2D and 3D Measurements

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VC.06 Sources, Sinks, Swirls and Singularities
Basics

B.1) Using a 2D integral to measure flow across closed curves

□B.1.a)

Explain this:
To calculate the net flow of a vector field
Field[x, y] = {m[x, y], n[x, y]}
across the boundary C of a region R, you have your choice:
→ You can go through the labor of parameterizing C, and then calculate
$\oint_C -n[x, y]\,dx + m[x, y]\,dy$,
→ or, if the field has no singularities inside R, you can put
divField[x, y] = D[m[x, y], x] + D[n[x, y], y]
and calculate the 2D integral
$\int\int_R \text{divField}[x, y]\,dx\,dy$.

□Answer:

Back in the lesson on 2D integrals (Lesson 2.05), you met up with the Gauss-Green formula. The Gauss-Green formula says that if C is the boundary curve of a region R, then you are guaranteed that

$$\oint_C -n[x, y]\,dx + m[x, y]\,dy = \int\int_R \text{divField}[x, y]\,dx\,dy.$$

Go with Field[x, y] = {m[x, y], n [x, y}, and put
divField[x, y] = D[m[x, y], x] + D[n[x, y], y]
and read off

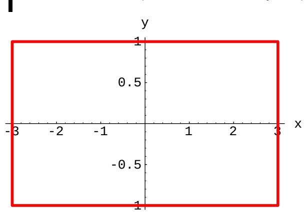
$$\oint_C -n[x, y]\,dx + m[x, y]\,dy = \int\int_R \text{divField}[x, y]\,dx\,dy.$$

Because $\oint_C -n[x, y]\,dx + m[x, y]\,dy$ measures the net flow of Field[x, y] = {m[x, y], n[x, y]} across C, you are guaranteed that
$\int\int_R \text{divField}[x, y]\,dx\,dy$
makes the same measurement.

□B.1.b)

Here's the rectang1e R with corners at {−3, −1}, {3, −1}, {3, 1}, and {−3, 1}:

```
Rplot = Show[Graphics[{Red, Thickness[0.01],
    Line[{{-3, -1}, {3, -1}, {3, 1}, {-3, 1}, {-3, -1}}]}],
  Axes → True, AxesLabel → {"x", "y"}, AspectRatio → 1/GoldenRatio];
```



Use a 2D integral to measure the net flow of the vector field
Field[x, y] = {x^3 + y, x − y}
across the boundary curve C of this rectangle.

□Answer:

Enter the vector field:

```
Clear[x, y, m, n, Field]
{m[x_, y_], n[x_, y_]} = {x^3 + y, x - y};
Field[x_, y_] = {m[x, y], n[x, y]};
```

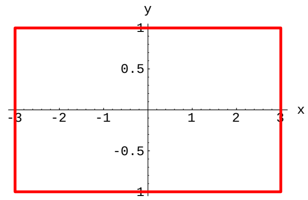
Calculate the divergence, divField[x, y]:

```
Clear[divField]
divField[x_, y_] = D[m[x, y], x] + D[n[x, y], y]
```

$-1 + 3x^2$

Take another look at R:

```
Show[Rplot];
```



Calculate $\int\int_R \text{divField}[x, y]\,dx\,dy$:

```
∫_{-1}^{1} ∫_{-3}^{3} divField[x, y] dx dy
```

96

Big time positive. This means that the net flow of this vector field across the boundary of this rectangle is from inside to outside. There must be a lot of sources of new fluid inside the rectangle.

□B.1.c.i)

Take a look at divField[x, y] for the vector field
Field[x, y] = {Sin[y] − x, Cos[x] − y}:

```
Clear[x, y, m, n, Field, divField]
{m[x_, y_], n[x_, y_]} = {Sin[y] - x, Cos[x] - y};
Field[x_, y_] = {m[x, y], n[x, y]};
divField[x_, y_] = D[m[x, y], x] + D[n[x, y], y]
```

−2

You look at this and note that
divField[x, y] < 0
no matter what {x, y} is, and then you say:
"Good, this tells me that the flow of this vector field across any closed curve is from outside to inside."
You are right.
Why are you right?

□Answer:

Take any closed curve C, and call R the region C encloses. You can calculate the net flow
$\oint_C -n[x, y]\,dx + m[x, y]\,dy$
of Field[x, y] = {m[x, y], n[x, y]} across C by calculating the 2D integral
$\int\int_R \text{divField}[x, y]\,dx\,dy$.
Because
divField[x, y] < 0
no matter what {x, y} you go with, you are guaranteed that
$\int\int_R \text{divField}[x, y]\,dx\,dy < 0$.
So:

$$\oint_C -n[x, y]\,dx + m[x, y]\,dy = \int\int_R \text{divField}[x, y]\,dx\,dy < 0$$

no matter what closed curve C you go with.
This tells you that the flow of this vector field across any closed curve is from outside to inside.
Good deal.

□B.1.c.ii)

Take a look at divField[x, y] for the vector field
Field[x, y] = {Sin[y] + x^5, Cos[x] + y^3}

```
Clear[x, y, m, n, Field, divField]
{m[x_, y_], n[x_, y_]} = {Sin[y] + x^5, Cos[x] + y^3};
Field[x_, y_] = {m[x, y], n[x, y]};
divField[x_, y_] = D[m[x, y], x] + D[n[x, y], y]
```

$5x^4 + 3y^2$

You look at this and note that

divField[x, y] > 0

unless {x, y} = {0, 0}, and then you say:
"Good, this tells me that the flow of this vector field across any closed curve is from inside to outside."
You are right.
Why are you right?

□ **Answer:**

Take any closed curve C, and call R the region C encloses. You can calculate the flow

$\oint_C -n[x, y]\,dx + m[x, y]\,dy$

of Field[x, y] = {m[x, y], n[x, y]} across C by calculating the 2D integral

$\int\int_R \text{divField}[x, y]\,dx\,dy.$

Because

divField[x, y] > 0

except at one point, you are guaranteed that

$\int\int_R \text{divField}[x, y]\,dx\,dy > 0.$

So:

$\oint_C -n[x, y]\,dx + m[x, y]\,dy$
$= \int\int_R \text{divField}[x, y]\,dx\,dy > 0$

no matter what closed curve C you go with.

This tells you that the flow of this vector field across any closed curve is from inside to outside.

DivField[x, y] is a really handy tool.

B.2) Sources, sinks, and the divergence of a vector field

□B.2.a) The meaning of the sign of

divField[x, y] = D[m[x, y], x] + D[n[x, y], y]

Given a vector field

Field[x, y] = {m[x, y], n[x, y]},

you calculate

divField[x, y] = D[m[x, y], x] + D[n[x, y], y].

How does the sign of

divField[x, y] = D[m[x, y], x] + D[n[x, y], y]

tell you whether {x, y} is a source of new fluid or a sink (drain) for old fluid?

□ **Answer:**

If divField[x_0, y_0] > 0, then the point {x_0, y_0} is a source of new fluid.

If divField[x_0, y_0] < 0, then the point {x_0, y_0} is a sink for old fluid.

Here's why:

Take a small circle C centered at {x_0, y_0}. Calculate the flow of Field[x, y] across C by calculating

$\oint_C -n[x, y]\,dx + m[x, y]\,dy$
$= \int\int_R \text{divField}[x, y]\,dx\,dy.$

Here's the kicker:

If

divField[x_0, y_0] > 0,

then it is positive for all {x, y}'s close to {x_0, y_0}. So if C is so small that

divField[x_0, y_0] > 0

at all {x, y}'s inside C, then you see that

$\oint_C -n[x, y]\,dx + m[x, y]\,dy$
$= \int\int_R \text{divField}[x, y]\,dx\,dy > 0.$

This means that if divField[x_0, y_0] > 0, then the net flow of Field[x, y] across small circles centered at {x_0, y_0} is from inside to outside.

The upshot:

If divField[x_0, y_0] > 0, then the point {x_0, y_0} is a source of new fluid.

Similarly, if divField[x_0, y_0] < 0, then the net flow of Field[x, y] across small circles centered at {x_0, y_0} is from outside to inside.

So:

If divField[x_0, y_0] < 0, then the point {x_0, y_0} is a sink for old fluid.

Check this out:

Go with Field[x, y] = {3 x y, y} and look at divField[x, y]:

```
Clear[x, y, Field, m, n, divField]
{m[x_, y_], n[x_, y_]} = {3 x y, y};
Field[x_, y_] = {m[x, y], n[x, y]};
divField[x_, y_] = D[m[x, y], x] + D[n[x, y], y]
```

$1 + 3y$

See whether the point {0, 2} is a source or a sink:

```
divField[0, 2]
```

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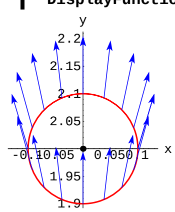
Positive. This tells you that the point {0, 2} is a source.

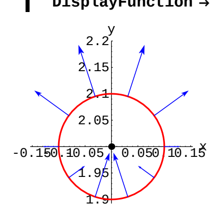
Take a look at this vector field on a small circle centered at {0, 2} to see whether this calculation agrees with reality:

```
point = {0, 2};
radius = 0.1;
pointplot = Graphics[{PointSize[0.04], Point[point]}];
Clear[x, y, t]
{x[t_], y[t_]} = point + radius {Cos[t], Sin[t]};
tlow = 0;
thigh = 2 π;
smallcircle = ParametricPlot[{x[t], y[t]}, {t, tlow, thigh},
  PlotStyle → {{Red, Thickness[0.01]}}, DisplayFunction → Identity];
scalefactor = 0.05;
fieldplot = Table[Arrow[Field[x[t], y[t]], Tail → {x[t], y[t]},
   ScaleFactor → scalefactor], {t, tlow, thigh, (thigh - tlow)/16}];
Show[pointplot, smallcircle, fieldplot,
 Axes → True, AxesLabel → {"x", "y"}, AspectRatio → Automatic,
 DisplayFunction → $DisplayFunction];
```



Confirm by looking at the normal components of the field vectors on the curve:

```
Clear[normal]
normal[t_] = {y'[t], -x'[t]};
Clear[normalcomponent]
normalcomponent[t_] = (Field[x[t], y[t]] . normal[t])/(normal[t] . normal[t]) normal[t];
jump = π/5;
actualflowacross =
 Table[Arrow[normalcomponent[t], Tail → {x[t], y[t]},
   ScaleFactor → scalefactor], {t, 0, 2 π - jump, jump}];
Show[pointplot, smallcircle, actualflowacross,
 Axes → True, AxesLabel → {"x", "y"}, AspectRatio → Automatic,
 DisplayFunction → $DisplayFunction];
```



Yessiree, Bob. The plot shows a lot more flow from inside to outside than from outside to inside.

Just as you would expect on a small circle centered at a source.

□B.2.b) Sources and sinks

Here's a vector field:

```
Clear[Field, m, n, x, y]
{m[x_, y_], n[x_, y_]} = {Sin[x] Cos[y], Sin[y] Cos[x]};
Field[x_, y_] = {m[x, y], n[x, y]}
```

```
{Cos[y] Sin[x], Cos[x] Sin[y]}
```

Give a sample plot of some of the sources and sinks in this vector field.

□Answer:

Here's divField[x, y]:

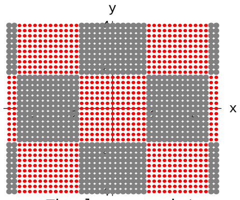
```
Clear[divField]
divField[x_, y_] = D[m[x, y], x] + D[n[x, y], y]
```

```
2 Cos[x] Cos[y]
```

A point {x, y} is a source if divField[x, y] > 0, and {x, y} is a sink if divField[x, y] < 0.

Here comes the plot:

```
sourcesandsinks = Show[Table[If[N[divField[x, y]] > 0,
    Graphics[{PointSize[0.015], Red, Point[{x, y}]}],
    Graphics[{PointSize[0.025], GrayLevel[0.5], Point[{x, y}]}]],
   {x, -5, 5, 0.25}, {y, -4, 4, 0.25}], Axes → True,
  AxesLabel → {"x", "y"}];
```



The larger points are sinks; the smaller points are sources.

Alternate squares of sources and sinks.

Think of the sources as little individual springs feeding the flow.

Think of the sinks as tiny little holes through which fluid seeps out as the flow goes by.

□B.2.c.i) All sources inside C

If every point inside a closed curve (like a deformed circle) C is a source of a given vector field, and if the vector field has no singularities inside C, then how do you know that the net flow of the given vector field across C is automatically from inside to outside?

□Answer:

If every point inside a closed curve C is a source of a given vector field, then

→ new fluid is oozing out of each point inside C, and

→ there there is no place within C to absorb excess outside-to-inside flow.

The result:

If every point inside a closed curve C is a source of a given vector field, then the flow of this vector field across C is automatically from inside to outside.

For example, look at:

```
Clear[Field, m, n, x, y, divField]
{m[x_, y_], n[x_, y_]} = {x^3 - y, y^3 + x};
Field[x_, y_] = {m[x, y], n[x, y]};
divField[x_, y_] = D[m[x, y], x] + D[n[x, y], y]
```

$3x^2 + 3y^2$

Unless {x, y} = {0, 0}, divField[x, y] > 0.

This tells you that all points {x, y} except {0, 0} are sources for Field[x, y] and the lone exception is not a sink.

This vector field has no singularities.

So:

On the basis of this information you can say with confidence and authority that the flow of this vector field across any closed curve is from inside to outside.

□B.2.c.ii) All sinks inside C

If every point inside a closed curve (like a deformed circle) C is a sink of a given vector field, and if the vector field has no singularities inside C, then how do you know that the net flow of the given vector field across C is automatically from outside to inside?

□Answer:

If every point inside a closed curve C is a sink of a given vector field, then

→ old fluid is soaking into each point inside C, and

→ there there is no place within C to generate excess inside-to-outside flow.

The result:

If every point inside a closed curve C is a sink of a given vector field, then the flow of this vector field across C is automatically from outside to inside.

For example, look at:

```
Clear[Field, m, n, x, y, divField]
{m[x_, y_], n[x_, y_]} = {y - x^3, x - y^7};
Field[x_, y_] = {m[x, y], n[x, y]};
divField[x_, y_] = D[m[x, y], x] + D[n[x, y], y]
```

$-3x^2 - 7y^6$

Unless {x, y} = {0, 0}, divField[x, y] < 0.

This tells you that all points {x, y} except {0, 0} are sinks for Field[x, y] and the lone exception is not a source.

This vector field has no singularities.

