, y]$ has a singularity (blow-up or blow-down).

T.3.a.ii)

Is the vector field

$$\text{Field}[x, y] = \{e^x \cos[y], -e^x \sin[y]\}$$

a gradient field?

Answer:

Look at:

```
Clear[m, n, x, y]
{m[x_, y_], n[x_, y_]} = {E^x Cos[y], -E^x Sin[y]}
{E^x Cos[y], -E^x Sin[y]}
```

Good; neither $m[x, y]$ nor $n[x, y]$ has any singularities.

Now go with the gradient test:

```
gradtest = D[m[x, y], y] - D[n[x, y], x]
0
```

Hot dog!

No doubt about it, this vector field is a gradient field.

T.3.a.iii)

Is the vector field

$$\text{Field}[x, y] = \{e^y \cos[x], -e^y \sin[x]\}$$

a gradient field?

Answer:

Look at:

```
Clear[m, n, x, y]
{m[x_, y_], n[x_, y_]} = {E^y Cos[x], -E^y Sin[x]}
{E^y Cos[x], -E^y Sin[x]}
```

Good; neither $m[x, y]$ nor $n[x, y]$ has any singularities.

Now go with the gradient test:

```
gradtest = D[m[x, y], y] - D[n[x, y], x]
2 E^y Cos[x]
```

This is not 0.

Absolutely no doubt about it, this vector field is NOT a gradient field.

T.3.a.iv) Singularities (blow-ups or blow-downs)

Is the vector field

$$\text{Field}[x, y] = \left\{-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right\}$$

a gradient field?

Answer:

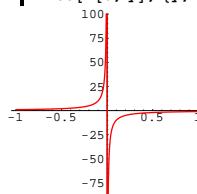
Look at:

```
Clear[m, n, Field, x, y]
{m[x_, y_], n[x_, y_]} = {-y/(x^2+y^2), x/(x^2+y^2)};
Field[x_, y_] = {m[x, y], n[x, y]}
{-y/(x^2+y^2), x/(x^2+y^2)}
```

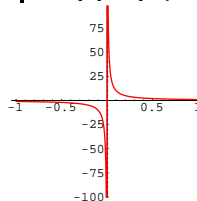
A big fat singularity at $\{x, y\} = \{0, 0\}$.

You can see this by plotting.

```
Plot[m[0, y], {y, -1, 1}, PlotStyle → Red, AspectRatio → 1];
```



```
Plot[n[x, 0], {x, -1, 1}, PlotStyle → Red, AspectRatio → 1];
```



This vector field fails the first part of the gradient test.

Now look at the second part of the gradient test:

```
| gradtest = Simplify[D[n[x, y], x] - D[m[x, y], y]]
0
```

This field passes the second part of the gradient test.

Whether this vector field is a genuine gradient field is in doubt.

Try something else by looking at

$$\oint_C m[x, y] dx + n[x, y] dy$$

where C is the circle of radius 1 centered at the singularity at (0, 0):

```
Clear[t]
{x[t_], y[t_]} = {Cos[t], Sin[t]};
tlow = 0;
thigh = 2 π;
NIntegrate[m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t], {t, tlow, thigh}]
6.28319
```

Not zero.

This tells you for sure that this vector field is not a gradient field.

□T.3.a.v)

Is every vector field a gradient field?

□Answer:

Hell no.

□T.3.b)

Here's a vector field:

```
Clear[Field, m, n, x, y]
{m[x_, y_], n[x_, y_]} = {x^2 + 2 x Sin[y], Sin[5 y] + x^2 Cos[y]};
Field[x_, y_] = {m[x, y], n[x, y]}
{x^2 + 2 x Sin[y], x^2 Cos[y] + Sin[5 y]}
```

No singularities.

Give it the second part of the gradient test:

```
| gradtest = D[m[x, y], y] - D[n[x, y], x]
0
```

Good.

Field[x, y] is definitely a gradient field.

Try to come up with a function f[x, y] so that
gradf[x, y] = Field[x, y].

□Answer:

The goal is to come up with a function f[x, y] with

$$\text{gradf}[x, y] = \text{Field}[x, y].$$

To do this, fix any point {a, b} you like and parameterize a line C (or other curve) running from {a, b} to the variable point {x, y}:

{0, 0} is usually a good choice for {a, b}.

```
Clear[t]
{a, b} = {0, 0};
fixedpoint = {a, b};
variablepoint = {x, y};
tlow = 0;
thigh = 1;
{x[t_], y[t_]} = fixedpoint + t (variablepoint - fixedpoint)
{t x, t y}
```

To get a function f[x, y] with

$$\text{gradf}[x, y] = \{m[x, y], n[x, y]\},$$

all you gotta do is set

$$f[x, y] = \int_C m[x, y] dx + n[x, y] dy$$

where C is the line (or other curve) running from the fixed point {a, b} to the variable point {x, y}:

```
Clear[f]
f[x_, y_] = Integrate[m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t], {t, tlow, thigh} dt
1/5 + x^3/3 - 1/5 Cos[5 y] + x^2 Sin[y]
```

Try it out:

```
Clear[gradf]
gradf[x_, y_] = {D[f[x, y], x], D[f[x, y], y]}
{x^2 + 2 x Sin[y], x^2 Cos[y] + Sin[5 y]}
```

Compare:

```
{m[x, y], n[x, y]}
{x^2 + 2 x Sin[y], x^2 Cos[y] + Sin[5 y]}
TrigExpand[gradf[x, y]] == TrigExpand[{m[x, y], n[x, y]}]
True
```

Great.

This tells you that

$$\text{gradf}[x, y] = \{m[x, y], n[x, y]\}$$

just as you wanted.

See what happens when you go with {a, b} = {1, π/2}:

```
Clear[x, y, t]
{a, b} = {1, π/2};
fixedpoint = {a, b};
variablepoint = {x, y};
tlow = 0;
thigh = 1;
{x[t_], y[t_]} = fixedpoint + t (variablepoint - fixedpoint)
{1 + t (-1 + x), π/2 + t (-π/2 + y)}
```

Set

$$f[x, y] = \int_C m[x, y] dx + n[x, y] dy$$

where C is the line running from {a, b} to {x, y}:

```
Clear[f]
f[x_, y_] = Integrate[m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t], dt
-4/3 + x^3/3 + x^2 Cos[1/2 (π - 2 y)] - 1/5 Sin[5/2 (π - 2 y)]
```

Looks bad; check whether it feels good:

```
Clear[gradf]
gradf[x_, y_] = {D[f[x, y], x], D[f[x, y], y]}
{x^2 + 2 x Cos[1/2 (π - 2 y)], Cos[5/2 (π - 2 y)] + x^2 Sin[1/2 (π - 2 y)]}
```

Compare:

```
{m[x, y], n[x, y]}
```

```
{x^2 + 2 x Sin[y], x^2 Cos[y] + Sin[5 y]}
TrigExpand[gradf[x, y]] == TrigExpand[{m[x, y], n[x, y]}]
True
```

It feels great!

Each time you change the fixed point {a, b}, you make a different function f[x, y] whose gradient is {m[x, y], n[x, y]}.

□T.3.c)

What is the value of

$$\int_C e^{-5xy} (3 \cos[3x] - 5y \sin[3x]) dx - 5x e^{-5xy} \sin[3x] dy$$

for any curve C running from {-0.7, 0} to {1.1, 0.4}?

□Answer:

Here's the vector field:

```
Clear[Field, m, n, x, y]
{m[x_, y_], n[x_, y_]} = {E^{-5xy} (3 Cos[3 x] - 5 y Sin[3 x]), -5 x E^{-5xy} Sin[3 x]};
Field[x_, y_] = {m[x, y], n[x, y]}
{E^{-5xy} (3 Cos[3 x] - 5 y Sin[3 x]), -5 E^{-5xy} x Sin[3 x]}
```

No singularities (because e^s > 0 no matter what s is).

Give it the second part of the gradient test:

```
| gradtest = Together[D[m[x, y], y] - D[n[x, y], x]]
0
```

Good. Now you know that

$$\text{Field}[x, y] = \{m[x, y], n[x, y]\}$$

is a gradient field.

This is really good news because this tells you that

$$\int_C e^{-5xy} (3 \cos[3x] - 5y \sin[3x]) dx - 5x e^{-5xy} \sin[3x] dy$$

calculates out to the same value no matter what curve C you go with as

long as C starts at $\{-0.7, 0\}$ and stops at $\{1.1, 0.4\}$. This information is quite a relief because now you know that you can calculate this integral by using any cheap curve C running from $\{-0.7, 0\}$ to $\{1.1, 0.4\}$.

One really cheap curve is the straight line parameterized by:

```
Clear[x, y, t]
start = {-0.7, 0};
end = {1.1, 0.4};
tlow = 0;
thigh = 1;
{x[t_], y[t_]} = start + t (end - start)
{-0.7 + 1.8 t, 0.4 t}
```

Here comes the calculation of

$$\int_C e^{-5xy} (3 \cos[3x] - 5y \sin[3x]) dx - 5x e^{-5xy} \sin[3x] dy$$

for any curve C running from $\{-0.7, 0\}$ to $\{1.1, 0.4\}$:

```
NIntegrate[m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t], {t, tlow, thigh}]
0.845731
```

Not hard at all.

T.4) Line integrals

□T.4.a)

What do folks mean when they talk about line integrals?