So:

On the basis of this information you can say with confidence and authority that the flow of this vector field across any closed curve is from outside to inside.

The region inside C is like a big vacuum sucking up fluid.

□B.2.c.iii) No sources or sinks inside C

If there are no sinks and there are no sources of a given vector field inside a closed curve (like a deformed circle) C, and if the vector field has no singularities inside C, then how do you know that the net flow of the given vector field across C is 0?

□**Answer:**

If there are no sinks and there are no sources of a given vector field inside C, and there are no singularities inside C, then no new fluid is injected and no old fluid is sucked up inside C.

So:

→ What flows from inside to outside must be replaced by equal outside to inside flow.

→ What flows from outside to inside must be replaced by equal inside to outside flow.

The result:

If there are no sinks and there are no sources of a given vector field inside C, and there are no singularities inside C, then the net flow of the vector field across C is 0.

For example, look at:

```
Clear[Field, m, n, x, y, divField]
{m[x_, y_], n[x_, y_]} = {Cos[x] Cosh[y], Sin[x] Sinh[y]};
Field[x_, y_] = {m[x, y], n[x, y]};
divField[x_, y_] = D[m[x, y], x] + D[n[x, y], y]
0
```

This tells you that this vector field has no sources or sinks.

This vector field has no singularities, so:

On the basis if this information you can say with confidence and authority that the net flow of this vector field across any closed curve is 0.

□B.2.d) Divergence

Why do most folks call divField[x, y] the divergence of a vector field Field[x, y]?

□**Answer:**

The name fits.

If divField[x, y] > 0, then new fluid is oozing out of the point {x, y} and diverges elsewhere.

If divField[x, y] < 0, then old fluid is sucked into the point {x, y} and converges onto this point.

If divField[x, y] = 0, then no new fluid is introduced, and no old fluid is sucked off as the flow passes by {x, y}.

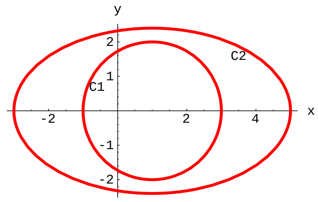
B.3) Flow-across-the-curve measurements in the presence of singularities

The presentation of this problem was heavily influenced by
The Feynman Lectures on Physics
by Richard P. Feynman, Robert B. Leighton, and Matthew Sands,
Addison-Wesley, 1964.

□B.3.a) Singularities

Here are two closed curves, each parameterized correctly in the counterclockwise way:

```
Clear[x1, y1, x2, y2, t]
tlow = 0;
thigh = 2 π;
{x1[t_], y1[t_]} = {1, 0} + 4 {Cos[t], 0.6 Sin[t]};
{x2[t_], y2[t_]} = {1, 0} + 2 {Cos[t], Sin[t]};

curves =
 ParametricPlot[{{x1[t], y1[t]}, {x2[t], y2[t]}}, {t, tlow, thigh},
  PlotStyle → {{Red, Thickness[0.01]}, {Red, Thickness[0.01]}},
  AxesLabel → {"x", "y"},
  Epilog → {Text["C1", {-0.6, 0.7}], Text["C2", {3.5, 1.6}]}];
```



Go with

Field[x, y] = {m[x, y], n[x, y]}
$= \{\frac{6x}{x^2+y^2}, \frac{6y}{x^2+y^2}\}$,

and look at the measurements

$$\oint_{C_1} -n[x, y]\,dx + m[x, y]\,dy,$$

and

$$\oint_{C_2} -n[x, y]\,dx + m[x, y]\,dy$$

of the flow of this vector field across each curve:

```
Clear[Field, m, n, x, y]
{m[x_, y_], n[x_, y_]} = {6 x/(x^2 + y^2), 6 y/(x^2 + y^2)};
Field[x_, y_] = {m[x, y], n[x, y]};

flowacrossC1 =
 NIntegrate[-n[x1[t], y1[t]] x1'[t] + m[x1[t], y1[t]] y1'[t],
  {t, tlow, thigh}]
37.6991
flowacrossC2 = NIntegrate[
  -n[x2[t], y2[t]] x2'[t] + m[x2[t], y2[t]] y2'[t], {t, tlow, thigh}]
37.6991
```

The curves are different, but the flow of this vector field across the one curve is the same as the flow of this vector field across the other. Was this an accident?

□**Answer:**

You got it right.

In mathematics, there are no accidents.

Take a look at the vector field.

```
Field[x, y]
{6 x/(x^2 + y^2), 6 y/(x^2 + y^2)}
```

Note the nasty singularity at {0, 0}.

```
Field[0, 0]
∞::indet : Indeterminate expression 0 ComplexInfinity encountered.
{Indeterminate, Indeterminate}
```

Look at divField[x, y]:

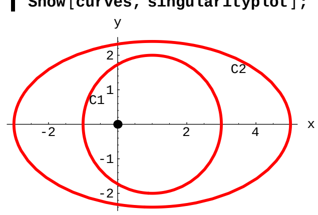
```
Clear[divField]
divField[x_, y_] = Together[D[m[x, y], x] + D[n[x, y], y]]
0
```

Ah-ha!

The vector field has no sources or sinks other than at the singularity at {0, 0}.

Check out the position of the singularity relative to the curves:

```
singularity = {0, 0};
singularityplot = Graphics[{PointSize[0.03], Point[singularity]}];
Show[curves, singularityplot];
```



The singularity is not between the curves.

And because there are no sources or sinks anywhere but at the singularity, you know that the same amount of fluid that flows across the inner curve also flows across the outer curve.

This is why when you measure the flow of the vector field

$$\text{Field}[x, y] = \{m[x, y], n[x, y]\}$$
$$= \left\{\frac{6x}{x^2+y^2}, \frac{6y}{x^2+y^2}\right\},$$

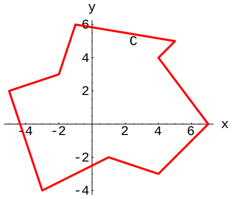
as above, you get

$$\oint_{C_1} -n[x, y]\,dx + m[x, y]\,dy$$
$$= \oint_{C_2} -n[x, y]\,dx + m[x, y]\,dy$$

□B.3.b) Avoiding grisly parameterizations by encapsulating the singularities.

Look at the following curve C:

```
Cplot = Graphics[{Thickness[0.01], Red, Line[{{7, 0}, {4, 4}, {5, 5},
     {-1, 6}, {-2, 3}, {-5, 2}, {-3, -4}, {1, -2}, {4, -3}, {7, 0}}]}];
label = Graphics[Text["C", {2.5, 5}]];
Show[Cplot, label, Axes → True, AxesLabel → {"x", "y"},
 AspectRatio → Automatic];
```



Calculate the flow of

$$\text{Field}[x, y] = \left\{\frac{2(x-2)}{(x-2)^2+(y-1)^2}, \frac{2(y-1)}{(x-2)^2+(y-1)^2}\right\}$$

across C without going to all the bother of parameterizing C.

□Answer:

This is definitely a case in which a little knowledge can save a lot of work. Only a bean-counting dweeb would find pleasure in parameterizing that silly polygonal curve C.

```
Clear[Field, m, n, x, y]
m[x_, y_] = 2 (x - 2) / ((x - 2)^2 + (y - 1)^2);
n[x_, y_] = 2 (y - 1) / ((x - 2)^2 + (y - 1)^2);
Field[x_, y_] = {m[x, y], n[x, y]}
```

$$\left\{\frac{2(-2+x)}{(-2+x)^2+(-1+y)^2}, \frac{2(-1+y)}{(-2+x)^2+(-1+y)^2}\right\}$$

Note the singularity at {2, 1}.

```
Field[2, 1]
∞::indet : Indeterminate expression 0 ComplexInfinity encountered.
{Indeterminate, Indeterminate}
```

Look at divField[x, y]:

```
Together[D[m[x, y], x] + D[n[x, y], y]]
0
```

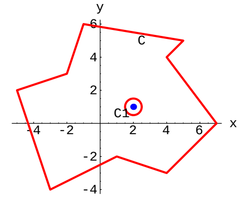
Good.

If there were no singularities inside C, then this information would be enough to tell you that the flow of this field across C is 0. But the nasty singularity (blow-up) at {2, 1} inside the curve C might be a sink or source.

This won't hold you back because you can pull off a pretty neat stunt: Encapsulate the singularity by centering a little circle C_1 around the singularity at {2, 1}, taking care that the little circle lies completely within C.

Here's one:

```
singularity = {2, 1};
Clear[x1, y1, t]
{x1[t_], y1[t_]} = singularity + 0.5 {Cos[t], Sin[t]};
C1plot = ParametricPlot[{x1[t], y1[t]}, {t, 0, 2 π},
   PlotStyle → {{Thickness[0.01], Red}}, DisplayFunction → Identity];
singularityplot =
 Graphics[{Blue, PointSize[0.03], Point[singularity]}];
extralabel = Graphics[Text["C1", {1.3, 0.6}]];
Show[Cplot, C1plot, singularityplot, label,
 extralabel, PlotRange → All, Axes → True, AxesLabel → {"x", "y"},
 AspectRatio → Automatic, DisplayFunction → $DisplayFunction];
```



Because

$$\text{divField}[x, y] = D[m[x, y], x] + D[n[x, y], y] = 0$$

at all points between the circle C_1 and the ugly curve C, and because there are no singularities between the two curves, you can be certain that that the flow of this vector field across the little circle C_1 is the same as the flow of this vector field across C. In other words,

$$\oint_{C} -n[x, y]\,dx + m[x, y]\,dy$$
$$= \oint_{C_1} -n[x, y]\,dx + m[x, y]\,dy.$$

The beauty of this is that you can calculate

$$\oint_{C_1} -n[x, y]\,dx + m[x, y]\,dy$$

with a couple flicks of your fingers:

```
flowacrossC1 = NIntegrate[
  -n[x1[t], y1[t]] x1′[t] + m[x1[t], y1[t]] y1′[t], {t, 0, 2 π}]
12.5664
```

The net flow across the little circle C_1 is from inside to outside.

There was a gushing source at the singularity.

The flow of the given vector field across C is the same as the flow of of this field across C_1:

```
flowacrossC = flowacrossC1
12.5664
```

And you did this without going to the trouble of parameterizing C.

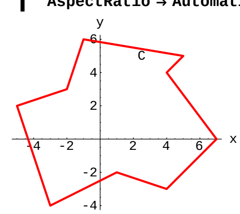
A little knowledge saved a lot of work.

Does anyone other than a wuss really want to parameterize that silly curve?

□B.3.c) Encapsulating two singularities

Stay with the same curve C as in part b):

```
Cplot = Graphics[{Thickness[0.01], Red, Line[{{7, 0}, {4, 4}, {5, 5},
     {-1, 6}, {-2, 3}, {-5, 2}, {-3, -4}, {1, -2}, {4, -3}, {7, 0}}]}];
label = Graphics[Text["C", {2.5, 5}]];

Show[Cplot, label, Axes → True, AxesLabel → {"x", "y"},
 AspectRatio → Automatic];
```



Go with:

```
Clear[Field, m, n, x, y]
m[x_, y_] = (x - 2) / ((x - 2)^2 + (y - 1)^2) - 2 (x + 3) / ((x + 3)^2 + (y + 2)^2);
n[x_, y_] = (y - 1) / ((x - 2)^2 + (y - 1)^2) - 2 (y + 2) / ((x + 3)^2 + (y + 2)^2);
Field[x_, y_] = {m[x, y], n[x, y]}
```

$$\left\{\frac{-2+x}{(-2+x)^2+(-1+y)^2} - \frac{2(3+x)}{(3+x)^2+(2+y)^2}, \frac{-1+y}{(-2+x)^2+(-1+y)^2} - \frac{2(2+y)}{(3+x)^2+(2+y)^2}\right\}$$

Calculate the flow of this vector field across C without breaking into a heavy sweat.

□Answer:

Look at the vector field again.

```
Field[x, y]
{ (-2 + x)/((-2 + x)^2 + (-1 + y)^2) - (2 (3 + x))/((3 + x)^2 + (2 + y)^2),
  (-1 + y)/((-2 + x)^2 + (-1 + y)^2) - (2 (2 + y))/((3 + x)^2 + (2 + y)^2)}
```

Note the singularities at at {2, 1} and at {−3, −2}.

```
{Field[2, 1], Field[-3, -2]}
{{Indeterminate, Indeterminate}, {Indeterminate, Indeterminate}}
```

Now look at:

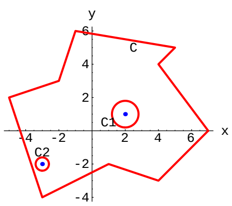
```
Together[D[m[x, y], x] + D[n[x, y], y]]
0
```

Good.

The vector field has no sources or sinks aside from the singularities. Encapsulate the singularities by centering a little circle C_1 at the singularity at {2, 1}, and centering another little circle C_2 at the singularity at {−3, −2} taking care that both little circles lie completely within C, and taking care that they do not invade each other's territory.

Here are two very acceptable circles:

```
singularity1 = {2, 1};
singularity2 = {-3, -2};
Clear[x1, y1, x2, y2, t]
{x1[t_], y1[t_]} = singularity1 + 0.8 {Cos[t], Sin[t]};
{x2[t_], y2[t_]} = singularity2 + 0.4 {Cos[t], Sin[t]};
littlecircles =
  ParametricPlot[{{x1[t], y1[t]}, {x2[t], y2[t]}}, {t, 0, 2 π},
   PlotStyle → {{Thickness[0.01], Red}, {Thickness[0.01], Red}},
   DisplayFunction → Identity];
singularityplot =
  {Graphics[{Blue, PointSize[0.02], Point[singularity1]}],
   Graphics[{Blue, PointSize[0.02], Point[singularity2]}]};
extralabels =
  {Graphics[Text["C1", {1, 0.5}]], Graphics[Text["C2", {-3, -1.3}]]};

Show[Cplot, littlecircles, singularityplot, label,
  extralabels, PlotRange → All, Axes → True, AxesLabel → {"x", "y"},
  AspectRatio → Automatic, DisplayFunction → $DisplayFunction];
```



Because

$$\text{divField}[x, y] = D[m[x, y], x] + D[n[x, y], y] = 0$$

at all points between the circles and the ugly curve C, and since there are no singularities between the circles and C, you can be certain that the net flow across C is given by

$$\oint_C -n[x, y]\,dx + m[x, y]\,dy$$
$$= \oint_{C_1} -n[x, y]\,dx + m[x, y]\,dy + \oint_{C_2} -n[x, y]\,dx + m[x, y]\,dy.$$

Here you go with just a couple flicks of your fingers:

```
flowacrossC1 = NIntegrate[
   -n[x1[t], y1[t]] x1′[t] + m[x1[t], y1[t]] y1′[t], {t, 0, 2 π}];
flowacrossC2 = NIntegrate[
   -n[x2[t], y2[t]] x2′[t] + m[x2[t], y2[t]] y2′[t], {t, 0, 2 π}];

flowacrossC = flowacrossC1 + flowacrossC2
-6.28319
```

The net flow of this vector field across C is from outside to inside. At least one of the singularities must have been a big time sink.