□Answer:

A line integral is the same thing as a path integral. This alternate terminology is in common use. This is unfortunate because many path integrals involve curves that are not lines.

T.5) Summary of main ideas

Calculus&Mathematica offers this summary to you
for your good use and enjoyment.
It comes from the home office to you.
'93

□T.5.a.i) Flow along

If a curve C is parameterized by $\{x[t], y[t]\}$ with $t_{low} \leq t \leq t_{high}$, then

$$\int_C m[x, y] dx + n[x, y] dy$$

$$\int_{t_{low}}^{t_{high}} \text{Field}[x[t], y[t]] \cdot \{x'[t], y'[t]\} dt$$

$$= \int_{t_{low}}^{t_{high}} (m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t]) dt$$

measures of the flow of a vector field

$\text{Field}[x, y] = \{m[x, y], n[x, y]\}$
along the curve C.

□T.5.a.ii)

If you have a closed curve C (like a deformed circle), and if

$$\oint_C m[x, y] dx + n[x, y] dy > 0,$$

then the flow of $\text{Field}[x, y] = \{m[x, y], n[x, y]\}$ along C is counterclockwise.

But if

$$\oint_C m[x, y] dx + n[x, y] dy < 0,$$

then the flow of $\text{Field}[x, y] = \{m[x, y], n[x, y]\}$ along C is clockwise.

□T.5.b.i) Flow across

If a curve C is parameterized by $\{x[t], y[t]\}$ with $t_{low} \leq t \leq t_{high}$, then

$$\int_C -n[x, y] dx + m[x, y] dy$$

$$= \int_{t_{low}}^{t_{high}} \text{Field}[x[t], y[t]] \cdot \{y'[t], -x'[t]\} dt$$

$$= \int_{t_{low}}^{t_{high}} (-n[x[t], y[t]] x'[t] + m[x[t], y[t]] y'[t]) dt$$

measures of the flow of a vector field

$\text{Field}[x, y] = \{m[x, y], n[x, y]\}$
across the curve C.

□T.5.b.ii)

If you have a closed curve C (like a deformed circle), and if

$$\oint_C -n[x, y] dx + m[x, y] dy > 0,$$

then the flow of $\text{Field}[x, y] = \{m[x, y], n[x, y]\}$ across C is from inside to outside.

But if

$$\oint_C -n[x, y] dx + m[x, y] dy < 0,$$

then the flow of $\text{Field}[x, y] = \{m[x, y], n[x, y]\}$ across C is from outside to inside.

□T.5.c.i) Gradient fields

A vector field

$$\text{Field}[x, y] = \{m[x, y], n[x, y]\}$$

is a gradient field if there is a function $f[x, y]$ with

$$\text{grad}f[x, y] = \text{Field}[x, y].$$

You can be sure a vector field

$$\text{Field}[x, y] = \{m[x, y], n[x, y]\}$$

is a gradient field if neither $m[x, y]$ nor $n[x, y]$ has singularities and

$$D[m[x, y], y] = D[n[x, y], x].$$

□T.5.c.ii)

If a vector field

$$\text{Field}[x, y] = \{m[x, y], n[x, y]\}$$

is a gradient field, then for any closed curve C, you are guaranteed that

$$\oint_C m[x, y] dx + n[x, y] dy = 0.$$

This tells you that the net flow of a gradient field along any closed curve is 0.

□T.5.c.iii)

If a vector field

$$\text{Field}[x, y] = \{m[x, y], n[x, y]\}$$

is a gradient field, then for any curve C, the value of the path integral

$$\int_C m[x, y] dx + n[x, y] dy$$

depends on the location of the starting point of C and the location of the end point of C but does not depend on the specific path C takes as it runs from its start to its end.

VC.05 Flow Measurements by Integrals Give it a Try!

Experience with the starred problems will be useful for understanding developments later in the course.

G.1) Flow along and flow across*

□G.1.a)

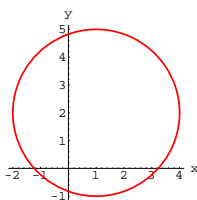
Here's a vector field:

```
Clear[x, y, m, n, Field]
{m[x_, y_], n[x_, y_]} = {x^2 - 2y, -y^2 + x};
Field[x_, y_] = {m[x, y], n[x, y]}
{x^2 - 2y, x - y^2}
```

Here's the circle C of radius 3 centered at $\{1, 2\}$ parameterized in the counterclockwise way:

```
Clear[t]
{x[t_], y[t_]} = {1, 2} + 3 {Cos[t], Sin[t]};
tlow = 0;
thigh = 2 Pi;

curveplot = ParametricPlot[{x[t], y[t]}, {t, tlow, thigh},
PlotStyle -> {{Thickness[0.01], Red}}, AspectRatio -> Automatic,
AxesLabel -> {"x", "y"}];
```



Calculate

$$\oint_C -n[x, y] dx + m[x, y] dy$$

and use your result to determine whether the net flow of this vector field across this curve is from inside to outside, outside to inside, or 0.

Calculate

$$\oint_C m[x, y] dx + n[x, y] dy$$

and use your result to determine whether the net flow of this vector field along this curve is clockwise, counterclockwise or 0.

□G.1.b)

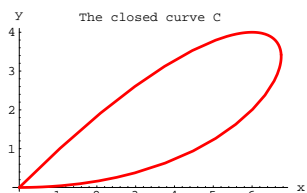
Here's a vector field:

```
Clear[x, y, m, n, Field]
{m[x_, y_], n[x_, y_]} = {0.5 x - 1.2 y, 1};
Field[x_, y_] = {m[x, y], n[x, y]}
{0.5 x - 1.2 y, 1}
```

Here's a parameterization and a plot of a closed curve C:

```
Clear[t]
{x[t_], y[t_]} = {3 t (3 - t), t (t - 3)^2};
tlow = 0;
thigh = 3;

curveplot = ParametricPlot[{x[t], y[t]}, {t, tlow, thigh}, PlotStyle ->
{{Thickness[0.01], Red}}, PlotLabel -> "The closed curve C",
AspectRatio -> Automatic, AxesLabel -> {"x", "y"}];
```



Is C parameterized in the counterclockwise or clockwise way?

Use a path integral to determine whether the net flow of this vector field across C is from outside to inside, inside to outside, or 0.

Use a path integral to determine whether the net flow of this vector field along C is clockwise, counterclockwise, or 0.

□G.1.c)

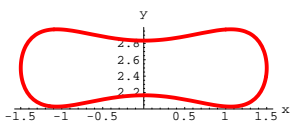
Here's a vector field:

```
Clear[x, y, m, n, Field]
{m[x_, y_], n[x_, y_]} = {x^3 - 3 x y^2, 3 x^2 y - y^3};
Field[x_, y_] = {m[x, y], n[x, y]}
{x^3 - 3 x y^2, 3 x^2 y - y^3}
```

Here's a parameterization and a plot of a closed curve C.

```
Clear[t]
{x[t_], y[t_]} = {3 Sin[t] Cos[t], Sin[t]^2 + 1/6 Cos[6 t] + 2};
tlow = 0;
thigh = pi;

curveplot = ParametricPlot[{x[t], y[t]}, {t, tlow, thigh},
PlotStyle -> {{Thickness[0.015], Red}}, AspectRatio -> Automatic,
AxesLabel -> {"x", "y"}];
```



Is the curve parameterized in the counterclockwise or clockwise way?

Use a path integral to determine whether the net flow of this vector

field across this curve is from outside to inside, inside to outside, or 0. Use a path integral to determine whether the net flow of this vector field along this curve is clockwise, counterclockwise, or 0.

□G.1.d.i) A gradient field

Here's the gradient field of the function

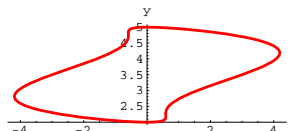
$$f[x, y] = e^{2x-y};$$

```
Clear[f, gradf, x, y, m, n, Field]
f[x_, y_] = E^{2 x - y};
gradf[x_, y_] = {D[f[x, y], x], D[f[x, y], y]};
{m[x_, y_], n[x_, y_]} = gradf[x, y];
Field[x_, y_] = {m[x, y], n[x, y]}
{2 E^{2 x - y}, -E^{2 x - y}}
```

Here's a parameterization and a plot of a closed curve C.

```
Clear[t]
{x[t_], y[t_]} = {6 Sin[t] Cos[t] (1 - 0.7 Sin[4 t]), 3 Sin[t]^2 + 2};
tlow = 0;
thigh = pi;

curveplot = ParametricPlot[{x[t], y[t]}, {t, tlow, thigh},
PlotStyle -> {{Thickness[0.01], Red}}, AspectRatio -> Automatic,
AxesLabel -> {"x", "y"}];
```



Explain how you know in advance that the net flow of this vector field along this curve is 0.

Use a path integral to determine whether the net flow of this vector field across this curve is from outside to inside, inside to outside, or 0.

□G.1.d.ii)

You already know that the net flow of a gradient field along a closed curve is guaranteed to be 0.

Is it true that the net flow of a gradient field across a closed curve is guaranteed to be 0?

G.2) Path integrals: Backward and forward*

□G.2.a.i)

Suppose C₁ and C₂ are the same physical curve, but the starting point of C₁ is the ending point of C₂, and the ending point of C₁ is the starting point of C₂.