VC.06 Sources, Sinks, Swirls and Singularities Tutorials

T.1) The pleasure of calculating path integrals

$$\oint_C m[x, y]\,dx + n[x, y]\,dy$$

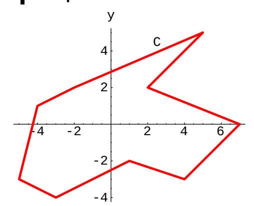
when

$$D[n[x, y], x] - D[m[x, y], y] = 0$$

□T.1.a.i) No singularity inside C

Here is a curve:

```
Cplot = Graphics[{Thickness[0.01], Red, Line[{{7, 0}, {2, 2}, {5, 5},
      {-2, 2}, {-4, 1}, {-5, -3}, {-3, -4}, {1, -2}, {4, -3}, {7, 0}}]}];
label = Graphics[Text["C", {2.5, 4.5}]];

Show[Cplot, label, Axes → True, AxesLabel → {"x", "y"},
 AspectRatio → Automatic];
```

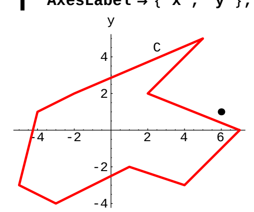


Here are two functions m[x, y] and n[x, y]:

```
Clear[m, n, x, y]
m[x_, y_] = x^2 + (2 (x - 6))/((x - 6)^2 + (y - 1)^4)
x^2 + (2 (-6 + x))/((-6 + x)^2 + (-1 + y)^4)
n[x_, y_] = y^2 + (4 (y - 1)^3)/((x - 6)^2 + (y - 1)^4)
(4 (-1 + y)^3)/((-6 + x)^2 + (-1 + y)^4) + y^2
```

Note the singularity at {6, 1} and plot it:

```
singularity = {6, 1};
singularityplot = Graphics[{PointSize[0.03], Point[singularity]}];

Show[Cplot, singularityplot, label, Axes → True,
 AxesLabel → {"x", "y"}, AspectRatio → Automatic];
```



Good.
The singularity is not inside the region enclosed by C.
Now check D[n[x, y], x] − D[m[x, y], y]:

```
Together[D[n[x, y], x] - D[m[x, y], y]]
0
```

Good.
Now, without further calculation, you can be sure that

$$\oint_C m[x, y]\,dx + n[x, y]\,dy = 0.$$

Why can you be sure of this?

□Answer:

Go with the m[x,y] and n[x,y] used above.

$$\oint_C m[x, y]\,dx + n[x, y]\,dy$$

measures the net flow of the vector field

$$\text{Field}[x, y] = \{n[x, y], -m[x, y]\}$$

across C.

Stop here and read again.
The vector field you are working with here is
Field[x, y] = {n[x, y], −m[x, y]}.

The divergence of THIS vector field is
D[n[x, y], x] – D[m[x, y], y].

This vector field has no singularities inside C, and because

divField[x, y] = D[n[x, y], x] – D[m[x, y], y] = 0,

this vector field has no sources or sinks within C. And because you can't get something out of nothing, the net flow of this vector field across C is 0.

So

$$\oint_C m[x, y]\,dx + n[x, y]\,dy = 0.$$

If this stings your brain, you could use the Gauss-Green formula to explain this.

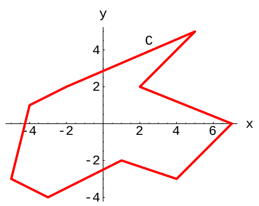
The Gauss-Green formula says that if C is the boundary curve of a region R, then you are guaranteed that

$$\oint_C m[x, y]\,dx + n[x, y]\,dy$$
$$= \int\int_R (D[n[x, y], x] - D[m[x, y], y])\,dx\,dy$$
$$= \int\int_R 0\,dx\,dy = 0.$$

☐T.1.a.ii) A singularity inside C

Here's the same curve:

```
Cplot = Graphics[{Thickness[0.01], Red, Line[{{7, 0}, {2, 2}, {5, 5},
      {-2, 2}, {-4, 1}, {-5, -3}, {-3, -4}, {1, -2}, {4, -3}, {7, 0}}]}];
label = Graphics[Text["C", {2.5, 4.5}]];

Show[Cplot, label, Axes → True, AxesLabel → {"x", "y"},
 AspectRatio → Automatic];
```



Here are two new functions m[x, y] and n[x, y]:

```
Clear[m, n, x, y]
m[x_, y_] = (1 - y)/((x + 2)^2 + (y - 1)^2)
```

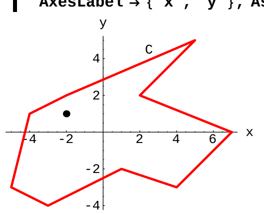
$$\frac{1-y}{(2+x)^2+(-1+y)^2}$$

```
n[x_, y_] = (x + 2)/((x + 2)^2 + (y - 1)^2)
```

$$\frac{2+x}{(2+x)^2+(-1+y)^2}$$

Note the singularity at {−2, 1} and plot it:

```
singularity = {-2, 1};
singularityplot = Graphics[{PointSize[0.03], Point[singularity]}];

Show[Cplot, singularityplot, label, Axes → True,
 AxesLabel → {"x", "y"}, AspectRatio → Automatic];
```



This time the singularity is inside the region enclosed by C.
Now check D[n[x, y], x] – D[m[x, y], y]:

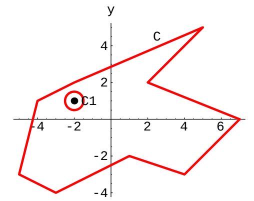
```
Together[D[n[x, y], x] - D[m[x, y], y]]
0
```

If there were no singularities inside C, then this information would be enough to tell you that $\oint_C m[x, y]\,dx + n[x, y]\,dy = 0$, but this does not tell you that

$$\oint_C m[x, y]\,dx + n[x, y]\,dy = 0$$

because of the nasty singularity (blow up) inside the curve C.
Encapsulate the singularity by centering a little circle C_1 at the singularity, taking care that the circle lies completely within C.
Here's one:

```
Clear[x1, y1, t]
{x1[t_], y1[t_]} = singularity + 0.5 {Cos[t], Sin[t]};
C1plot = ParametricPlot[{x1[t], y1[t]}, {t, 0, 2 π},
  PlotStyle → {{Thickness[0.01], Red}}, DisplayFunction → Identity];
singularityplot = Graphics[{PointSize[0.03], Point[singularity]}];
extralabel = Graphics[Text["C1", {-1.2, 1}]];

Show[Cplot, C1plot, singularityplot, label,
 extralabel, PlotRange → All, Axes → True, AxesLabel → {"x", "y"},
 AspectRatio → Automatic, DisplayFunction → $DisplayFunction];
```



Now you can be sure that

$$\oint_C m[x, y]\,dx + n[x, y]\,dy$$
$$= \oint_{C_1} m[x, y]\,dx + n[x, y]\,dy$$

which is given by:

```
NIntegrate[m[x1[t], y1[t]] x1'[t] + n[x1[t], y1[t]] y1'[t], {t, 0, 2 π}]
6.28319
```

Explain why you can be sure that this calculation is correct.

☐**Answer:**

Go with the m[x, y] and n[x, y] used above.

$$\oint_C m[x, y]\,dx + n[x, y]\,dy$$

measures the net flow of the vector field

Field[x, y] = {n[x, y], −m[x, y]}

across C. This vector field has no singularities between C and C_1, and because

divField[x, y] = D[n[x, y], x] – D[m[x, y], y] = 0,

this vector field has no sources or sinks between C and C_1. And because you can't get something out of nothing, the net flow of this vector field across C is the same as the net flow of this vector field across C_1.

So

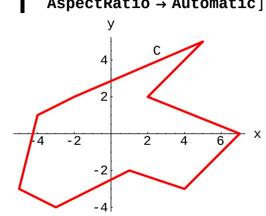
$$\oint_C m[x, y]\,dx + n[x, y]\,dy$$
$$= \oint_{C_1} m[x, y]\,dx + n[x, y]\,dy.$$

You've got a hard time using the Gauss-Green formula to explain this directly, because the Gauss-Green formula can fail when C encloses a singularity.

☐T.1.a.iii) Two singularities inside C

Here is the same curve again:

```
Cplot = Graphics[{Thickness[0.01], Red, Line[{{7, 0}, {2, 2}, {5, 5},
      {-2, 2}, {-4, 1}, {-5, -3}, {-3, -4}, {1, -2}, {4, -3}, {7, 0}}]}];
label = Graphics[Text["C", {2.5, 4.5}]];

Show[Cplot, label, Axes → True, AxesLabel → {"x", "y"},
 AspectRatio → Automatic];
```



Here are two new functions m[x, y] and n[x, y]:

```
Clear[m, n, x, y]
m[x_, y_] = 3 (1 - y)/((x + 2)^2 + (y - 1)^2) + (y + 1)/((x + 2)^2 + (y + 1)^2)
```

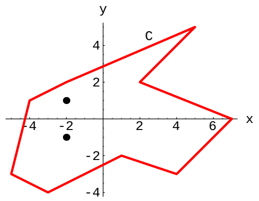
$$\frac{3\,(1-y)}{(2+x)^2+(-1+y)^2}+\frac{1+y}{(2+x)^2+(1+y)^2}$$

```
n[x_, y_] = 3 (x + 2)/((x + 2)^2 + (y - 1)^2) - (x + 2)/((x + 2)^2 + (y + 1)^2)
```

$$\frac{3\,(2+x)}{(2+x)^2+(-1+y)^2}-\frac{2+x}{(2+x)^2+(1+y)^2}$$

Note the singularities at {−2, 1} and {−2, −1} and plot them:

```
singularity1 = {-2, 1};
singularity2 = {-2, -1};
singularityplot = {Graphics[{PointSize[0.03], Point[singularity1]}],
  Graphics[{PointSize[0.03], Point[singularity2]}]};

Show[Cplot, singularityplot, label, Axes → True,
 AxesLabel → {"x", "y"}, AspectRatio → Automatic];
```



The singularities are inside the region enclosed by C.
Now check D[n[x, y], x] – D[m[x, y], y]:

```
Together[D[n[x, y], x] - D[m[x, y], y]]
0
```

If there were no singularities inside C, then this information would be enough to tell you that $\oint_C m[x, y]\,dx + n[x, y]\,dy$ is 0 . But this does not tell you that

$$\oint_C m[x, y]\,dx + n[x, y]\,dy = 0$$

because of the nasty singularities inside the curve C.
Encapsulate the singularities by centering a little circle C_1 at one singularity and centering another little circle C_2 at the other singularity, taking care that both little circles lie completely within C, across C. This vector field has no singularities between C and the little circles, and because

divField[x, y] = D[n[x, y], x] – D[m[x, y], y] = 0,

this vector field has no sources or sinks between C and the little circles. And because you can't get something out of nothing, the net flow of this vector field across C is the same as the net flow of this vector field across one little circle added to the net flow of this vector field across the other little circle.

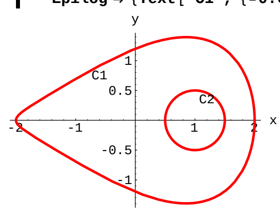
So

$$\oint_C m[x, y]\,dx + n[x, y]\,dy$$
$$= \oint_{C_1} m[x, y]\,dx + n[x, y]\,dy + \oint_{C_2} m[x, y]\,dx + n[x, y]\,dy.$$

□T.1.b) Path integrals

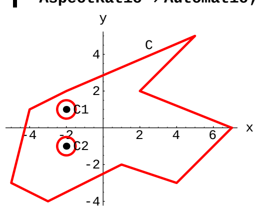
Here are two new curves both correctly parameterized in the counterclockwise way:

```
Clear[x1, y1, x2, y2, t]
tlow = 0;
thigh = 2 π;
{x1[t_], y1[t_]} = 2 {Cos[t], 0.6 Sin[t] + 0.2 Sin[2 t]};
{x2[t_], y2[t_]} = {1, 0} + 0.5 {Cos[t], Sin[t]};

curves =
 ParametricPlot[{{x1[t], y1[t]}, {x2[t], y2[t]}}, {t, tlow, thigh},
  PlotStyle → {{Red, Thickness[0.01]}, {Red, Thickness[0.01]}},
  AxesLabel → {"x", "y"}, PlotRange → All,
  Epilog → {Text["C1", {-0.6, 0.75}], Text["C2", {1.2, 0.35}]}];
```



and that C_1 and C_2 don't touch or invade each other's territory.
Here are two suitable circles:

```
Clear[x1, y1, x2, y2, t]
{x1[t_], y1[t_]} = singularity1 + 0.5 {Cos[t], Sin[t]};
{x2[t_], y2[t_]} = singularity2 + 0.5 {Cos[t], Sin[t]};
littles = ParametricPlot[{{x1[t], y1[t]}, {x2[t], y2[t]}}, {t, 0, 2 π},
  PlotStyle → {{Thickness[0.01], Red}}, DisplayFunction → Identity];
singularityplot = {Graphics[{PointSize[0.03], Point[singularity1]}],
  Graphics[{PointSize[0.03], Point[singularity2]}]};
extralabels =
 {Graphics[Text["C1", {-1.2, 1}]], Graphics[Text["C2", {-1.2, -1}]]};

Show[Cplot, littles, singularityplot, label, extralabels,
 PlotRange → All, Axes → True, AxesLabel → {"x", "y"},
 AspectRatio → Automatic, DisplayFunction → $DisplayFunction];
```



Now you can be sure that

$$\oint_C m[x, y]\,dx + n[x, y]\,dy$$
$$= \oint_{C_1} m[x, y]\,dx + n[x, y]\,dy + \oint_{C_2} m[x, y]\,dx + n[x, y]\,dy,$$

which is given by:

```
NIntegrate[m[x1[t], y1[t]] x1′[t] + n[x1[t], y1[t]] y1′[t], {t, 0, 2 π}] +
 NIntegrate[m[x2[t], y2[t]] x2′[t] + n[x2[t], y2[t]] y2′[t], {t, 0, 2 π}]
12.5664
```

Explain why you can be sure that this calculation is correct.

□**Answer:**

Go with the m[x, y] and n[x, y] used above.

$$\oint_C m[x, y]\,dx + n[x, y]\,dy$$

measures the net flow of the vector field

Field[x, y] = {n[x, y], −m[x, y]}

Given functions m[x, y] and n[x, y], what do you check to be sure that

$$\oint_{C_1} m[x, y]\,dx + n[x, y]\,dy$$
$$= \oint_{C_2} m[x, y]\,dx + n[x, y]\,dy$$

without going to all the bother of making the individual calculations?

□**Answer:**

If no singularities of {m[x, y], n[x, y]} pop up between C_1 and C_2, and if

D[n[x, y], x] – D[m[x, y], y] = 0

at all points between C_1 and C_2,

then you can be sure that

$$\oint_{C_1} m[x, y]\,dx + n[x, y]\,dy$$
$$= \oint_{C_2} m[x, y]\,dx + n[x, y]\,dy.$$

T.2) Using a 2D integral to measure flow along closed curves

This problem is a copy, paste and edit job of B.2)

□T.2.a)

Explain this:
To calculate the net flow of a vector field

Field[x, y] = {m[x, y], n[x, y]}

along the boundary C of a region R, you have your choice:
→ You can go to all the labor of parameterizing C and then calculate

$$\oint_C m[x, y]\,dx + n[x, y]\,dy.$$

→ Or if the field has no singularities inside R, you can put

rotField[x, y] = D[n[x, y], x] – D[m[x, y], y]

and calculate the 2D integral

$$\int\int_R \text{rotField}[x, y]\,dx\,dy.$$

□**Answer:**

Back in the lesson on 2D integrals, you met up with the Gauss-Green formula. The Gauss-Green formula says that if C is the boundary curve of a region R, then you are guaranteed that

$$\oint_C m[x, y]\,dx + n[x, y]\,dy$$
$$= \int\int_R D[n[x, y], x] - D[m[x, y], y]\,dx\,dy.$$

Go with Field[x, y] = {m[x, y], n[x, y]} and put

rotField[x, y] = D[n[x, y], x] − D[m[x, y], y],

and read off

$$\oint_C m[x, y]\,dx + n[x, y]\,dy$$
$$= \int\int_R \text{rotField}[x, y]\,dx\,dy.$$

Because

$$\oint_C m[x, y]\,dx + n[x, y]\,dy$$

measures the flow of Field[x, y] = {m[x, y], n[x, y]} along C, you are guaranteed that

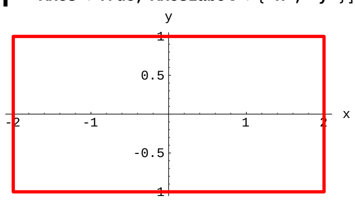
$$= \int\int_R \text{rotField}[x, y]\,dx\,dy$$

makes the same measurement.

□T.2.b)

Here is the rectangle R with corners at {−2, −1}, {2, −1}, {2, 1}, and {−2, 1}:

```
Rplot = Show[Graphics[{Red, Thickness[0.01],
      Line[{{-2, -1}, {2, -1}, {2, 1}, {-2, 1}, {-2, -1}}]}],
   Axes → True, AxesLabel → {"x", "y"}];
```



Use a 2D integral to measure the net flow of the vector field

Field[x, y] = {x + y^2, x − y^2}

along the boundary curve C of this rectangle.

□Answer:

Enter the field:

```
Clear[x, y, m, n, Field]
{m[x_, y_], n[x_, y_]} = {x + y^2, x - y^2};
Field[x_, y_] = {m[x, y], n[x, y]}
```

{x + y^2, x − y^2}

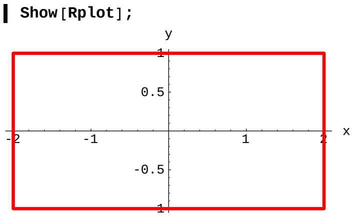
Calculate rotField[x, y]:

```
Clear[rotField]
rotField[x_, y_] = D[n[x, y], x] - D[m[x, y], y]
```

1 − 2 y

Take another look at R:

```
Show[Rplot];
```



Calculate $\int\int_R$ rotField[x, y] $dx\,dy$:

```
∫_{-1}^{1} ∫_{-2}^{2} rotField[x, y] dx dy
```

8

Positive.

This tells you that the net flow of this vector field along the boundary of this rectangle is counterclockwise.

□T.2.c.i)

Take a look at rotField[x, y] for the vector field

Field[x, y] = {Sin[x] + y, Cos[y] − x}

```
Clear[x, y, m, n, Field, rotField]
{m[x_, y_], n[x_, y_]} = {Sin[x] + y, Cos[y] - x};
Field[x_, y_] = {m[x, y], n[x, y]};
rotField[x_, y_] = D[n[x, y], x] - D[m[x, y], y]
```

−2

You look at this and note that

rotField[x, y] < 0

no matter what {x, y} is. And then you say:

"Good, this tells me that the flow of this vector field along any closed curve is clockwise."

You are right.

Why are you right?

□Answer:

Take any closed curve C and call R the region C encloses. You can calculate the flow

$$\oint_C m[x, y]\,dx + n[x, y]\,dy$$

of Field[x, y] = {m[x, y], n[x, y]} along C by calculating the 2D integral

$$\int\int_R \text{rotField}[x, y]\,dx\,dy\,.$$

Because

rotField[x, y] < 0

no matter what {x, y} you go with, you are guaranteed that

$$\int\int_R \text{rotField}[x, y]\,dx\,dy < 0.$$

So

$$\oint_C m[x, y]\,dx + n[x, y]\,dy$$
$$= \int\int_R \text{rotField}[x, y]\,dx\,dy < 0$$

no matter what closed curve C you go with.

This tells you that the flow of this vector field along any closed curve is clockwise.

□T.2.c.ii)

Take a look at rotField[x, y] for the vector field

Field[x, y] = {Sin[x] − y^5, Cos[y] + x^3}

```
Clear[x, y, m, n, Field, rotField]
{m[x_, y_], n[x_, y_]} = {Sin[x] - y^5, Cos[y] + x^3};
Field[x_, y_] = {m[x, y], n[x, y]};

rotField[x_, y_] = D[n[x, y], x] - D[m[x, y], y]
```

3 x^2 + 5 y^4

You look at this and note that

rotField[x, y] > 0

unless {x, y} = {0, 0}. And then you say:

"Good, this tells me that the net flow of this vector field along any closed curve is counterclockwise."

You are right.

Why are you right?

□Answer:

Take any closed curve C, and call R the region C encloses. You can calculate the flow

$$\oint_C m[x, y]\,dx + n[x, y]\,dy$$

of Field[x, y] = {m[x, y], n[x, y]} along C by calculating the 2D integral

$$\int\int_R \text{rotField}[x, y]\,dx\,dy.$$

Because

rotField[x, y] > 0

except at one point, you are guaranteed that

$$\int\int_R \text{rotField}[x, y]\,dx\,dy > 0.$$

So

$$\oint_C m[x, y]\,dx + n[x, y]\,dy$$

$= \int \int_R$ rotField[x, y] dx dy > 0

no matter what closed curve C you go with.

This tells you that the net flow of this vector field along any closed curve is counterclockwise.

T.3) Rotation (swirl) of a vector field

This problem is a copy, paste, and edit job of B.2).

□T.3.a) Fingering a vector field to get the meaning of the sign of rotField[x, y] = D[n[x, y], x] – D[m[x, y], y]

Given a vector field Field[x, y] = {m[x, y], n[x, y]}, lick the tip of your index finger and touch it to a point {x_0, y_0} while the vector field is in full flow. What does the sign of

rotField[x_0, y_0]

tell you about the about the swirl your finger feels?
Illustrate with a plot.

□Answer:

If rotField[x_0, y_0] > 0, then you feel Field[x, y] swirling around {x_0, y_0} in the counterclockwise way.

If rotField[x_0, y_0] < 0, then you fell Field[x, y] swirling around {x_0, y_0} in the clockwise way.

Here's why:

Take a small circle C centered at {x_0, y_0}. Calculate the flow of Field[x, y] along C by calculating

$\oint_C$ m[x, y] dx + n[x, y] dy = $\int\int_R$ rotField[x, y] dx dy.

Here's the kicker:

If

rotField[x_0, y_0] > 0,

then it is positive for all {x, y}'s close to {x_0, y_0}, so if C is so small that

rotField[x_0, y_0] > 0

at all {x, y}'s inside C, then you see that

$\oint_C$ m[x, y] dx + n[x, y] dy = $\int\int_R$ rotField[x, y] dx dy > 0.

The upshot:

If rotField[x_0, y_0] > 0, then the net flow of Field[x, y] along small circles centered at {x_0, y_0} is counterclockwise.

Similarly, if rotField[x_0, y_0] < 0, then the net flow of Field[x, y] along small circles centered at {x_0, y_0} is clockwise.

Check this out for the following vector field at {0, 0}:

```
Clear[x, y, m, n, Field, rotField]
m[x_, y_] = x + 3 y^2;
n[x_, y_] = y + 6 x;
Field[x_, y_] = {m[x, y], n[x, y]};

rotField[x_, y_] = D[n[x, y], x] - D[m[x, y], y]
6 - 6 y
```

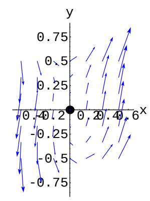
Look at:

```
rotField[0, 0]
6
```

Positive.

This means Field[x, y] swirls around {0, 0} in the counterclockwise way. Take a look at what the vector field is doing near {0, 0}:

```
fingerpoint = {0, 0};
h = 0.5;
scalefactor = 0.1;
fieldplot =
 Table[Arrow[Field@@(fingerpoint + {x, y}), Tail → fingerpoint + {x, y},
   ScaleFactor → scalefactor], {x, -h, h, h/3}, {y, -h, h, h/3}];
fingerpointplot = Graphics[{PointSize[0.07], Point[fingerpoint]}];

Show[fingerpointplot, fieldplot, Axes → Automatic,
 AspectRatio → Automatic, PlotRange → All, AxesLabel → {"x", "y"}];
```



Lick the tip of your index finger, put the tip of your finger at the point, and feel the counterclockwise swirl.

□T.3.b) Clockwise versus counterclockwise

Here's a vector field:

```
Clear[Field, m, n, x, y]
{m[x_, y_], n[x_, y_]} = {Sin[x] Cos[2 y], Sin[y] Cos[2 x]};

Field[x_, y_] = {m[x, y], n[x, y]}
{Cos[2 y] Sin[x], Cos[2 x] Sin[y]}
```

Give a sample plot of some of the points {x, y} about which this vector field swirls in the counterclockwise way.

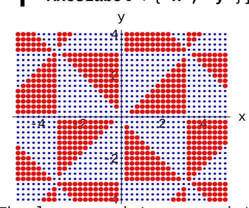
□Answer:

Here's rotField[x, y]:

```
Clear[rotField]
rotField[x_, y_] = D[n[x, y], x] - D[m[x, y], y]
-2 Sin[2 x] Sin[y] + 2 Sin[x] Sin[2 y]
```

The vector field swirls around {x, y} in the counterclockwise way if rotField[x, y] > 0. Here comes the plot:

```
Show[Table[If[N[rotField[x, y]] > 0,
    Graphics[{PointSize[0.02], Red, Point[{x, y}]}],
    Graphics[{PointSize[0.01], Blue, Point[{x, y}]}]],
  {x, -5, 5, 0.25}, {y, -4, 4, 0.25}], Axes → True,
 AxesLabel → {"x", "y"}];
```



The larger points are points at which the swirl is counterclockwise; the smaller points are points at which the swirl is either 0 or clockwise.

□T.3.c) Rotation (swirl) at a point

Why do most folks call rotField[x, y] the rotation of a vector field Field[x, y]?

□Answer:

The name fits.

If rotField[x, y] > 0, then the fluid swirls around {x, y} in the counterclockwise way.

If rotField[x, y] < 0, then the fluid swirls around {x, y} in the clockwise way.

If rotField[x, y] = 0, then the fluid has no swirl at all as it passes by {x, y}.

T.4) Summary of main ideas.

Calculus&Mathematica offers this summary to you
for your good use and enjoyment.
It comes from the home office to you.
'99

□T.4.a.i) Sources versus sinks

If

$\text{Field}[x, y] = \{m[x, y], n[x, y]\}$

is a vector field, then a point {x, y} is:

→ a source of new fluid if

$\text{divField}[x, y] = D[m[x, y], x] + D[n[x, y], y] > 0$

→ a sink for old fluid if

$\text{divField}[x, y] = D[m[x, y], x] + D[n[x, y], y] < 0$

→ neither a source nor a sink if

$\text{divField}[x, y] = D[m[x, y], x] + D[n[x, y], y] = 0.$

□T.4.a.ii) Counterclockwise versus clockwise swirl

If

$\text{Field}[x, y] = \{m[x, y], n[x, y]\}$

is a vector field, then a the swirl about a point {x, y} is:

→ counterclockwise provided

$\text{rotField}[x, y] = D[n[x, y], x] - D[m[x, y], y] > 0$

→ clockwise provided

$\text{rotField}[x, y] = D[n[x, y], x] - D[m[x, y], y] < 0.$

□T.4.b.i) Flow across

If C is a closed curve and R is the region inside C, then the net flow of a vector field

$\text{Field}[x, y] = \{m[x, y], n[x, y]\}$

across C is measured by

$$\oint_C -n[x, y]\,dx + m[x, y]\,dy = \int\int_R \text{divField}[x, y]\,dx\,dy$$

provided there are no singularities inside C.

(Here $\text{divField}[x, y] = D[m[x, y], x] + D[n[x, y], y]$.)

If divField[x, y] = 0 at all points inside C, and there are no singularities inside C, then the net flow of Field[x, y] across C is 0.

If divField[x, y] > 0 at all points inside C, and there are no singularities inside C, then the net flow of Field[x, y] across C is from inside to outside.

If divField[x, y] < 0 at all points inside C, and there are no singularities inside C, then the net flow of Field[x, y] across C is from outside to inside.