Express

$$\int_{C_2} m[x, y] dx + n[x, y] dy$$

in terms of

$$\int_{C_1} m[x, y] dx + n[x, y] dy.$$

□G.2.a.ii)

Calculate the path integral

$$\int_{C_1} x^2 y dx - 3 x y dy,$$

where C₁ starts at (-1, 3), runs to {1, 0} on a straight line, and then follows the parabola y = 3(x - 1)² to {2, 3} where it stops.

Then calculate the path integral

$$\int_{C_2} x^2 y dx - 3 x y dy,$$

where C₂ starts at {2, 3}, runs to {1, 0} on the parabola y = 3(x - 1)², and then follows the straight line from {1, 0} to (-1, 3) where it stops.

□Tip:

After you've gone to the trouble of calculating

$$\int_{C_1} x^2 y dx - 3 x y dy,$$

you should be able to write down the value of

$$\int_{C_2} x^2 y dx - 3 x y dy$$

with no extra labor.

□G.2.b)

Here's a parameterization of the ellipse

$$\left(\frac{x-1}{3}\right)^2 + \left(\frac{y+2}{2}\right)^2 = 1:$$

```

Clear[x, y, t]
tlow = 0;
thigh = 2 π;
{x[t_], y[t_]} = {1, -2} + {3 Sin[t], 2 Cos[t]}
{1 + 3 Sin[t], -2 + 2 Cos[t]}

```

Call this ellipse C.

Now look at this calculation:

```

NIntegrate[y[t]^2 x'[t] + (x[t] y[t] + 1) y'[t], {t, tlow, thigh}]
-37.6991

```

Does this result calculate

$$\oint_C y^2 dx + (xy + 1) dy?$$

If not, how do you modify this result to get the value of

$$\oint_C y^2 dx + (xy + 1) dy?$$

G.3) Calculations and interpretations*

Many different notations for path integrals are in regular use in science. In this problem, you will meet some of them.

Go with a given vector field

$$\text{Field}[x, y] = \{m[x, y], n[x, y]\},$$

a curve C, and a direction for C via a parameterization

$$P[t] = \{x[t], y[t]\}, \text{tlow} \leq t \leq \text{thigh}.$$

□Integrals measuring flow – across – C

All the following four integrals calculate out to the same value, and all measure flow of

$$\text{Field}[x, y] = \{m[x, y], n[x, y]\}$$

across C:

i) $\int_C -n[x, y] dx + m[x, y] dy;$

ii) $\int_{\text{low}}^{\text{high}} \text{Field}[x[t], y[t]] \cdot \{y'[t], -x'[t]\} dt;$

iii) $\int_C \text{Field}[x[t], y[t]] \cdot \text{unitnormal} ds.$

This last integral is with respect to length measured on the curve from the start of the curve. The way to see that the last integral is the same as the others is to notice that

$$ds = \sqrt{x'[t]^2 + y'[t]^2} dt$$

and

$$\text{unitnormal}[t] = \frac{\{y'[t], -x'[t]\}}{\sqrt{x'[t]^2 + y'[t]^2}}.$$

When you transform to the t variable, you get

$$\text{Field} \cdot \text{unitnormal} ds \longleftrightarrow \text{Field}[x[t], y[t]] \cdot \{y'[t], -x'[t]\} dt.$$

□G.3.a.i)

Calculate

$$\oint_C \text{Field} \cdot \text{unitnormal} ds$$

in the case in which

$$\text{Field}[x, y] = \{x^2 y^2, x y^2\}$$

and the curve C is the ellipse $x^2 + 2y^2 = 1$.

Give an interpretation of the result as a flow-along measurement and illustrate with a plot.

□Tip:

Remember \oint_C demands a counterclockwise parameterization.

□Integrals measuring flow – along – C

All the following four integrals calculate out to the same value, and all measure flow of

$$\text{Field}[x, y] = \{m[x, y], n[x, y]\}$$

along C:

i) $\int_C m[x, y] dx + n[x, y] dy;$

ii) $\int_{\text{low}}^{\text{high}} \text{Field}[x[t], y[t]] \cdot \{x'[t], y'[t]\} dt;$

iii) $\int_C \text{Field}[x[t], y[t]] \cdot dP,$

where $dP = \{x'[t], y'[t]\} dt;$

iv) $\int_C \text{Field}[x[t], y[t]] \cdot \text{unittan} ds.$

This last integral is with respect to length measured on the curve from the start of the curve. The way to see that the last integral is the same as the others, notice that

$$ds = \sqrt{x'[t]^2 + y'[t]^2} dt$$

and

$$\text{unittan}[t] = \frac{\{x'[t], y'[t]\}}{\sqrt{x'[t]^2 + y'[t]^2}}.$$

When you transform to the t variable, you get

$$\text{Field} \cdot \text{unittan} ds \longleftrightarrow \text{Field}[x[t], y[t]] \cdot \{x'[t], y'[t]\} dt.$$

□G.3.a.ii)

Calculate

$$\oint_C \text{Field} \cdot dP$$

in the case in which

$$\text{Field}[x, y] = \{x^2 y^2, x y^2\}$$

and the curve C is the ellipse $x^2 + 2y^2 = 1$.

Give an interpretation of the result.

□G.3.a.iii)

Calculate

$$\oint_C \text{Field} \cdot \text{unitnormal} ds$$

in the case in which

$$\text{Field}[x, y] = \{x^2 y^2, x y^2\}$$

and the curve C is the ellipse $x^2 + 2y^2 = 1$.

Give an interpretation of the result.

□G.3.a.iv)

Calculate

$$\int_C \text{Field} \cdot \text{unitnormal} ds$$

in the case in which

$$\text{Field}[x, y] = \{x^2 y^2, x y^2\}$$

and the curve C is the top half of the ellipse $x^2 + 2y^2 = 1$ starting on the far right and ending on the far left.

Give an interpretation of the result.

□G.3.b)

Calculate

$$\oint_C (-5y) dx + x dy$$

where C is the circle

$$x^2 + (y - 2)^2 = 9$$

and interpret the result in two ways:

→ As a flow – along – C measurement of a certain vector field and

→ As a flow – across – C measurement of another vector field.

□Tip:

Remember \oint_C demands a counterclockwise parameterization.

□G.3.c)

Calculate

$$\oint_C \text{Sin}[y] dx + \text{Cos}[x] dy$$

where C is the circle

$$x^2 + (y - 1)^2 = 0.8$$

and give two interpretations of the measurement.

G.4) Water*

If you like the vector fields plotted in this problem and would like to be able to design flows for special situations for yourself, study the topic called "conformal mapping" in a complex variables course.

For an incompressible fluid flow, like water flow, the net flow across any closed curve C without loops must be 0 unless there are sources (spigots) or sinks (drains) inside C.

In addition, for water flow, the net flow along a closed curve C can't be positive because you can't have more water at the end of a trip around a closed curve than you had at the beginning.

Also the net flow along a closed curve C can't be negative because you can't have less water at the end of a trip around a closed curve than you had at the beginning.

The upshot:

For water flow without sources or sinks, you know that net flow across any closed curve C without loops must be 0. And you know that net flow along any closed curve C must be 0.

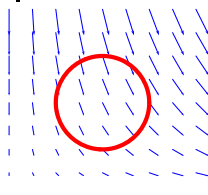
Now look at the vector field

$$\text{Field}[x, y] = \{0.07 x, -0.14 y\}$$

shown with the circle C of radius 1 centered at {2, 2}:

```
Clear[Field, m, n, x, y];
{m[x_, y_], n[x_, y_]} = {0.07 x, -0.14 y};
Field[x_, y_] = {m[x, y], n[x, y]};
vectorfieldplot = Table[
  Arrow[Field[x, y], Tail -> {x, y}], {x, 0, 4, 0.5}, {y, 0, 4, 0.5}];
{x[t_], y[t_]} = {2, 2} + {Cos[t], Sin[t]};
tlow = 0;
thigh = 2 pi;
```

```
Cplot = ParametricPlot[{x[t], y[t]}, {t, tlow, thigh},
  PlotStyle -> {{Thickness[0.02], Red}}, DisplayFunction -> Identity];
Show[vectorfieldplot, Cplot, AxesLabel -> {"x", "y"},
  DisplayFunction -> $DisplayFunction];
```



This looks like water flowing around a corner.

Check whether it can be by looking at

$$\oint_C m[x, y] dx + n[x, y] dy:$$

```
Chop[Integrate[m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t] dt,
  {t, tlow, thigh}]]
0
```

Good; the flow of this vector field along this curve is 0.

Now look at

$$\oint_C -n[x, y] dx + m[x, y] dy:$$

```
Chop[Integrate[-n[x[t], y[t]] x'[t] + m[x[t], y[t]] y'[t] dt,
  {t, tlow, thigh}]]
-0.219911
```

Whoops!

The flow across this curve is from outside to inside.

The must be a weak sink (drain) inside C.

The upshot:

This vector field cannot be a model for water flow without sources (spigots) or sinks (drains).

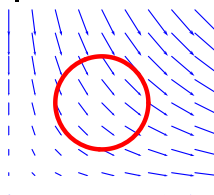
□G.4.a)

Look at the vector field

$$\text{Field}[x, y] = \{0.14 x, -0.14 y\}$$

shown with the circle C of radius 1 centered at {2, 2}:

```
Clear[Field, m, n, x, y];
{m[x_, y_], n[x_, y_]} = 0.14 {x, -y};
Field[x_, y_] = {m[x, y], n[x, y]};
vectorfieldplot = Table[
  Arrow[Field[x, y], Tail -> {x, y}], {x, 0, 4, 0.5}, {y, 0, 4, 0.5}];
{x[t_], y[t_]} = {2, 2} + {Cos[t], Sin[t]};
tlow = 0;
thigh = 2 pi;
Cplot = ParametricPlot[{x[t], y[t]}, {t, tlow, thigh},
  PlotStyle -> {{Thickness[0.02], Red}}, DisplayFunction -> Identity];
Show[vectorfieldplot, Cplot, AxesLabel -> {"x", "y"},
  DisplayFunction -> $DisplayFunction];
```



This also looks like water flowing around a corner.

Go with the curve C plotted above and report on whether the flow measurements

$$\oint_C m[x, y] dx + n[x, y] dy$$

$$\oint_C -n[x, y] dx + m[x, y] dy$$

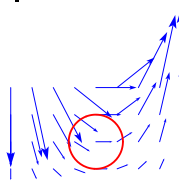
tell you that this vector field cannot be a model for water flow without sources (spigots) or sinks (drains).

□G.4.b.i)

Look at this vector field shown with the circle C of radius 0.5 centered at {0, 1}:

```
Clear[Field, m, n, x, y];
{m[x_, y_], n[x_, y_]} = {0.2 Cos[x] Cosh[y], 0.4 Sin[x] Sinh[y]};
Field[x_, y_] = {m[x, y], n[x, y]};
vectorfieldplot = Table[
  Arrow[Field[x, y], Tail -> {x, y}], {x, -pi/2, pi/2, pi/8}, {y, 0, 2, 0.5}];
{x[t_], y[t_]} = {0, 1} + 0.5 {Cos[t], Sin[t]};
tlow = 0;
```

```
thigh = 2 pi;
Cplot = ParametricPlot[{x[t], y[t]}, {t, tlow, thigh},
  PlotStyle -> {{Thickness[0.01], Red}}, DisplayFunction -> Identity];
Show[vectorfieldplot, Cplot, AxesLabel -> {"x", "y"},
  DisplayFunction -> $DisplayFunction];
```



This looks like water flowing down on the left toward the x-axis and then flowing up on the right.

Go with the curve C plotted above and report on whether the flow measurements

$$\oint_C m[x, y] dx + n[x, y] dy$$

$$\oint_C -n[x, y] dx + m[x, y] dy$$

tell you that this vector field cannot be a model for water flow without sources (spigots) or sinks (drains).

□Tip:

Use NIntegrate with AccuracyGoal -> 2.

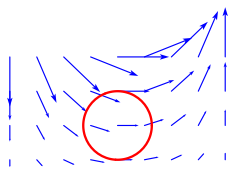
□G.4.b.ii)

Look at this vector field shown with the circle C of radius 0.5 centered at {0, 1}:

```
Clear[Field, m, n, x, y];
{m[x_, y_], n[x_, y_]} = {0.2 Cos[x] Cosh[y], 0.2 Sin[x] Sinh[y]};
Field[x_, y_] = {m[x, y], n[x, y]};
vectorfieldplot = Table[
  Arrow[Field[x, y], Tail -> {x, y}], {x, -pi/2, pi/2, pi/8}, {y, 0, 2, 0.5}];
{x[t_], y[t_]} = {0, 1} + 0.5 {Cos[t], Sin[t]};
tlow = 0;
thigh = 2 pi;
Cplot = ParametricPlot[{x[t], y[t]}, {t, tlow, thigh},
  PlotStyle -> {{Thickness[0.01], Red}}, DisplayFunction -> Identity];
```



```
Show[vectorfieldplot, Cplot, AxesLabel -> {"x", "y"},
DisplayFunction -> $DisplayFunction];
```



Again, this looks like water flowing down on the left toward the x-axis and then flowing up on the right.

Go with the curve C plotted above and report on whether the flow measurements

$$\oint_C m[x, y] dx + n[x, y] dy, \text{ and}$$

$$\oint_C -n[x, y] dx + m[x, y] dy$$

tell you that this vector field cannot be a model for water flow without sources (spigots) or sinks (drains).

Tip:

Use NIntegrate with AccuracyGoal -> 2.

G.5) Sources and sinks*

The simplest way to spot a source of new fluid or a drain of old fluid at a point {a, b} is to center a circle C[r] of very small radius r at {a, b} and then to calculate the flow - across - C[r] measurement

$$\oint_{C[r]} -n[x, y] dx + m[x, y] dy.$$

If this measurement is positive for ALL very small r's, then you can be sure that the point {a, b} is a source of new fluid.

If this measurement is negative for ALL very small r's, then you can be sure that the point {a, b} is a drain of old fluid.

Try this out on

$$\text{Field}[x, y] = \{3x^2, 4y^4\};$$

```
Clear[Field, m, n, x, y]
{m[x_, y_], n[x_, y_]} = {3x^2, 4y^4};
Field[x_, y_] = {m[x, y], n[x, y]}
{3x^2, 4y^4}
```

Center a circle C[r] of radius r at {a, b} = {2, 1} and calculate the flow - across - C[r] measurement

$$\oint_{C[r]} -n[x, y] dx + m[x, y] dy:$$

```
{a, b} = {2, 1};
Clear[x, t]
tlow = 0;
thigh = 2π;
{x[t_], y[t_]} = {a, b} + r {Cos[t], Sin[t]};
∫_{tlow}^{thigh} (-n[x[t], y[t]] x'[t] + m[x[t], y[t]] y'[t]) dt
28πr^2 + 12πr^4
```

This is positive no matter what r you go with.

The upshot:

→ {a, b} = {2, 1} is a source for new fluid.

Now go with cleared values of {a, b} and center a circle C[r] of very small radius r at {a, b} and calculate the flow - across - C[r] measurement

$$\oint_{C[r]} -n[x, y] dx + m[x, y] dy:$$

```
Clear[a, b, r, t]
tlow = 0;
thigh = 2π;
{x[t_], y[t_]} = {a, b} + r {Cos[t], Sin[t]};
∫_{tlow}^{thigh} (-n[x[t], y[t]] x'[t] + m[x[t], y[t]] y'[t]) dt
6aπr^2 + 16b^3πr^2 + 12bπr^4
```

This tells you that:

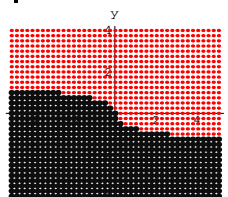
→ {a, b} is a source of new fluid if $(3a + 8b^3) > 0$,

and

→ {a, b} is a sink (drain) for old fluid if $(3a + 8b^3) < 0$.

Here's a sample plot of some of the sources and sinks in this vector field:

```
sourcesandsinks = Show[Table[
If[3a + 8b^3 > 0, Graphics[{PointSize[0.015], Red, Point[{a, b]}]},
Graphics[{PointSize[0.025], GrayLevel[0.05], Point[{a, b]}]}],
{a, -5, 5, 0.25}, {b, -4, 4, 0.25}], Axes -> True,
AxesLabel -> {"x", "y"}];
```



The larger points are sinks; the smaller points are sources.

The sinks are in the lower part of the plot.

Think of the sources as little individual springs feeding the flow.

Think of the sinks as little holes through which fluid seeps out as the flow goes by.

G.5.a.i)

Give a sample plot of some of the sources and sinks in the vector field $\text{Field}[x, y] = \{x^2, y^3\}$.

G.5.a.ii)

Give a sample plot of some of the sources and sinks in the vector field $\text{Field}[x, y] = \{3x, -x^2y^3\}$.

G.5.b) Singularity source

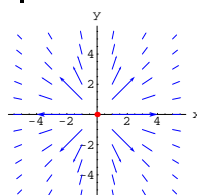
Here's a look at the vector field

$$\text{Field}[x, y] = 3 \left\{ \frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right\};$$

Note the singularity at {0, 0}.

```
Clear[Field, m, n, x, y]
{m[x_, y_], n[x_, y_]} = 3 {x/(x^2 + y^2), y/(x^2 + y^2)};
Field[x_, y_] = {m[x, y], n[x, y]};
fieldplot =
Table[Arrow[Field[x, y], Tail -> {x, y}], {x, -5, 5, 2}, {y, -5, 5, 1}];
singularity = {0, 0};
singularityplot =
```

```
Graphics[{Red, PointSize[0.03], Point[singularity]}];
Show[fieldplot, singularityplot,
Axes -> True, AxesLabel -> {"x", "y"}];
```



Go with the circle C of radius 2 centered at the singularity at {0, 0} and look at the calculation

$$\oint_C -n[x, y] dx + m[x, y] dy$$

of the flow of this vector field across C :

```
Clear[x, y, t]
r = 2;
{x[t_], y[t_]} = r {Cos[t], Sin[t]};
tlow = 0;
thigh = 2π;
∫_{tlow}^{thigh} (-n[x[t], y[t]] x'[t] + m[x[t], y[t]] y'[t]) dt
6π
```

Now go with the circle C₁ of radius 1 centered at the singularity at {0, 0} and look at this calculation

$$\oint_{C_1} -n[x, y] dx + m[x, y] dy$$

of the flow of this vector field across C₁:

```
Clear[x, y, t]
r = 1;
{x[t_], y[t_]} = r {Cos[t], Sin[t]};
tlow = 0;
thigh = 2π;
∫_{tlow}^{thigh} (-n[x[t], y[t]] x'[t] + m[x[t], y[t]] y'[t]) dt
6π
```

Now go with any circle C_r of radius r centered at the singularity at $\{0, 0\}$ and look at this calculation

$\oint_{C_r} -n[x, y] dx + m[x, y] dy$
of the flow of this vector field across C_r :

```
Clear[r, x, y, t]
{x[t_], y[t_]} = r {Cos[t], Sin[t]};
tlow = 0;
thigh = 2 π;
∫_{tlow}^{thigh} (-n[x[t], y[t]] x'[t] + m[x[t], y[t]] y'[t]) dt
```

No matter what positive radius you go with, the flow of this vector field across the circle of radius r centered at the singularity is 6π .
What clue does this give you about the location of the only source of new fluid in this flow?