□T.4.b.ii) Flow along

If C is a closed curve and R is the region inside C, then the net flow of a vector field

$\text{Field}[x, y] = \{m[x, y], n[x, y]\}$

along C is measured by

$$\oint_C m[x, y]\,dx + n[x, y]\,dy = \int\int_R \text{rotField}[x, y]\,dx\,dy$$

provided there are no singularities inside C.

(Here $\text{rotField}[x, y] = D[n[x, y], x] - D[m[x, y], y]$.)

If rotField[x, y] = 0 at all points inside C, and there are no singularities inside C, then the net flow of Field[x, y] along C is 0.

If rotField[x, y] > 0 at all points inside C, and there are no singularities inside C, then the net flow of Field[x, y] along C is counterclockwise.

If divField[x, y] < 0 at all points inside C, and there are no singularities inside C, then the net flow of Field[x, y] along C is clockwise.

□T.4.c.i) Singularities and flow-across measurements

Given a vector field Field[x, y] = {m[x, y], n[x, y]},
if C_1 and C_2 are two closed curves with C_1 running inside C_2(C_1 is allowed to touch C_2) and if

→ there are no singularities of Field[x, y] between C_1 and C_2, and

→ $\text{DivField}[x, y] = D[m[x, y], x] + D[n[x, y], y] = 0$

at all points {x, y} between C_1 and C_2,

then

$$\oint_{C_1} -n[x, y]\,dx + m[x, y]\,dy = \oint_{C_2} -n[x, y]\,dx + m[x, y]\,dy,$$

so that the net flow of Field[x, y] across C_1 equals the net flow of Field[x, y] across C_2.

□T.4.c.ii) Singularities and path integrals

Given functions m[x, y] and n[x, y], if

→ C_1 and C_2 are two closed curves with C_1 running inside C_2(C_1 is allowed to touch C_2), and if

→ neither m[x, y] nor n[x, y] has a singularity anywhere between C_1 and C_2, and if

→ $D[n[x, y], x] - D[m[x, y], y] = 0,$

then

$$\oint_{C_1} m[x, y]\,dx + n[x, y]\,dy = \oint_{C_2} m[x, y]\,dx + n[x, y]\,dy.$$

VC.06 Sources, Sinks, Swirls and Singularities
Give them a Try!

Experience with the starred problems will be useful for understanding developments later in the course.

G.1) Sources, sinks and swirls*

□G.1.a.i)

Go with

$\text{Field}[x, y] = \{e^x \text{Sin}[y], e^x \text{Cos}[y]\}.$

Calculate

divField[x, y]

and use the result to say why the net flow of Field[x, y] across any closed curve is 0.

□G.1.a.ii)

Go with

$\text{Field}[x, y] = \{x, y\}.$

Calculate

divField[x, y]

and use the result to say why the net flow of Field[x, y] across any closed curve is from inside to outside.

□G.1.b.i)

Go with

$\text{Field}[x, y] = \{e^x \text{Sin}[y], e^x \text{Cos}[y]\}.$

Calculate

rotField[x, y]

and use the result to say why the net flow of Field[x, y] along any closed curve is 0.

□G.1.b.ii)

Go with

$\text{Field}[x, y] = \{y, -x\}.$

Calculate

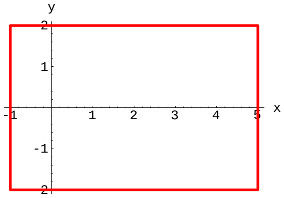
rotField[x, y]

and use the result to say why the net flow of Field[x, y] along any closed curve is clockwise.

□**G.1.c)**

Here is the rectang1e R with corners at
$\{-1, -2\}, \{5, -2\}, \{5, 2\}$, and $\{-1, 2\}$:

```
Rplot = Show[Graphics[{Red, Thickness[0.01],
    Line[{{-1, -2}, {5, -2}, {5, 2}, {-1, 2}, {-1, -2}}]}],
  Axes → True, AxesLabel → {"x", "y"}];
```



Use the formula

$\oint_C -n[x, y]\, dx + m[x, y]\, dy$

$= \int\int_R \text{divField}[x, y]\, dx\, dy$

to measure the net flow of the vector field

Field[x, y] = $\{x^2 + 2\, y^2, x^2 - 2\, y^2\}$

across the boundary curve C of R. Is the net flow of this vector field across C from outside to inside, or is it from inside to outside?

Next, use the formula

$\oint_C m[x, y]\, dx + n[x, y]\, dy$

$= \int\int_R (D[n[x, y], x] - D[m[x, y], y])\, dx\, dy.$

to measure the net flow of the vector field

Field[x, y] = $\{x^2 + 2\, y^2, x^2 - 2\, y^2\}$

across the boundary curve C of R. Is the net flow of this vector field along C counterclockwise or clockwise?

□**G.1.d.i)**

Here's a vector field:

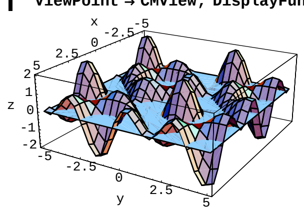
```
Clear[Field, m, n, x, y]
{m[x_, y_], n[x_, y_]} = {Cos[x] Cos[2 y], Cos[y] Cos[2 x]};

Field[x_, y_] = {m[x, y], n[x, y]};
```

Give a sample plot of some of the sources in this vector field.

□**G.1.d.ii)**

Go with the same vector field as in part i) immediately above and look at this plot:

```
Clear[Field, m, n, x, y]
{m[x_, y_], n[x_, y_]} = {Cos[x] Cos[2 y], Cos[y] Cos[2 x]};
Field[x_, y_] = {m[x, y], n[x, y]};
Clear[divField]
divField[x_, y_] = D[m[x, y], x] + D[n[x, y], y];
divplot = Plot3D[divField[x, y], {x, -5, 5},
  {y, -5, 5}, PlotPoints → {25, 25}, DisplayFunction → Identity];
xyplane =
 Graphics3D[Polygon[{{-5, -5, 0}, {-5, 5, 0}, {5, 5, 0}, {5, -5, 0}}]];

Show[divplot, xyplane, AxesLabel → {"x", "y", "z"},
 ViewPoint → CMView, DisplayFunction → $DisplayFunction];
```



How is this plot related to the plot you did in part d.i) immediately above?

□**G.1.e.i)**

Here's another vector field:

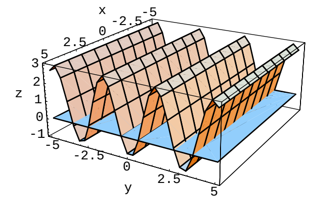
```
Clear[Field, m, n, x, y]
{m[x_, y_], n[x_, y_]} = {Sin[2 y], x};
Field[x_, y_] = {m[x, y], n[x, y]}
{Sin[2 y], x}
```

Give a sample plot of some of the points at which this vector field swirls in the counterclockwise direction.

□**G.1.e.ii)**

Go with the same vector field as in part e.i) immediately above and look at this plot:

```
Clear[Field, m, n, x, y]
{m[x_, y_], n[x_, y_]} = {Sin[2 y], x};
Field[x_, y_] = {m[x, y], n[x, y]};
Clear[rotField]
rotField[x_, y_] = D[n[x, y], x] - D[m[x, y], y];
rotplot = Plot3D[rotField[x, y], {x, -5, 5},
  {y, -5, 5}, PlotPoints → {10, 30}, DisplayFunction → Identity];
xyplane =
 Graphics3D[Polygon[{{-5, -5, 0}, {-5, 5, 0}, {5, 5, 0}, {5, -5, 0}}]];

Show[rotplot, xyplane, AxesLabel → {"x", "y", "z"},
 ViewPoint → CMView, DisplayFunction → $DisplayFunction];
```



How is this plot related to the plot you did in part e.i) immediately above?

G.2) Singularity sources, sinks and swirls*

□**G.2.a) Singularity sources and sinks**

Given a vector field

Field[x, y] = {m[x, y], n[x, y]},

you can calculate

divField[x, y] = D[m[x, y], x] + D[n[x, y], y]

to look for sources and sinks.

If you locate a source at {x, y} and {x, y} is not a singularity, you can expect new fluid to be slowly oozing into the flow at {x, y}.

If you locate a sink at {x, y} and {x, y} is not a singularity, you can expect old fluid to be slowly soaking out of the flow at {x, y}.

The vivid sources and sinks are often found at singularities; in fact if you envision a sink at a singularity to be a black hole, you are thinking correctly.

To detect a source or a sink at a singularity, you center a small circle C[r] of radius r at the singularity and calculate

$\oint_{C[r]} -n[x, y]\, dx + m[x, y]\, dy.$

→ If

$\lim_{r \to 0} \oint_{C[r]} -n[x, y]\, dx + m[x, y]\, dy > 0,$

you have located a singularity source (a gusher) at the singularity.

→ If

$\lim_{r \to 0} \oint_{C[r]} -n[x, y]\, dx + m[x, y]\, dy < 0,$

you have located a singularity sink (a black hole) at the singularity.

→ If

$\lim_{r \to 0} \oint_{C[r]} -n[x, y]\, dx + m[x, y]\, dy = 0,$

you have located a singularity that is neither a source nor a sink.

Try it out:

The point {a, b} is a singularity of the 2D electric field coming from a point charge of strength 2 placed at {a, b}:

```
Clear[ElectricField, a, b, m, n, q, x, y]
{m[x_, y_], n[x_, y_]} = q {x - a, y - b} / ((x - a)^2 + (y - b)^2);
ElectricField[x_, y_] = {m[x, y], n[x, y]}
```

$\left\{\frac{q\,(-a+x)}{(-a+x)^2+(-b+y)^2},\ \frac{q\,(-b+y)}{(-a+x)^2+(-b+y)^2}\right\}$

Center a circle C[r] of radius r at {a, b} and calculate

$\oint_{C[r]} -n[x, y]\,dx + m[x, y]\,dy$:

```
singularity = {a, b};
Clear[xr, yr, r, t]
{xr[t_], yr[t_]} = singularity + r {Cos[t], Sin[t]};

∫_0^(2π) (-n[xr[t], yr[t]] xr'[t] + m[xr[t], yr[t]] yr'[t]) dt
```

$2\pi q$

This tells you that the singularity at {a, b} is a source of new juice if q > 0 (positive charge at {a, b}) and is a sink for old juice if q < 0 (negative charge at {a, b}).

□G.2.a.i)

Does the electric field above have sources or sinks other than at the singularity?

□Tip:

Look at:

```
Together[D[m[x, y], x] + D[n[x, y], y]]
```

0

□G.2.a.ii)

Here's a vector field related to the electric field:

```
Clear[Field, a, b, m, n, x, y]
{m[x_, y_], n[x_, y_]} = 5 {x - a, y - b} / Sqrt[(x - a)^2 + (y - b)^2];
Field[x_, y_] = {m[x, y], n[x, y]}
```

$\left\{\frac{5\,(-a+x)}{\sqrt{(-a+x)^2+(-b+y)^2}},\ \frac{5\,(-b+y)}{\sqrt{(-a+x)^2+(-b+y)^2}}\right\}$

Note the singularity at {a, b}.
Now look at the following information:

```
singularity = {a, b};
Clear[xr, yr, r, t]
{xr[t_], yr[t_]} = singularity + r {Cos[t], Sin[t]};

∫_0^(2π) (-n[xr[t], yr[t]] xr'[t] + m[xr[t], yr[t]] yr'[t]) dt
```

$10\pi\sqrt{r^2}$

How does this help to tell you that the singularity at {a, b} is neither a singularity source nor a singularity sink?

□G.2.a.iii)

Stay with the same vector field as in part ii) above and look at divField[x, y] together with

$(x - a)^2 + (y - b)^2$:

```
Clear[Field, a, b, m, n, x, y]
{m[x_, y_], n[x_, y_]} = 5 {x - a, y - b} / Sqrt[(x - a)^2 + (y - b)^2];
Field[x_, y_] = {m[x, y], n[x, y]};
{Together[D[m[x, y], x] + D[n[x, y], y]], Expand[(x - a)^2 + (y - b)^2]}
```

$\left\{\frac{5}{\sqrt{(-a+x)^2+(-b+y)^2}},\ a^2+b^2-2\,a\,x+x^2-2\,b\,y+y^2\right\}$

How does this help to tell you that all points {x, y} other than the singularity at {a, b} are sources?
How does this tell you that the big-time sources of new fluid for this field are packed near the singularity?

□G.2.a.iv)

Here's another vector field related to the electric field:

```
Clear[Field, a, b, q, m, n, x, y]
{m[x_, y_], n[x_, y_]} = 7 {x - a, y - b} / ((x - a)^2 + (y - b)^2)^(3/2);
Field[x_, y_] = {m[x, y], n[x, y]}
```

$\left\{\frac{7\,(-a+x)}{((-a+x)^2+(-b+y)^2)^{3/2}},\ \frac{7\,(-b+y)}{((-a+x)^2+(-b+y)^2)^{3/2}}\right\}$

Determine whether {a, b} is a singularity source, a singularity sink, or neither.
Also determine whether there are sources or sinks other than at the singularity.

□G.2.b.i) Singularity swirls

When you pull the plug in a bathtub, you see a good example of a singularity swirl (and a singularity sink).
To detect a singularity swirl, you center a small circle C[r] of radius r at the singularity and calculate a limit of a certain path integral. If this limit is positive, then you have located a counterclockwise singularity swirl. If this limit is negative, then you have located a clockwise singularity swirl.
If this limit is 0, then you have located singularity that has no swirling effect at all.
What limit do you look at?

□G.2.b.ii)

Here's a vector field with a singularity at {0, 0}:

```
Clear[Field, m, n, x, y]
{m[x_, y_], n[x_, y_]} = {y, -x} / (x^2 + y^2);
Field[x_, y_] = {m[x, y], n[x, y]}
```

$\left\{\frac{y}{x^2+y^2},\ -\frac{x}{x^2+y^2}\right\}$

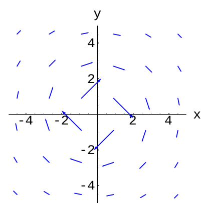
Here is rotField[x, y]:

```
rotField[x, y] = Together[D[n[x, y], x] - D[m[x, y], y]]
```

0

This field has no swirl around any point other than possibly the singularity at {0, 0}.
Look at a plot of this vector field.

```
scalefactor = 2;
fieldplot =
  Table[Arrow[Field[x, y], Tail → {x, y}, ScaleFactor → scalefactor],
    {x, -4.5, 4.5, 9/5}, {y, -4.5, 4.5, 9/5}];
Show[fieldplot, Axes → True, AxesLabel → {"x", "y"}];
```



Big-time clockwise singularity swirl around the singularity at {0, 0}.
Test your answer to part b.i) immediately above to see whether your limit agrees with reality.

G.3) Agree or disagree*

Indicate your agreement or disagreement with each of the following statements and paragraphs. When you disagree, say why you disagree. When you agree, feel free to say why you agree, but you are under no obligation to do so.

□G.3.a)

If Field[x, y] = {m[x, y], n[x, y]} has no sinks, sources, or singularities within a closed curve C, then the net flow of Field[x, y] across C must be 0.

□G.3.b)

If Field[x, y] = {m[x, y], n[x, y]} has no singularities within a closed curve C, and rotField[x, y] = 0 at all points within C, then the net flow of Field[x, y] along C is 0.

□G.3.c)

If Field[x, y] = {m[x, y], n[x, y]} has no singularities and has the property that rotField[x, y] = 0 at all points {x, y}, then Field[x, y] is guaranteed to be a gradient field.

□Tip:

Go back one lesson and look up the two parts of the gradient test.

□G.3.d)

Here is a vector field:

```
Clear[m, n, x, y]
{m[x_, y_], n[x_, y_]} = {x^3, y^3};
Field[x_, y_] = {m[x, y], n[x, y]}
```

$\{x^3, y^3\}$

No singularities anywhere.
Look at divField[x, y]:

```
D[m[x, y], x] + D[n[x, y], y]
```

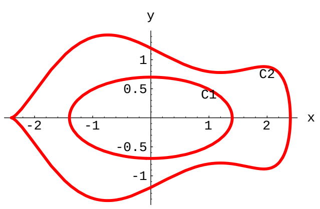
$3 x^2 + 3 y^2$

This is always positive except when {x, y} = {0, 0}.
As a result, every {x, y} other than {0, 0} is a source for Field[x, y].
Consequently, if C is any closed curve (like a deformed circle), then the net flow of Field[x, y] across C is from inside to outside.
After all, it cannot be from outside to inside because there are no sinks inside C to absorb the excess fluid.

□G.3.e)

Here are two curves:

```
Clear[x1, y1, x2, y2, t]
{x1[t_], y1[t_]} = 0.7 {2 Cos[t], Sin[t]};
{x2[t_], y2[t_]} = 1.2 {2 Cos[t], Sin[t] + 1/4 Sin[4 t]};
curveplots =
 ParametricPlot[{{x1[t], y1[t]}, {x2[t], y2[t]}}, {t, 0, 2 π},
  PlotStyle → {{Thickness[0.01], Red}}, DisplayFunction → Identity];
label =
 {Graphics[Text["C1", {1, 0.4}]], Graphics[Text["C2", {2, 0.75}]]};

Show[curveplots, label, PlotRange → All,
 Axes → True, AxesLabel → {"x", "y"}, AspectRatio → Automatic,
 DisplayFunction → $DisplayFunction];
```



The inner curve is C_1.

Suppose Field[x, y] = {m[x, y], n[x, y]} has the properties that
→ all the singularities of Field[x, y] are inside C_1,
→ $\oint_{C_1} -n[x, y]\,dx + m[x, y]\,dy > 0$, and
→ divField[x, y] = D[m[x, y], x] + D[n[x, y], y] > 0
at all points {x, y} between C_1 and C_2.
This tells you that all points {x, y} between C_1 and C_2 are sources for the flow corresponding to Field[x, y]. As a result,

$$\oint_{C_1} -n[x, y]\,dx + m[x, y]\,dy < \oint_{C_2} -n[x, y]\,dx + m[x, y]\,dy$$

so that the flow of Field[x, y] across C_2 must be greater than the flow of Field[x, y] across C_1.

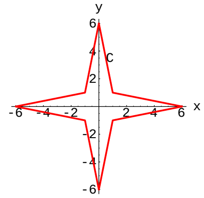
□G.3.f)

Suppose Field[x, y] = {m[x, y], n[x, y]} has exactly two singularities in the region R enclosed by a closed curve C. Also suppose that C_1 is a closed curve running totally within R. If Field[x, y] has no sinks, sources, or singularities between C and C_1, then the net flow of Field[x, y] across C equals the net flow of Field[x, y] across C_1.

□G.3.g)

Here's a closed curve C:

```
Cplot = Graphics[{Thickness[0.01], Red, Line[{{6, 0}, {1, 1}, {0, 6},
      {-1, 1}, {-6, 0}, {-1, -1}, {0, -6}, {1, -1}, {6, 0}}]}];
label = Graphics[Text["C", {0.8, 3.5}]];

Show[Cplot, label, Axes → True, AxesLabel → {"x", "y"},
 AspectRatio → Automatic];
```



And here's a vector field:

```
Clear[m, n, x, y]
{m[x_, y_], n[x_, y_]} = {(-y + (x/4 - 1))/(x^2 + y^2), x/(x^2 + y^2)};
Field[x_, y_] = {m[x, y], n[x, y]}
```

$\left\{\frac{-1 + \frac{x}{4} - y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\right\}$

Note the singularity at {0, 0}.
This tells you that you can calculate the flow of Field[x, y] across C by calculating

$$\oint_{C_1} -n[x, y]\,dx + m[x, y]\,dy$$

where C_1 is a small circle centered at {0, 0}.

□Tip:

Before you leap on this one, check to see whether there are sources or sinks between the singularity and C.

```
Together[D[m[x, y], x] + D[n[x, y], y]]
```

$\frac{8 x - x^2 + y^2}{4 (x^2 + y^2)^2}$

G.4) Flow calculations in the presence of singularities*

□G.4.a)

How can you tell without evaluating any path integral that the flow of the vector field

$$\text{Field}[x, y] = \left\{\frac{2 x}{x^2 + y^2}, \frac{2 y}{x^2 + y^2}\right\}$$

across the circle $x^2 + y^2 = r^2$ is the same as its flow across any ellipse $(\frac{x}{a})^2 + (\frac{y}{b})^2 = 1$?

□Tip:

```
Clear[m, n, x, y]
{m[x_, y_], n[x_, y_]} = {2 x/(x^2 + y^2), 2 y/(x^2 + y^2)}
```

$\left\{\frac{2 x}{x^2 + y^2}, \frac{2 y}{x^2 + y^2}\right\}$

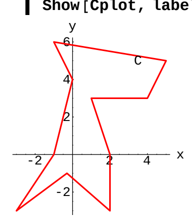
```
Together[D[m[x, y], x] + D[n[x, y], y]]
```

0

□G.4.b.i)

Look at the following curve C:

```
Cplot = Graphics[
   {Thickness[0.01], Red, Line[{{2, 0}, {1, 3}, {4, 3}, {5, 5}, {-1, 6},
      {0, 4}, {-1, 0}, {-3, -3}, {-.3, -1}, {2, -3}, {2, 0}}]}];
label = Graphics[Text["C", {3.5, 5}]];

Show[Cplot, label, Axes → True, AxesLabel → {"x", "y"}];
```



Why would only a dweeb measure the net flow of

$$\text{Field}[x, y] = \left\{-\frac{4 x}{x^2 + (y - 2)^2}, -\frac{4 (y - 2)}{x^2 + (y - 2)^2}\right\}$$

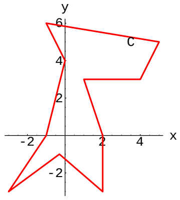
across C by parameterizing C and calculating the path integral?

Remove yourself from the dweeb class by measuring the net flow of this vector field across C by measuring the net flow of this vector field across a convenient substitute curve.
Is the net flow of this vector field across C from inside to outside, or is it from outside to inside?

□G.4.b.ii)

Go with the same curve C as above:

```
Show[Cplot, label, Axes → True, AxesLabel → {"x", "y"}];
```



You are asked to measure the net flow of

$$\text{Field}[x, y] = \left\{\frac{4\,(x+2)}{(x+2)^2+(y-2)^2}, \frac{4\,(y-2)}{(x+2)^2+(y-2)^2}\right\}$$

across C.
You weren't born yesterday; so you look at:

```
Clear[x, y, m, n, Field, divField]
{m[x_, y_], n[x_, y_]} = {4 (x + 2)/((x + 2)^2 + (y - 2)^2), 4 (y - 2)/((x + 2)^2 + (y - 2)^2)};
Field[x_, y_] = {m[x, y], n[x, y]};
divField[x_, y_] = Together[D[m[x, y], x] + D[n[x, y], y]]
```

```
0
```

And then you announce that the net flow of this vector field across C is 0. You are right.
Why are you right?

G.5) 2D electric fields, dipole fields, and Gauss's law in physics*

□G.5.a)

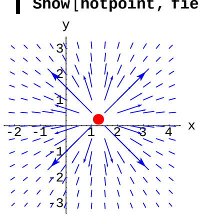
When you put a charge proportional to a number q at a point {a, b}, the resulting two-dimensional electrical field is:

```
Clear[x, y, a, b, q, Electfield]
Electfield[x_, y_, a_, b_, q_] = q {x - a, y - b}/((x - a)^2 + (y - b)^2)
```

$$\left\{\frac{q\,(-a+x)}{(-a+x)^2+(-b+y)^2}, \frac{q\,(-b+y)}{(-a+x)^2+(-b+y)^2}\right\}$$

Here's a plot in the case {a, b} = {1.25, 0.25} and q = 2:

```
a = 1.25;
b = 0.25;
q = 2;
hotpoint = Graphics[{PointSize[0.06], Red, Point[{a, b}]}];
scalefactor = 0.4;
fieldplot = Table[Arrow[Electfield[x, y, a, b, q], Tail → {x, y},
    ScaleFactor → scalefactor], {x, -2, 4, 0.5}, {y, -3, 3, 0.5}];

Show[hotpoint, fieldplot, Axes → True, AxesLabel → {"x", "y"}];
```



Describe what you see.

□G.5.b)

Here are the divergence and the rotation of the electric field resulting from a point charge of strength q placed at {a, b}:

```
Clear[x, y, a, b, q, Electfield, m, n, divElectfield, rotElectfield]
Electfield[x_, y_, a_, b_, q_] = q {x - a, y - b}/((x - a)^2 + (y - b)^2)
```

$$\left\{\frac{q\,(-a+x)}{(-a+x)^2+(-b+y)^2}, \frac{q\,(-b+y)}{(-a+x)^2+(-b+y)^2}\right\}$$

```
{m[x_, y_], n[x_, y_]} = Electfield[x, y, a, b, q];
divElectfield[x_, y_] = Simplify[D[m[x, y], x] + D[n[x, y], y]]
```

```
0
```

```
rotElectfield[x_, y_] = Simplify[D[n[x, y], x] - D[m[x, y], y]]
```

```
0
```

Does this electric field have sources or sinks at points other than {a, b}?
How do you know?
What is the net flow of this electric field across any closed curve without loops that does not wrap around the singularity at {a, b}?

□G.5.c)

Here is the measurement of the flow of the electric field in across any circle of radius r centered at {a, b}.

Electrical folks call the net flow measurement of an electric field across a curve C by the name "flux across C."

```
Clear[x, y, a, b, q, Electfield, m, n]
Electfield[x_, y_, a_, b_, q_] = q {x - a, y - b}/((x - a)^2 + (y - b)^2);
{m[x_, y_], n[x_, y_]} = Electfield[x, y, a, b, q];
{x[t_], y[t_]} = {a, b} + r {Cos[t], Sin[t]};

Integrate[(-n[x[t], y[t]] x'[t] + m[x[t], y[t]] y'[t]), {t, 0, 2 π}]
```

$2\pi q$

Usually when the divergence of a vector field is 0, then its flow across any closed curve is 0. But this is not true here for closed curves wrapping around {a, b} because of the singularity (blow up) of the electric field at {a, b}.
What is the flow of the electric field across any closed curve without loops that wraps around the singularity at {a, b}?

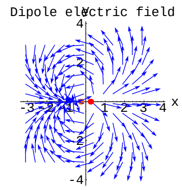
□G.5.d) Dipoles

A dipole can be approximated by two charges of the same magnitude but opposite sign separated by a small distance.
Dipoles are especially important in atomic theory. The great scientist Richard Feynman explained it this way:

"Although an atom or molecule remains neutral in an external electronic field, there is a tiny separation of its positive and negative charges and it becomes a microscopic dipole."
.....................The Feynman Lectures

Here is a plot of the dipole electrical field resulting from a positive charge at {0.25, 0} and the opposite charge of the same magnitude at {−0.25, 0}.

```
Clear[x, y, a, b, q, Electfield, m, n];
Electfield[x_, y_, a_, b_, q_] = q {x - a, y - b}/((x - a)^2 + (y - b)^2);
{aplus, bplus} = {0.25, 0};
{aminus, bminus} = {-0.25, 0};
q = 2;
singularitysource =
 Graphics[{PointSize[0.04], Red, Point[{aplus, bplus}]}];
singularitysink =
 Graphics[{PointSize[0.04], CadmiumOrange, Point[{aminus, bminus}]}];
dipoleplot = Table[Arrow[(Electfield[x, y, aplus, bplus, q] +
     Electfield[x, y, aminus, bminus, -q]),
   Tail → {x, y}, ScaleFactor → Normalize, HeadSize -> 0.3],
  {x, -3.1, 2.9, 0.5}, {y, -3.1, 2.9, 0.5}];

Show[singularitysource, singularitysink, dipoleplot, Axes → True,
 AxesLabel → {"x", "y"}, PlotLabel -> "Dipole electric field"];
```



The dipole electric field pictured above is given by the sum of the individual electric fields:

```
Clear[x, y, a, b, q, Electfield, m, n]
Electfield[x_, y_, a_, b_, q_] = q {x - a, y - b} / ((x - a)^2 + (y - b)^2);
{aplus, bplus} = {0.25, 0};
{aminus, bminus} = {-0.25, 0};
Clear[Dipole]
Dipole[x_, y_] =
 Electfield[x, y, aplus, bplus, q] + Electfield[x, y, aminus, bminus, -q]
```

$$\left\{\frac{q\,(-0.25+x)}{(-0.25+x)^2+y^2}-\frac{q\,(0.25+x)}{(0.25+x)^2+y^2},\ \frac{q\,y}{(-0.25+x)^2+y^2}-\frac{q\,y}{(0.25+x)^2+y^2}\right\}$$

Report on:
→ divDipole[x, y]
→ rotDipole[x, y]
→ The flow of Dipole[x, y] across the square with corners at {−1, −1}, {1, −1}, {1, 1} and {−1, 1}.
→ The flow of Dipole[x, y] across any closed curve without loops enclosing both of the singularites at {aplus, bplus} and {aminus, bminus}.