G.6) Gradient fields are where the mathematical action is*

G.6.a.i)

The curve C_1 runs on a straight line starting at $\{0, 0\}$ and ending at $\{1, 2\}$. Here's a parameterization of C_1 with t running from 0 to 1:

```
Clear[x1, y1, t]
{x1[t_], y1[t_]} = {0, 0} + t {1, 2}
{t, 2 t}
```

The curve C_2 also starts at $\{0, 0\}$ and ends at $\{1, 2\}$, but C_2 runs on the parabola

$$y = 2x^2$$

Here's a parameterization of C_2 with t running from 0 to 1:

```
Clear[x2, y2, t]
{x2[t_], y2[t_]} = {t, 2 t^2}
{t, 2 t^2}
```

The curve C_3 also starts at $\{0, 0\}$ and ends at $\{1, 2\}$ but C_3 runs on the sine curve

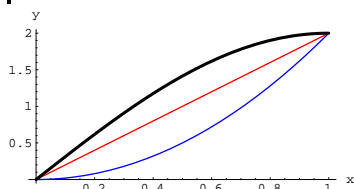
$$y = 2 \sin\left[\frac{\pi x}{2}\right]$$

Here's a parameterization of C_3 with t running from 0 to 1:

```
Clear[x3, y3, t]
{x3[t_], y3[t_]} = {t, 2 Sin[π t / 2]}
{t, 2 Sin[π t / 2]}
```

Take a look:

```
ParametricPlot[{{x1[t], y1[t]}, {x2[t], y2[t]}, {x3[t], y3[t]}},
{t, 0, 1}, PlotStyle -> {{Red}, {Blue}, {Thickness[0.01]}},
AxesLabel -> {"x", "y"}, AspectRatio -> 1/2];
```



Here are calculations of the three path integrals

$$\int_{C_1} y \sin[xy] dx + x \sin[xy] dy,$$

$$\int_{C_2} y \sin[xy] dx + x \sin[xy] dy, \text{ and}$$

$$\int_{C_3} y \sin[xy] dx + x \sin[xy] dy$$

in order:

```
Clear[m, n, x, y]
{m[x_, y_], n[x_, y_]} = {y Sin[x y], x Sin[x y]};
NIntegrate[m[x1[t], y1[t]] x1'[t] + n[x1[t], y1[t]] y1'[t], {t, 0, 1}]
1.41615
NIntegrate[m[x2[t], y2[t]] x2'[t] + n[x2[t], y2[t]] y2'[t], {t, 0, 1}]
1.41615
NIntegrate[m[x3[t], y3[t]] x3'[t] + n[x3[t], y3[t]] y3'[t], {t, 0, 1}]
1.41615
```

Here are calculations of the three path integrals

$$\int_{C_1} x \sin[xy] dx + y \sin[xy] dy,$$

$$\int_{C_2} x \sin[xy] dx + y \sin[xy] dy, \text{ and}$$

$$\int_{C_3} x \sin[xy] dx + y \sin[xy] dy$$

in order:

```
Clear[m, n, x, y]
{m[x_, y_], n[x_, y_]} = {x Sin[x y], y Sin[x y]};
NIntegrate[m[x1[t], y1[t]] x1'[t] + n[x1[t], y1[t]] y1'[t], {t, 0, 1}]
1.77018
NIntegrate[m[x2[t], y2[t]] x2'[t] + n[x2[t], y2[t]] y2'[t], {t, 0, 1}]
1.86313
NIntegrate[m[x3[t], y3[t]] x3'[t] + n[x3[t], y3[t]] y3'[t], {t, 0, 1}]
1.59888
```

Explain how you could have predicted in advance that the first three integrals would have calculated out to the same value.

Explain why most knowledgeable folks would have been quite surprised if the second three path integrals had all calculated out to be equal.

G.6.a.ii)

Given specific functions $m[x, y]$ and $n[x, y]$, what do you look for in order to recognize whether

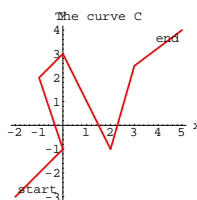
$$\int_{C_1} m[x, y] dx + n[x, y] dy = \int_{C_2} m[x, y] dx + n[x, y] dy$$

provided C_1 and C_2 are two curves starting at the same point and ending at the same point?

G.6.b.i) Substitute curves

Here's a curve C running from its start at $\{-2, -3\}$ to its end at $\{5, 4\}$:

```
start = {-2, -3};
end = {5, 4};
curve = Graphics[{Red, Thickness[0.01],
Line[{start, {0, -1}, {-1, 2}, {0, 3}, {2, -1}, {3, 2.5}, end}]}];
labels = {Graphics[Text["start", start, {-1, -1}],
Graphics[Text["end", end, {1, 1}]}];
Show[curve, labels, Axes -> True, AxesLabel -> {"x", "y"},
PlotLabel -> "The curve C"];
```



Some yo-yo asks you how to calculate

$$\int_C y e^{-xy} dx + x e^{-xy} dy.$$

At first your stomach tightens up because you don't want to go to the trouble of parameterizing that silly curve C .

Then your anxiety turns into a smile as you look at

$$m[x, y] = y e^{-xy} \text{ and } n[x, y] = x e^{-xy}.$$

And then you say: "I don't need to use this silly curve because I can use a simple substitute curve C_1 with the same start and the same end, and then I'll calculate:"

$$\int_{C_1} y e^{-xy} dx + x e^{-xy} dy$$

Come up with the substitute curve C_1 , make the calculation, and explain how you know that your calculation is correct.

G.6.b.ii)

Go with the same curve C as in part i) immediately above.

If you had been asked to calculate

$$\int_C x e^{-xy} dx + y e^{-xy} dy,$$

would you have even considered the use of a substitute curve? Why or why not?

G.6.c.i)

Here's a function $f[x, y]$:

```
Clear[f, x, y]
f[x_, y_] = x^2 y
```

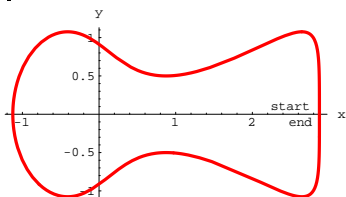
Here's the gradient field of this function:

```
Clear[gradf, m, n, Field]
gradf[x_, y_] = {D[f[x, y], x], D[f[x, y], y]};
{m[x_, y_], n[x_, y_]} = gradf[x, y];
Field[x_, y_] = {m[x, y], n[x, y]}
{2 x y, x^2}
```

Here's a closed curve C:

```
Clear[t]
{x[t_], y[t_]} = {1 + 2 Cos[t] - 1/8 Cos[4 t], Sin[t] + 1/2 Sin[3 t]};
tlow = 0;
thigh = 2 Pi;

curveplot = ParametricPlot[{x[t], y[t]}, {t, tlow, thigh},
PlotStyle -> {{Thickness[0.01], Red}}, AxesLabel -> {"x", "y"},
Epilog -> {Text["start", {x[tlow], y[tlow]}, {1.3, -1}],
Text["end", {x[thigh], y[thigh]}, {1.3, 1}]}];
```



Look at this calculation of $\int_C m[x, y] dx + n[x, y] dy$ for

```
{m[x, y], n[x, y]} = gradf[x, y].
pathintegral = Integrate[m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t], {t, tlow, thigh}]
0
```

Why should you have been able to predict this result before you did the calculation?

G.6.c.ii)

Here's a function f[x, y]:

```
Clear[f, x, y]
f[x_, y_] = Sin[x] Cos[y];
```

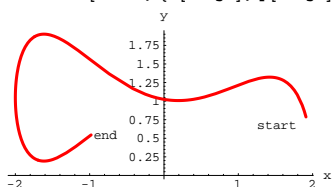
Here's the gradient field of this function:

```
Clear[gradf, m, n, Field]
gradf[x_, y_] = {D[f[x, y], x], D[f[x, y], y]};
{m[x_, y_], n[x_, y_]} = gradf[x, y];
Field[x_, y_] = {m[x, y], n[x, y]}
{Cos[x] Cos[y], -Sin[x] Sin[y]}
```

Here's a curve C:

```
Clear[t]
{x[t_], y[t_]} = {2 Cos[t], t/3 + Sin[t] + 1/2 Sin[3 t]};
tlow = 0.3;
thigh = 4.2;

curveplot = ParametricPlot[{x[t], y[t]}, {t, tlow, thigh},
PlotStyle -> {{Thickness[0.01], Red}}, AxesLabel -> {"x", "y"},
Epilog -> {Text["start", {x[tlow], y[tlow]}, {1.3, 1}],
Text["end", {x[thigh], y[thigh]}, {-1, 0}]}];
```



Look at this calculation of $\int_C m[x, y] dx + n[x, y] dy$ for

```
{m[x, y], n[x, y]} = gradf[x, y]
as above.
pathintegral = NIntegrate[m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t], {t, tlow, thigh}]
-1.37579
```

Now look at this:

```
fchange = f[x[thigh], y[thigh]] - f[x[tlow], y[tlow]]
-1.37579
```

Compare:

```
{pathintegral, fchange}
{-1.37579, -1.37579}
```

Why did this happen?

G.6.d)

Take:

```
Clear[f, x, y]
f[x_, y_] = Sin[x^2 y]
Sin[x^2 y]
```

Look at gradf[x, y]:

```
{D[f[x, y], x], D[f[x, y], y]}
{2 x y Cos[x^2 y], x^2 Cos[x^2 y]}
```

If C is any curve starting at {0, 1.2} and ending at {2.7, -5.1}, then

$$\int_C 2 x y \text{Cos}[x^2 y] dx + x^2 \text{Cos}[x^2 y] dy = f[2.7, -5.1] - f[0, 1.2]$$

```
{f[2.7, -5.1] - f[0, 1.2]}
0.496977
```

Do you agree or disagree? Why?

G.6.e)

Do you think that

$$\text{Field}[x, y] = \{y + x^2, x - \text{Sin}[y]\}$$

is a gradient field?

If you do, then come up with a function f[x, y] with

$$\text{gradf}[x, y] = \{y + x^2, x - \text{Sin}[y]\}.$$

G.6.f)

Suppose you know for sure that Field[x, y] = {m[x, y], n[x, y]} is a gradient field.

Explain how you know that the flow of Field[x, y] along any one curve starting at {x₀, y₀} and ending at {x₁, y₁} is the same as the flow of Field[x, y] along any other curve starting at {x₀, y₀} and ending at {x₁, y₁}.

G.7) Work and how the physicists measure it

What's work for some folks is fun for other folks. Trig identities come to mind; they always seem to be work to the math student but seem to be fun to the math teacher.

The physicists have their own technical notion of work.

The physicists envision a vector field

$$\text{Field}[x, y] = \{m[x, y], n[x, y]\}$$

to represent the force (push) on an object positioned at {x, y}. In this interpretation, the vector field Field[x, y] is called a force field.

Next, the physicists say that if you push an object along a curve C parameterized by

$$\{x[t], y[t]\} \text{ with } tlow \leq t \leq thigh,$$

then the work done by the force field Field[x, y] for you during the duration of the trip is measured by

$$\begin{aligned} \int_C m[x, y] dx + n[x, y] dy &= \int_{tlow}^{thigh} (m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t]) dt \\ &= \int_{tlow}^{thigh} \text{Field}[\{x[t], y[t]\}] \cdot \{x'[t], y'[t]\} dt. \end{aligned}$$

This might not be what your own notion of work is, but the physicists have a pretty good reason for using that word for this measurement.

Think of it this way:

If, at a point on the trip

$$\text{Field}[x[t], y[t]] \cdot \{x'[t], y'[t]\} > 0,$$

then at this point the force field Field[x, y] is working this much to push the object and you do no work at all.

But if

$$\text{Field}[x[t], y[t]] \cdot \{x'[t], y'[t]\} < 0,$$

then at this point the force field Field[x, y] is against your efforts to advance the object; you are working this much and the force field Field[x, y] does no work at all.

With this in mind, you can think of

$$\int_C m[x, y] dx + n[x, y] dy = \int_{tlow}^{thigh} \text{Field}[\{x[t], y[t]\}] \cdot \{x'[t], y'[t]\} dt$$

as a measurement of the force field's work – your work.

→ If $\int_C m[x, y] dx + n[x, y] dy = \int_{low}^{high} \text{Field}[[x[t], y[t]]. \{x'[t], y'[t]\} dt = 0,$

then the force field did the same amount of work that you did during the duration of the trip.

→ If $\int_C m[x, y] dx + n[x, y] dy = \int_{low}^{high} \text{Field}[[x[t], y[t]]. \{x'[t], y'[t]\} dt > 0,$

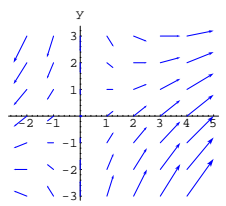
then the force field did most of the work during the duration of the object's trip.

→ If $\int_C m[x, y] dx + n[x, y] dy = \int_{low}^{high} \text{Field}[[x[t], y[t]]. \{x'[t], y'[t]\} dt < 0,$

then you did most of the work during the duration of the object's trip.

□G.7.a)

Is there a difference between net flow – along – the – curve measurements and work?



An object starts at {3.3, 0} and moves through this force field one time around the ellipse

$$\left(\frac{x-1}{2.3}\right)^2 + \left(\frac{y}{1.3}\right)^2 = 1$$

Which way should the object go (counterclockwise or clockwise) to make the force field do most of the work?

□G.7.c)

Comment on the statement:

If a given force field

$$\text{Field}[x, y] = \{m[x, y], n[x, y]\}$$

is a gradient field, and you are pushing an object from a start point $\{x_0, y_0\}$ to an end point $\{x_1, y_1\}$, you might as well push it on the line C starting at $\{x_0, y_0\}$ and ending at $\{x_1, y_1\}$, because if C_1 is any other curve starting at $\{x_0, y_0\}$ and ending at $\{x_1, y_1\}$, then

$$\int_C m[x, y] dx + n[x, y] dy = \int_{C_1} m[x, y] dx + n[x, y] dy$$

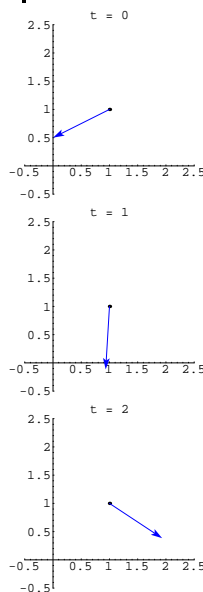
□G.7.d)

Here is a force field that changes as time t(measured in minutes) changes:

```
Clear[Field, m, n, x, y, t, td]
{m[x_, y_, t_], n[x_, y_, t_]} =
-x {Cos[π t/3], Sin[π t/3]} - 1/2 {-Sin[π t/3], Cos[π t/3]};
Field[x_, y_, t_] = {m[x, y, t], n[x, y, t]}
{-x Cos[π t/3] + 1/2 Sin[π t/3], -1/2 Cos[π t/3] - x Sin[π t/3]}
```

Here is how this vector field looks at $\{x, y\} = \{1, 1\}$ for $t = 0, 1, 2, 3, 4, 5,$ and 6 minutes:

```
Table[Show[Graphics[{PointSize[0.02], Point[{1, 1}]}],
Arrow[Field[1, 1, t], Tail -> {1, 1}], Axes -> True,
AxesOrigin -> {0, 0}, PlotRange -> {{-0.5, 2.5}, {-0.5, 2.5}},
PlotLabel -> "t = "<> ToString[t]],
{t, 0, 6}];
```



□Full answer given as a gesture of friendship:

Mathematically there is no difference because they are both measured by the same formula

$$\int_C m[x, y] dx + n[x, y] dy.$$

The difference is in the interpretation.

When you are talking about flow – along – the – curve measurements, then you envision

$$\text{Field}[x, y] = \{m[x, y], n[x, y]\}$$

as the velocity vector at $\{x, y\}$ of a fluid flow.

The fluid is flowing and the curve is just sitting there.

When you are talking about work, then you envision

$$\text{Field}[x, y] = \{m[x, y], n[x, y]\}$$

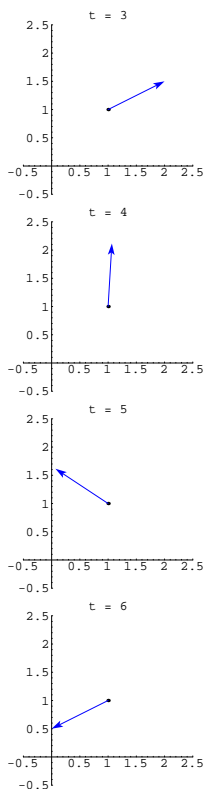
as the force on an object at $\{x, y\}$ moving on a curve.

This time the force field is just sitting there and the object is moving on the curve.

□G.7.b)

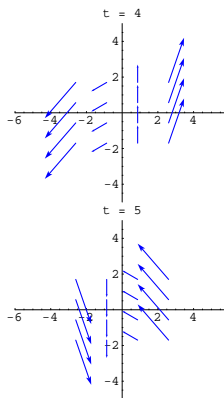
Here is a force field:

```
Clear[Field, m, n, x, y]
m[x_, y_] = x/4;
n[x_, y_] = (x-y)/5;
Field[x_, y_] = {m[x, y], n[x, y]};
forcefieldplot =
Table[Arrow[Field[x, y], Tail -> {x, y}], {x, -2, 4}, {y, -3, 3}];
Show[forcefieldplot, Axes -> Automatic, AxesLabel -> {"x", "y"}];
```



Animate, running forward at a moderate speed.