□G.5.e.i) Lots of point charges

You can build intricate electric fields by taking n different points
$\{a_1, b_1\}, \{a_2, b_2\}, \ldots, \{a_n, b_n\}$
and placing point charges of strength
q_1 at $\{a_1, b_1\}$,
q_2 at $\{a_2, b_2\}$,
… , and
q_n at $\{a_n, b_n\}$.
The strengths $q_1, q_2, \ldots$, and q_n can be positive (singularity sources) or negative (singularity sinks).
The combined electric field resulting from this placement of point charges is the sum of the individual electric fields resulting from the point charges of strength q_i at $\{a_i, b_i\}$.
As you can imagine, the formula for the resulting vector field can be complicated as all get-out. Nevertheless you can explain why this complicated electric field has no sources or sinks other than at the singularities at
$\{a_1, b_1\}, \{a_2, b_2\}, \ldots, \{a_n, b_n\}$.
Give your explanation.

□G.5.e.ii) Gauss's law in physics

Gauss's law in physics says that if you take n different points
$\{a_1, b_1\}, \{a_2, b_2\}, \ldots, \{a_n, b_n\}$
inside a closed curve C and you place electrical charges of strength
q_1 at $\{a_1, b_1\}$,
q_2 at $\{a_2, b_2\}$,
… , and
q_n at $\{a_n, b_n\}$,
then the flux (= flow) of the resulting electric field across C is simply
$2\pi(q_1 + q_2 + q_3 + \ldots + q_n)$.
Explain where Gauss's law comes from.

□Tip:

Look at this:

```
Clear[ElectricField, a, b, k, m, n, q, x, y]
{m_k[x_, y_], n_k[x_, y_]} = q_k {x - a_k, y - b_k} / ((x - a_k)^2 + (y - b_k)^2);
ElectricField_k[x_, y_] = {m_k[x, y], n_k[x, y]}
```

$$\left\{\frac{(x-a_k)\,q_k}{(x-a_k)^2+(y-b_k)^2},\ \frac{(y-b_k)\,q_k}{(x-a_k)^2+(y-b_k)^2}\right\}$$

and look at the flow-across measurement

$$\oint_{C_i} -n[x, y]\,dx + m[x, y]\,dy$$

where C_i is a small circle centered at $\{a_i, b_i\}$:

```
Clear[r, t]
{x[t_], y[t_]} = {a_k, b_k} + r_k {Cos[t], Sin[t]};
Integrate[(-n_k[x[t], y[t]] x'[t] + m_k[x[t], y[t]] y'[t]), {t, 0, 2π}]
```

$2\pi q_k$

G.6) The Laplacian $\frac{\partial^2 f[x,y]}{\partial x^2} + \frac{\partial^2 f[x,y]}{\partial y^2}$ and steady-state heat*

Here is a cleared function and its gradient field :

```
Clear[x, y, z, f, gradf, m, n, Field, divField, rotField]
gradf[x_, y_] = {D[f[x, y], x], D[f[x, y], y]};
{m[x_, y_], n[x_, y_]} = gradf[x, y];
Field[x_, y_] = {m[x, y], n[x, y]}
```

$\{f^{(1,0)}[x, y],\ f^{(0,1)}[x, y]\}$

Here is the rotation of this gradient field:

```
rotField[x_, y_] = D[n[x, y], x] - D[m[x, y], y]
```

0

□G.6.a.i)

Is it true that all gradient fields are irrotational (have no swirls)?

□G.6.a.ii)

Here is the divergence of this gradient field:

```
divField[x_, y_] = D[m[x, y], x] + D[n[x, y], y]
```

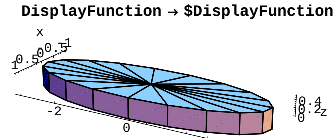
$f^{(0,2)}[x, y] + f^{(2,0)}[x, y]$

Folks like to call
$$\frac{\partial^2 f[x,y]}{\partial x^2} + \frac{\partial^2 f[x,y]}{\partial y^2}$$
= D[f[x, y], {x, 2}] + D[f[x, y], {y, 2}]
= divField[x, y]
by the name Laplacian of f[x, y].
How do you use the sign of
$$\frac{\partial^2 f[x,y]}{\partial x^2} + \frac{\partial^2 f[x,y]}{\partial y^2}$$
to determine whether a point {x, y} is a source or a sink of the gradient field of f[x, y]?
How do you check the Laplacian
$$\frac{\partial^2 f[x,y]}{\partial x^2} + \frac{\partial^2 f[x,y]}{\partial y^2}$$
to learn whether a given gradient field
Field[x, y] = gradf[x, y]
is free of sources and sinks at points other than singularities?

□G.6.b.i)

Here's a concrete slab:

```
Clear[x, y, t, r, z]
{x[t_], y[t_]} = {Cos[t], 3 Sin[t]};
tlow = 0;
thigh = 2 π;
zlow = 0;
zhigh = 0.5;
bottom =
 ParametricPlot3D[{r x[t], r y[t], 0}, {r, 0, 1}, {t, tlow, thigh},
  PlotPoints → {2, Automatic}, DisplayFunction → Identity];
top =
 ParametricPlot3D[{r x[t], r y[t], 0.5}, {r, 0, 1}, {t, tlow, thigh},
  PlotPoints → {2, Automatic}, DisplayFunction → Identity];
side =
 ParametricPlot3D[{x[t], y[t], z}, {z, zlow, zhigh}, {t, tlow, thigh},
  PlotPoints → {2, Automatic}, DisplayFunction → Identity];

Show[top, bottom, side, AxesLabel → {"x", "y", "z"},
 ViewPoint → CMView, BoxRatios → Automatic, Boxed → False,
 DisplayFunction → $DisplayFunction];
```



You heat this slab any way you like, and then you apply perfect insulation to the top and bottom. At the same time, you apply heating pads to the side to keep the temperature along each vertical segment through points

{x[t], y[t], z}(0 < z < 0.5)

at the same temperature, but your heating pad allows for the temperature to vary as t varies.
In other words, different vertical line segments on the sides are kept at possibly different temperatures, but the temperature doesn't vary along any one vertical line segment on the side.
You leave for a long time and wait for the temperature inside the slab to settle into its steady state condition. After you come back, the temperature at a point {x, y, z} inside the slab will not vary as z varies but the temperature at a point {x, y, z} probably will vary as x and y vary.
Put

temp[x, y] = steady state temperature at {x, y, z}.

In the steady state, no point inside the slab and not on the side or the top or the bottom can be a source of new heat flow or a sink for old heat.
Why does this tell you that

$$\frac{\partial^2 \text{temp}[x,y]}{\partial x^2} + \frac{\partial^2 \text{temp}[x,y]}{\partial y^2} = 0$$

at each point {x, y} with {x, y, z} inside but not on the outer surface of the slab?

□G.6.b.ii) Hot spots

Assume $\{x_0, y_0\}$ is a hot spot inside the slab in the sense that

$\text{temp}[x_0, y_0] > \text{temp}[x, y]$ for nearby {x, y}.

→ Why do you think that the net flow of the gradient field of temp[x, y] across small circles centered at $\{x_0, y_0\}$ must be from outside to inside?
→ What does the fact that

$$\frac{\partial^2 \text{temp}[x,y]}{\partial x^2} + \frac{\partial^2 \text{temp}[x,y]}{\partial y^2} = 0$$

tell you the flow of the gradient field of temp[x, y] across these same small circles actually is?

Say how your responses to the last two questions give you an explanation of why no such hot spot can exist.

□Tip:

Remember gradtemp[x, y] points in the direction of greatest instantaneous increase of temp[x, y].

□G.6.b.iii) Cold spots

Can there be $\{x_0, y_0\}$, a cold spot, inside the slab in the sense that

$\text{temp}[x_0, y_0] < \text{temp}[x, y]$ for nearby {x, y}?

□G.6.b.iv)

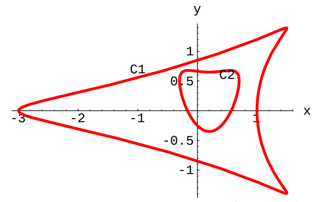
How do you know that both the hottest and coldest spots of the slab will be found on the sides?

G.7) Calculating $\oint_C$ m[x, y] dx + n[x, y] dy in the presence of singularities

□G.7.a)

Here are two curves C_1 and C_2:

```
Clear[x1, y1, x2, y2, t]
tlow = 0;
thigh = 2 π;
{x1[t_], y1[t_]} = 2 {Cos[t] - 1/2 Cos[2 t], 0.6 Sin[t] + 0.2 Sin[2 t]};
{x2[t_], y2[t_]} = {0.2, 0.5} + 0.5 {Cos[t], Sin[t] (1 - 0.7 Sin[t])};

curves =
 ParametricPlot[{{x1[t], y1[t]}, {x2[t], y2[t]}}, {t, tlow, thigh},
  PlotStyle → {{Red, Thickness[0.01]}, {Red, Thickness[0.01]}},
  AxesLabel → {"x", "y"}, PlotRange → All,
  Epilog → {Text["C1", {-1, 0.7}], Text["C2", {0.5, 0.6}]}];
```



You can be sure that the flow of a vector field

Field[x, y] = {m[x, y], n[x, y]}

across C_1 is the same as the flow of Field[x, y] across C_2 if there are no singularities between C_1 and C_2, and

divField[x, y] = D[m[x, y], x] + D[n[x, y], y] = 0

at all points {x, y} between C_1 and C_2, because this condition guarantees that no extra fluid comes into the flow and no old fluid goes out of the flow between C_1 and C_2.
How is this fact related to the fact that, given functions m[x, y] and n[x, y], you can be sure that

$$\oint_{C_1} m[x, y]\, dx + n[x, y]\, dy$$
$$= \oint_{C_2} m[x, y]\, dx + n[x, y]\, dy$$

if neither m[x, y] nor n[x, y] has singularities between C_1 and C_2, and

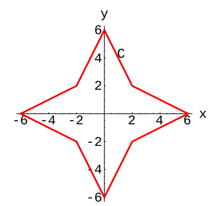
D[n[x, y], x] – D[m[x, y], y] = 0

at all points {x, y} between C_1 and C_2?

□G.7.b.i)

Here's a curve:

```
Cplot = Graphics[{Thickness[0.01], Red, Line[{{6, 0}, {2, 2}, {0, 6},
     {-2, 2}, {-6, 0}, {-2, -2}, {0, -6}, {2, -2}, {6, 0}}]}];
label = Graphics[Text["C", {1.2, 4.3}]];

Show[Cplot, label, Axes → True, AxesLabel → {"x", "y"},
 AspectRatio → Automatic];
```



Go with:

```
Clear[m, n, x, y]
{m[x_, y_], n[x_, y_]} = {-y/(x^2 + y^2), x/(x^2 + y^2)};
```

Note the singularity at {0, 0} and look at:

```
Together[D[n[x, y], x] - D[m[x, y], y]]
0
```

Go with the curve C plotted above.
Does the fact that

D[n[x, y], x] – D[m[x, y], y] = 0

tell you that you can calculate

$$\oint_C m[x, y]\, dx + n[x, y]\, dy$$

by calculating

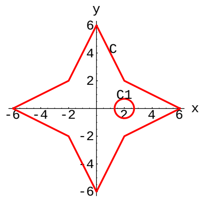
$$\oint_{C_1} m[x, y]\, dx + n[x, y]\, dy$$

for a well-chosen substitute closed curve C_1?

□G.7.b.ii)

Continue to go with the C, the m[x, y], and the n[x, y] as set above.
If you take C_1 to be the circle of radius 0.7 centered at {2, 0} like this:

```
Clear[x1, y1, t]
{x1[t_], y1[t_]} = {2, 0} + 0.7 {Cos[t], Sin[t]};
littlecircle = ParametricPlot[{x1[t], y1[t]}, {t, 0, 2 π},
   PlotStyle → {{Thickness[0.01], Red}}, DisplayFunction → Identity];
extralabel = Graphics[Text["C1", {2, 1}]];

Show[Cplot, littlecircle, label, extralabel, PlotRange → All,
 Axes → True, AxesLabel → {"x", "y"}, AspectRatio → Automatic,
 DisplayFunction → $DisplayFunction];
```



and if you say that

$\oint_C m[x, y]\, dx + n[x, y]\, dy$
$= \oint_{C_1} m[x, y]\, dx + n[x, y]\, dy$,

then you would be wrong.
Why would you be wrong?

□G.7.b.iii)

Continue to go with the C, the m[x, y], and the n[x, y] as set above.
Only a dweeb would calculate

$\oint_C m[x, y]\, dx + n[x, y]\, dy$

by parameterizing C. Take a good choice of a little circle C_1, and calculate

$\oint_C m[x, y]\, dx + n[x, y]\, dy$

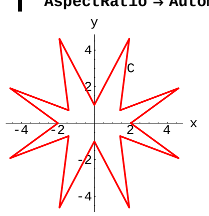
correctly by calculating

$\oint_{C_1} m[x, y]\, dx + n[x, y]\, dy$.

□G.7.c.i)

Here's a new curve C:

```
Cplot = Graphics[{Thickness[0.01], Red, Line[{{2 Cos[0 π/8], Sin[0 π/8]},
    5 {Cos[π/8], Sin[π/8]}, {2 Cos[2 π/8], Sin[2 π/8]},
    5 {Cos[3 π/8], Sin[3 π/8]}, {2 Cos[4 π/8], Sin[4 π/8]},
    5 {Cos[5 π/8], Sin[5 π/8]}, {2 Cos[6 π/8], Sin[6 π/8]},
    5 {Cos[7 π/8], Sin[7 π/8]}, {2 Cos[8 π/8], Sin[8 π/8]},
    5 {Cos[9 π/8], Sin[9 π/8]}, {2 Cos[10 π/8], Sin[10 π/8]},
    5 {Cos[11 π/8], Sin[11 π/8]}, {2 Cos[12 π/8], Sin[12 π/8]},
    5 {Cos[13 π/8], Sin[13 π/8]}, {2 Cos[14 π/8], Sin[14 π/8]},
    5 {Cos[15 π/8], Sin[15 π/8]}, {2 Cos[16 π/8], Sin[16 π/8]}}]}];
label = Graphics[Text["C", {2, 3}]];
Show[Cplot, label, Axes → True, AxesLabel → {"x", "y"},
 AspectRatio → Automatic];
```



This time go with:

```
Clear[m, n, x, y]
{m[x_, y_], n[x_, y_]} = {-y/(x^2 + y^2) + 3 (y + 4)/(x^2 + (y + 4)^2), x/(x^2 + y^2) - 3 x/(x^2 + (y + 4)^2)}
```

$\left\{-\frac{y}{x^2+y^2} + \frac{3\,(4+y)}{x^2+(4+y)^2},\ \frac{x}{x^2+y^2} - \frac{3\,x}{x^2+(4+y)^2}\right\}$

Check:

```
Together[D[n[x, y], x] - D[m[x, y], y]]
```

0

Go with the curve C plotted above and calculate

$\oint_C m[x, y]\, dx + n[x, y]\, dy$

by integrating over a substitute curve.