At time t , $\text{Field}[x, y, t]$ is the vector $\{-x, -\frac{1}{2}\}$ rotated by $\frac{\pi t}{3}$ radians. The effect is that this time-dependent force field is like a revolving force beacon at each point $\{x, y\}$. Here's an idea of how the whole vector field does its dance:



Animate.

You are making plans to send an object on a trip around the ellipse $x^2 + (\frac{y}{2})^2 = 1$ in the counterclockwise direction through this force field. The procedure is to start the clock now, wait t_d seconds, and then start the motion so that at time t the object is at the position $\{x[t], y[t]\} = \{\text{Cos}[t - t_d], 2 \text{Sin}[t - t_d]\}$ for

$$t_d \leq t \leq t_d + 2\pi.$$

Your job is to select a delay time t_d with $0 \leq t_d \leq 10$ to make the work done by the time-dependent force field as large as you can.

Tip:

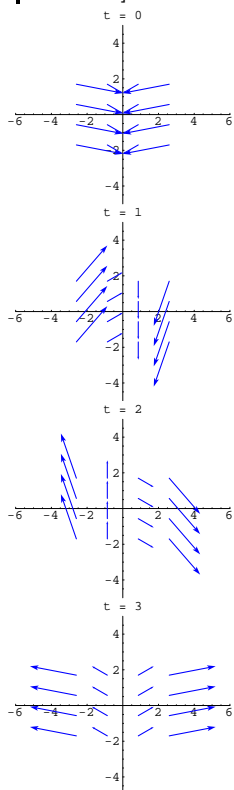
To calculate work of the time-dependent force field as a function of t_d , you integrate

$$m[x[t], y[t], t] x'[t] + n[x[t], y[t], t] y'[t],$$

using appropriate limits.

Once you get a formula for work $[t_d]$, then you can plot to estimate what the best t_d is.

```
Table[Show[Table[Arrow[Field[x, y, t], Tail -> {x, y}],
  {x, -2.6, 2.6, 5.2/3}, {y, -1.7, 1.7, 3.4/3}], Axes -> True,
  PlotLabel -> "t = "<>ToString[t], PlotRange -> {{-6, 6}, {-5, 5}},
  {t, 0, 5}];
```



G.7.e

You are moving along the x-axis, starting at $\{0, 0\}$ and ending at $\{s, 0\}$, under a constant force $F = \{d_1, d_2\}$ at all points.

Explain the formula

$$\text{work done by the force field} = d_1 s.$$

Given that physicists measure force in newtons and distance in meters, say why physicists measure work in newton-meters (which they call joules).

For your information and enjoyment:

$$\text{power} = \text{joules/second}$$

One unit of power is also called a watt. This is where the electric meter on the side of your house gets the measurement of kilowatt hour.

A kilowatt hour is 1000 watts times 3600 seconds

$$= 3.6 \cdot 10^6 \text{ (joules/second) seconds}$$

$$= 3.6 \cdot 10^6 \text{ joules.}$$

Every time you pay your power bill, you are paying for units of work done by electricity in keeping your computer running.

G.7.f

Write a few words on what you think is the difference between the everyday English language definition of work and the technical definition of work as used by the physicists. To get started, think about this:

According to the physicists, if there is no change of position, then there is no work.

If you must hold a heavy old computer in your arms while you stand in place for one hour, the physicists would say that you did no work.

Do you agree?

G.8) Spin fields

Start with a vector field

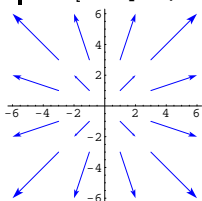
$$\text{Field}[x, y] = \{m[x, y], n[x, y]\}$$

and make what some folks call the spin field

$$\text{spinField}[x, y] = \{-n[x, y], m[x, y]\}.$$

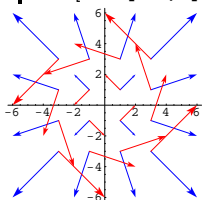
Here's a vector field:

```
Clear[x, y, m, n, Field, spinField]
{m[x_, y_], n[x_, y_]} = {x, y};
Field[x_, y_] = {m[x, y], n[x, y]};
spinField[x_, y_] = {-n[x, y], m[x, y]};
fieldplot =
  Table[Arrow[Field[x, y], Tail -> {x, y}], {x, -3, 3, 2}, {y, -3, 3, 2}];
Show[fieldplot, Axes -> True];
```



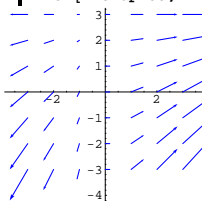
Here's the same vector field plotted together with its spin field:

```
spinfieldplot = Table[Arrow[spinField[x, y],
  Tail -> {x, y}, VectorColor -> Red], {x, -3, 3, 2}, {y, -3, 3, 2}];
Show[fieldplot, spinfieldplot, Axes -> True];
```



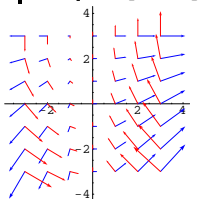
Again:

```
Clear[x, y, m, n, Field, spinField]
{m[x_, y_], n[x_, y_]} = {0.3 x + 0.2, 0.2 x (1 - Y/3)};
Field[x_, y_] = {m[x, y], n[x, y]};
spinField[x_, y_] = {-n[x, y], m[x, y]};
fieldplot =
  Table[Arrow[Field[x, y], Tail -> {x, y}], {x, -3, 3, 1}, {y, -3, 3, 1}];
Show[fieldplot, Axes -> True];
```



Here's the same vector field plotted together with its spin field:

```
spinfieldplot = Table[Arrow[spinField[x, y],
  Tail -> {x, y}, VectorColor -> Red], {x, -3, 3, 1}, {y, -3, 3, 1}];
Show[fieldplot, spinfieldplot, Axes -> True];
```



Run some more of these of your own choice to get a feeling for the relationship between a vector field and its spin field.

□G.8.a)

Try to describe the geometric relationship between a vector field and its spin field.

□Tip:

Think rotation.

□G.8.b)

True or false:

Trajectories of a vector field

$$\text{Field}[x, y] = \{m[x, y], n[x, y]\}$$

cross the trajectories of the spin field

$$\text{SpinField}[x, y] = \{-n[x, y], m[x, y]\}$$

at right angles.

Try to explain your response.

□G.8.c.i)

Explain why:

If C is a closed curve directed in the counterclockwise way, then the measurement of the flow of a vector field

$$\text{Field}[x, y] = \{m[x, y], n[x, y]\}$$

across C is the same as the measurement of the flow of its spin field

$$\text{spinField}[x, y] = \{-n[x, y], m[x, y]\}$$

along C.

Consequently, if the net flow of

$$\text{Field}[x, y] = \{m[x, y], n[x, y]\}$$

across C is from inside to outside, then the net flow of

$$\text{spinField}[x, y] = \{-n[x, y], m[x, y]\}$$

along C is counterclockwise.

□G.8.c.ii)

In what way is the measurement of the net flow of a vector field

$$\text{Field}[x, y] = \{m[x, y], n[x, y]\}$$

along C related to the measurement of the net flow of its spin field

$$\text{spinField}[x, y] = \{-n[x, y], m[x, y]\}$$

across C?

□G.8.d.i)

True or false:

If a vector field

$$\text{Field}[x, y] = \{m[x, y], n[x, y]\}$$

is a gradient field, then you are guaranteed that its spin field

$$\text{spinField}[x, y] = \{-n[x, y], m[x, y]\}$$

is also a gradient field.

□Tip:

$$\text{Field}[x, y] = \{m[x, y], n[x, y]\} = \{2xy, x^2\}$$

is a gradient field because

$$D[x^2, x] = D[2xy, y] = 2x.$$

Is $\text{spinField}[x, y] = \{-x^2, 2xy\}$ also a gradient field?

□G.8.d.ii)

Suppose $\text{Field}[x, y]$ is a function and

$$\text{Field}[x, y] = \{Df[x, y], x\}, D[f[x, y], y]$$

is its gradient field, and that neither $D[f[x, y], x]$ nor $D[f[x, y], y]$ has any singularities.

What condition on

$$\frac{\partial^2 f[x, y]}{\partial x^2} + \frac{\partial^2 f[x, y]}{\partial y^2}$$

$$= D[f[x, y], \{x, 2\}] + D[f[x, y], \{y, 2\}]$$

guarantees that the spin field

$$\text{spinField}[x, y] = \{-D[f[x, y], y], D[f[x, y], x]\}$$

passes the gradient test?

G.9) "Calculus Cal" screws up again

You remember Calculus Cal, that lab pest who talks a lot but produces nothing. What most people don't know is that last semester Cal dropped Calculus&Mathematica because he found that it was not hard to slide through the traditional course with a C. Besides, Cal never understood much anyway; so the traditional course emphasizing rote methods over concepts suits him just fine. On the other hand, Cal still comes to the Mathematica lab because he likes to press buttons and likes the chance to be a pest.

Today, Cal is in the lab doing his homework for the traditional course. Of course, he has hardly looked at his text or at the Basics and Tutorials of this lesson. Here are Cal's halfbaked responses to some problems:

Your job is to do them right.

□G.9.a)

→ Cal's problem:

Calculate the path integral

$$\int_C y \cos[x] dx + 2x y dy$$

where C is the line starting at {0, 0} and ending at $\{\frac{\pi}{2}, 4\}$

→ Cal's solution:

$$\begin{aligned} &\int_C y \cos[x] dx + 2x y dy \\ &= y \sin[x] \Big|_{(0,0)}^{(\pi/2,4)} + x y^2 \Big|_{(0,0)}^{(\pi/2,4)} \\ &= (4 - 0) + ((\frac{\pi}{2}) 16 - 0) = 8\pi + 4. \end{aligned}$$

□G.9.b)

→ Cal's problem:

Given $m[x, y] = x y$ and $n[x, y] = \frac{x^2}{2}$

Calculate the path integral

$$\int_C m[x, y] dx + n[x, y] dy$$

where C is the line starting at {0, 0} and ending at {1, 2}.

→ Cal's solution:

$$D[m[x, y], y] = x$$

and

$$D[n[x, y], x] = x.$$

So

$$D[m[x, y], y] = D[n[x, y], x],$$

and this tells me that $\{m[x, y], n[x, y]\}$ is a gradient field.

Therefore

$$\int_C m[x, y] dx + n[x, y] dy = 0.$$

□G.9.c)

→ Cal's problem:

Given $m[x, y] = x^2 + y$ and $n[x, y] = y^2 - x$

Calculate the path integral

$$\oint_C m[x, y] dx + n[x, y] dy$$

where C is the circle of radius 3 centered at {1, 2}.

→ Cal's solution:

```
Clear[m, n, x, y, t]
m[x_, y_] = x^2 + y;
n[x_, y_] = y^2 - x;
tlow = 0;
thigh = 4 π;
{x[t_], y[t_]} = {1, 2} + 3 {Sin[t], Cos[t]};
∫tlowthigh (m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t]) dt
36 π
```

□G.9.d)

→ Cal's problem:

Given

$$m[x, y] = \frac{x^2 - y + y^2}{x^2 + y^2}$$

and

$$n[x, y] = \frac{x + x^2 + y^2}{x^2 + y^2}$$

Calculate the path integral

$$\oint_C m[x, y] dx + n[x, y] dy$$

where C is the circle of radius 1 centered at {0, 0}.

→ Cal's solution:

Look at the gradient test:

```
Clear[m, n, x, y]
m[x_, y_] = (x^2 - y + y^2)/(x^2 + y^2);
n[x_, y_] = (x + x^2 + y^2)/(x^2 + y^2);
Together[D[m[x, y], y] - D[n[x, y], x]]
0
```

This guarantees that $\{m[x, y], n[x, y]\}$ is a gradient field and so

$$\oint_C m[x, y] dx + n[x, y] dy = 0.$$

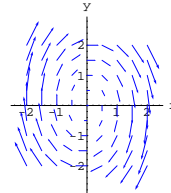
G.10) Force fields and their trajectories

A vector field is a function that spits out vectors. You make a 2D vector field by taking two regular functions $m[x, y]$ and $n[x, y]$ and throwing them into the two slots:

```
Clear[Field, m, n, x, y];
m[x_, y_] = 0.2 y;
n[x_, y_] = -0.4 x;
Field[x_, y_] = {m[x, y], n[x, y]}
{0.2 y, -0.4 x}
```

You plot a vector field by plotting the vector $\text{Field}[x, y]$ with its tail at $\{x, y\}$ for a selection of points $\{x, y\}$:

```
vectorfieldplot = Table[
  Arrow[Field[x, y], Tail -> {x, y}], {x, -2, 2, 0.5}, {y, -2, 2, 0.5}];
Show[vectorfieldplot, Axes -> True, AxesLabel -> {"x", "y"}];
```



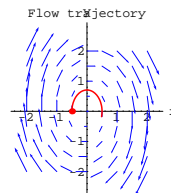
If you go with the usual interpretation of this vector field as velocity vectors of fluid flow and you drop a cork into the flow at $\{-0.5, 0\}$, then the cork advances on a trajectory of the vector field:

```
{a, b} = {-0.5, 0};
starterpoint = {a, b};

Clear[Derivative, x, y, t]
endt = 12;
approximations =
NDSolve[{x'[t] == m[x[t], y[t]], y'[t] == n[x[t], y[t]],
  x[0] == a, y[0] == b}, {x[t], y[t]}, {t, 0, endt}];
```

```
Clear[trajectory]
trajectory[t_] =
{x[t] /. approximations[[1]], y[t] /. approximations[[1]]};
trajectoryplot = ParametricPlot[trajectory[t], {t, 0, endt},
  PlotStyle -> {Red, Thickness[0.01]}, DisplayFunction -> Identity];
starterplot = Graphics[{Red, PointSize[0.04], Point[starterpoint]}];

Show[vectorfieldplot, starterplot, trajectoryplot,
  PlotRange -> All, PlotLabel -> "Flow trajectory", Axes -> True,
  AxesLabel -> {"x", "y"}, DisplayFunction -> $DisplayFunction];
```



On the other hand, if you interpret this vector field as a force field, and you put an object of mass 4 into this force field at $\{-0.5, 0\}$, and you give it an initial velocity of $\{0, 0\}$, then the path of this object isn't the trajectory you see above.

Reason:

One of Newton's laws says

$$\text{force} = \text{mass acceleration.}$$

So the object moves on a curve parameterized by $\{x[t], y[t]\}$ with

$$\text{Field}[x[t], y[t]] = \text{mass } \{x''[t], y''[t]\}$$

with

$$\{x[0], y[0]\} = \{-0.5, 0\} \text{ and } \{x'[0], y'[0]\} = \{0, 0\}.$$

Here's a look:

```
endt = 12;
mass = 4;
{a, b} = {-0.5, 0};

Clear[Derivative, x, y, t]
approximations = NDSolve[
  {m[x[t], y[t]] == mass x''[t], n[x[t], y[t]] == mass y''[t], x[0] == a,
  x'[0] == 0, y[0] == b, y'[0] == 0}, {x[t], y[t]}, {t, 0, endt}];

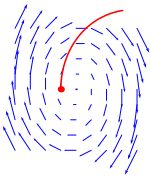
Clear[forcetrajectory]
forcetrajectory[t_] =
```



```
{x[t] /. approxsolutions[[1]], y[t] /. approxsolutions[[1]]};
forcetrajectoryplot = ParametricPlot[forcetrajectory[t], {t, 0, endt},
PlotStyle -> {{Red, Thickness[0.01]}}, DisplayFunction -> Identity];
newstarterplot = Graphics[{Red, PointSize[0.04], Point[{a, b}]}];
```

```
Show[vectorfieldplot, newstarterplot, forcetrajectoryplot,
PlotRange -> All, PlotLabel -> "Force trajectory",
DisplayFunction -> $DisplayFunction];
```

Force trajectory



$$\{x'[0], y'[0]\} = \{-v, 0\}$$

for a selection of v's like $v = 0, 1, 3$, and 4 .
Report on your results.

□G.10.a)

Experiment with different masses and report what you observe.

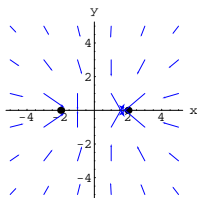
Does lowering the mass make the trajectory more or less responsive to the desires of the force field?

□G.10.b)

Here's a new force field shown with its two singularities:

```
Clear[Field, m, n, x, y]
q = -2.5;
qq = -1.5;
{x0, y0} = {2, 0};
{x1, y1} = {-2, 0};
{m[x_, y_], n[x_, y_]} = {
  q {x - x0, y - y0} / ((x - x0)^2 + (y - y0)^2) +
  qq {x - x1, y - y1} / ((x - x1)^2 + (y - y1)^2);
Field[x_, y_] = {m[x, y], n[x, y]};

forcefieldplot = Show[Graphics[{PointSize[0.04], Point[{x0, y0}]}],
Graphics[{PointSize[0.04], Point[{x1, y1}]}],
Table[ArrowField[x, y], Tail -> {x, y}], {x, -5, 5, 2}, {y, -5, 5, 2}],
Axes -> True, AxesLabel -> {"x", "y"}];
```



Here's what happens when you put an object of mass 0.9 into this force field at $\{3, 4\}$ and you give it an initial velocity of

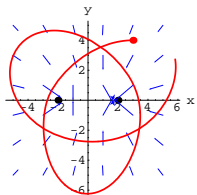
$$\{x'[0], y'[0]\} = \{-2, 0\}:$$

```
mass = 0.9;
{a, b} = {3, 4};
endt = 25;

Clear[Derivative, x, y, t]
approxsolutions = NDSolve[
{m[x[t], y[t]] == mass x''[t], n[x[t], y[t]] == mass y''[t], x[0] == a,
x'[0] == -2, y[0] == b, y'[0] == 0}, {x[t], y[t]}, {t, 0, endt}];

Clear[forcetrajectory]
forcetrajectory[t_] =
{x[t] /. approxsolutions[[1]], y[t] /. approxsolutions[[1]]};
forcetrajectoryplot = ParametricPlot[forcetrajectory[t], {t, 0, endt},
PlotStyle -> {{Red, Thickness[0.01]}}, DisplayFunction -> Identity];
starterplot = Graphics[{Red, PointSize[0.04], Point[{a, b}]}];

Show[forcefieldplot, starterplot, forcetrajectoryplot,
PlotRange -> All, Axes -> True, AxesLabel -> {"x", "y"},
DisplayFunction -> $DisplayFunction];
```



Quite an electric trip.

Experiment with what happens when you keep everything the same but start by giving it an initial velocity of