□Tip:

You might want to calculate

$\oint_C m[x, y]\, dx + n[x, y]\, dy$

by calculating

$\oint_C m_1[x, y]\, dx + n_1[x, y]\, dy +$
$\oint_C m_2[x, y]\, dx + n_2[x, y]\, dy$

where:

```
{m1[x_, y_], n1[x_, y_]} = {-y/(x^2 + y^2), x/(x^2 + y^2)}
```

$\left\{-\frac{y}{x^2+y^2},\ \frac{x}{x^2+y^2}\right\}$

and

```
{m2[x_, y_], n2[x_, y_]} = {3 (4 + y)/(x^2 + (4 + y)^2), -3 x/(x^2 + (4 + y)^2)}
```

$\left\{\frac{3\,(4+y)}{x^2+(4+y)^2},\ -\frac{3\,x}{x^2+(4+y)^2}\right\}$

Check:

```
{m[x, y], n[x, y]} == {m1[x, y], n1[x, y]} + {m2[x, y], n2[x, y]}
```

True

□G.7.d) Path dependence in the presence of singularities

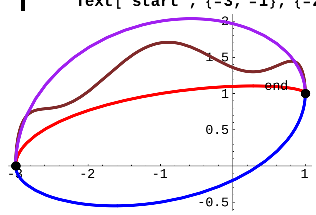
Here are two functions m[x, y] and n[x, y]:

```
Clear[m, n, x, y]
{m[x_, y_], n[x_, y_]} = {-y/(x^2 + y^2), x/(x^2 + y^2)}
```

$\left\{-\frac{y}{x^2+y^2},\ \frac{x}{x^2+y^2}\right\}$

Here are four curves all parameterized to start at {−3, 0} and end at {1, 0}:

```
Clear[x1, y1, x2, y2, x3, y3, x4, y4, t]
tlow = 0;
thigh = π;
{x1[t_], y1[t_]} = {-1, 0} + {2 Cos[π + t], Sin[t]/2 + t/π};
{x2[t_], y2[t_]} = {-1, 0} + {2 Cos[π + t], Sin[t] + 1/5 Sin[5 t] + t/π};
{x3[t_], y3[t_]} = {-1, 0} + {2 Cos[π + t], 1.5 Sin[t] + t/π};
{x4[t_], y4[t_]} = {-1, 0} + {2 Cos[π + t], -Sin[t] + t/π};
curveplots = ParametricPlot[{{x1[t], y1[t]}, {x2[t], y2[t]},
    {x3[t], y3[t]}, {x4[t], y4[t]}}, {t, tlow, thigh},
   PlotStyle → {{Thickness[0.01], Red}, {Thickness[0.01], Brown},
     {Thickness[0.01], Purple}, {Thickness[0.01], Blue}},
   Epilog → {{PointSize[0.03], Point[{-3, 0}]},
     {PointSize[0.03], Point[{1, 1}]},
     Text["start", {-3, -1}, {-2, -1}], Text["end", {1, 1}, {2, -1}]}];
```



Look at these calculations of

$\oint_{C_1} m[x, y]\, dx + n[x, y]\, dy$,
$\oint_{C_2} m[x, y]\, dx + n[x, y]\, dy$,
$\oint_{C_3} m[x, y]\, dx + n[x, y]\, dy$, and
$\oint_{C_4} m[x, y]\, dx + n[x, y]\, dy$:

```
C1pathintegral = NIntegrate[
  m[x1[t], y1[t]] x1'[t] + n[x1[t], y1[t]] y1'[t], {t, tlow, thigh}]
```

-2.35619

```
C2pathintegral = NIntegrate[
  m[x2[t], y2[t]] x2'[t] + n[x2[t], y2[t]] y2'[t], {t, tlow, thigh}]
```

-2.35619

```
C3pathintegral = NIntegrate[
  m[x3[t], y3[t]] x3'[t] + n[x3[t], y3[t]] y3'[t], {t, tlow, thigh}]
```

-2.35619

```
C4pathintegral = NIntegrate[
  m[x4[t], y4[t]] x4'[t] + n[x4[t], y4[t]] y4'[t], {t, tlow, thigh}]
```

3.92699

Compare them:

```
{C1pathintegral, C2pathintegral, C3pathintegral, C4pathintegral}
```

{-2.35619, -2.35619, -2.35619, 3.92699}

Was this an accident?
If not, try to explain why

$\oint_{C_1} m[x, y]\, dx + n[x, y]\, dy$
$= \oint_{C_2} m[x, y]\, dx + n[x, y]\, dy$

$= \oint_{C_3} m[x, y]\, dx + n[x, y]\, dy,$

but there was no reason to expect

$\oint_{C_4} m[x, y]\, dx + n[x, y]\, dy$

to agree with the others.

□ Tip:

You might begin your answer by looking at:

```
Together[D[n[x, y], x] - D[m[x, y], y]]
0
```

By the way, the vector field {m[x, y], n[x, y]} shows what can happen when a vector field passes the second part of the gradient test without passing the first part.

In fact, if {m[x, y], n[x, y]} were a genuine gradient field, then all four integrals would have calculated out to the same value.

G.8) Water and electricity*

□ G.8.a) Electricity

As you probably know, vector fields

Field[x, y] = {m[x, y], n[x, y]}

have an important interpretation in the realm of electricity.

In the electrical interpretation, the instantaneous voltage drop as you leave {x, y} in the direction of a unit vector U is given by

U . Field[x, y].

This quantity can be positive or negative.

As a result, the integral

$\oint_C \text{unittan} . \text{Field}\, ds = \oint_C m[x, y]\, dx + n[x, y]\, dy$

is a measure of acculumated voltage drop in the circuit given by one trip around a closed curve C that does not pass through a singularity.

But the voltage at your starting point on C is the same as the voltage at your ending point (because the ending point is the starting point).

So for this electrical interpretation,

$\oint_C \text{unittan} . \text{Field}\, ds = \oint_C m[x, y]\, dx + n[x, y]\, dy = 0$

for any closed curve C not passing through a singularity.

In other words, the net flow of

Field[x, y] = {m[x, y], n[x, y]}

along any closed curve not passing through a singularity must be 0.

Vector fields that are suitable for this electrical interpretation are often called irrotational fields.

□ G.8.a.i)

If Field[x, y] = {m[x, y], n[x, y]} has no singularities, then what condition on rotField[x, y] signals that Field[x, y] is suitable for the electrical interpretation?

□ G.8.a.ii)

If Field[x, y] is the gradient field of some function f[x, y], then is there anything to disqualify Field[x, y] from being suitable for the electric interpretation?

□ G.8.a.iii)

Here is a vector field:

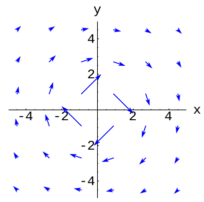
```
Clear[Field1, m1, n1, x, y]
{m1[x_, y_], n1[x_, y_]} = {y, -x}/(x^2 + y^2);
Field1[x_, y_] = {m1[x, y], n1[x, y]}
{y/(x^2 + y^2), -x/(x^2 + y^2)}
```

Here is rotField1[x, y]:

```
rotField1[x, y] = Together[D[n1[x, y], x] - D[m1[x, y], y]]
0
```

Look at a plot of this vector field.

```
scalefactor = 2;
field1plot = Table[Arrow[Field1[x, y], Tail → {x, y},
    VectorColor → Blue, ScaleFactor → scalefactor, HeadSize → 0.4],
  {x, -4.5, 4.5, 9/5}, {y, -4.5, 4.5, 9/5}];

Show[field1plot, Axes → True, AxesLabel → {"x", "y"}];
```



In spite of the fact that

rotField1[x, y] = 0

at all points {x, y} other than the singularity at {0, 0}, this vector field cannot be an electric field. Why?

□ Tip:

Look at

$\oint_C m1[x, y]\, dx + n1[x, y]\, dy$

where C is the circle of radius 1 centered at {0, 0}.

□ G.8.a.iv)

Here's another vector field:

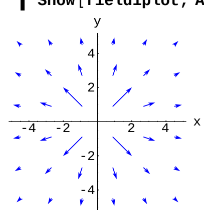
```
Clear[Field2, m2, n2, x, y]
{m2[x_, y_], n2[x_, y_]} = {x, y}/(x^2 + y^2);
Field2[x_, y_] = {m2[x, y], n2[x, y]}
{x/(x^2 + y^2), y/(x^2 + y^2)}
```

Here is rotField2[x, y]:

```
rotField2[x, y] = Together[D[n2[x, y], x] - D[m2[x, y], y]]
0
```

Look at a plot of this vector field.

```
scalefactor = 2;
field1plot = Table[Arrow[Field2[x, y], Tail → {x, y},
    VectorColor → Blue, ScaleFactor → scalefactor, HeadSize → 0.4],
  {x, -4.5, 4.5, 9/5}, {y, -4.5, 4.5, 9/5}];
Show[field1plot, Axes → True, AxesLabel → {"x", "y"}];
```



In spite of the singularity at {0, 0}, how do you know that this vector field qualifies as a model of electricity?

□ Tip:

Because

rotField2[x, y] = D[n2[x, y], x] − D[m2[x, y], y] = 0

except at the singularity at {0, 0}, you know that if C is a closed curve with {0, 0} not inside C, then

$\oint_C m2[x, y]\, dx + n2[x, y]\, dy = 0.$

What is the value of

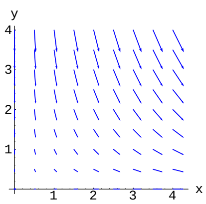
$$\oint_C m2[x, y]\, dx + n2[x, y]\, dy$$

for a closed curve C with {0, 0} inside C?

G.8.b.i) Water

For an incompressible fluid flow, like water flow, the net flow across any closed curve C without loops must be 0 unless there are sources (spigots) or sinks (drains) inside C.
The upshot:
If the flow of Field[x, y] is to model sinkless and sourceless water flow, then Field[x, y] must have no singularities and
divField[x, y] = 0 at all points {x, y}.
Also for water flow, the net flow of Field[x, y] along any closed curve C has to be 0, because when you go around a closed curve one time, you have the same amount of water at the end of the trip as you had at the beginning. This means that if the flow of Field[x, y] is to model sinkless and sourceless water flow, then
divField[x, y] = 0 and rotField[x, y] = 0
at all points {x, y}.
For these reasons, fluid dynamics professionals agree that any vector field Field[x, y] with no singularities and with
divField[x, y] = rotField[x, y] = 0
at all points {x, y} is a model for sinkless and sourceless water flow.
Look at a plot of the vector field
Field[x, y] = {0.07 x, −0.14 y}:

```
Clear[Field, m, n, x, y];
{m[x_, y_], n[x_, y_]} = {0.07 x, -0.14 y};
Field[x_, y_] = {m[x, y], n[x, y]};
fieldplot = Table[Arrow[Field[x, y], Tail → {x, y}, VectorColor → Blue],
  {x, 0, 4, 0.5}, {y, 0, 4, 0.5}];

Show[fieldplot, AxesLabel → {"x", "y"}, Axes → True,
 DisplayFunction → $DisplayFunction];
```



This looks like water flowing around a corner.
Check whether this flow can be sourceless and sinkless water flow around a corner by looking at
divField[x, y] and rotField[x, y]:

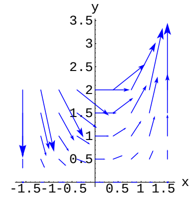
```
Clear[divField, rotField]
divField[x_, y_] = D[m[x, y], x] + D[n[x, y], y]
-0.07
rotField[x_, y_] = D[n[x, y], x] - D[m[x, y], y]
0
```

Drats.
Every point {x, y} is a sink for this vector field.
Consequently this vector field is not a model for water flow around a corner without sources and sinks.
Your job is to set a constant k so that
Field[x, y] = {0.07 x, −k y}
is a model for water flow around a corner.
Once you have your k, plot your vector field as above.

G.8.b.ii)

Look at this vector field:

```
Clear[Field, m, n, x, y];
{m[x_, y_], n[x_, y_]} = {0.2 Cos[x] Cosh[y], 0.4 Sin[x] Sinh[y]};
Field[x_, y_] = {m[x, y], n[x, y]};
fieldplot = Table[Arrow[Field[x, y], Tail → {x, y}, VectorColor → Blue],
  {x, -π/2, π/2, π/8}, {y, 0, 2, 0.5}];

Show[fieldplot, AxesLabel → {"x", "y"}, Axes → True,
 DisplayFunction → $DisplayFunction];
```



This looks like water flowing down on the left toward the x-axis, and then flowing up on the right.
Check whether this flow can be a model for sourceless and sinkless water flow down and up by looking at
divField[x, y] and rotField[x, y].
If it checks out, do nothing more; otherwise come up with a constant k so that
Field[x, y] = {0.2 Cos[x] Cosh[y], k Sin[x] Sinh[y]}
is a model for sourceless and sinkless water flow down and up.
Plot your model as above.

G.8.b.iii) Gradient fields

When you have a vector field
Field[x, y] = {m[x, y], n[x, y]}
with no singularities, and
rotField[x, y] = D[n[x, y], x] − D[m[x, y], y] = 0
at all points {x, y}, then Field[x, y] passes the gradient test.
The upshot:
All vector fields without singularities that model sinkless and sourceless water flow must be gradient fields.
But not all gradient fields can model sourceless and sinkless water flow.
Here is a cleared gradient field:

```
Clear[x, y, m, n, f, Field, gradf]
gradf[x_, y_] = {D[f[x, y], x], D[f[x, y], y]};
{m[x_, y_], n[x_, y_]} = gradf[x, y];
Field[x_, y_] = {m[x, y], n[x, y]}
{f^(1,0)[x, y], f^(0,1)[x, y]}
```

As predicted, rotField[x, y] = 0 at all points {x, y}:

```
D[n[x, y], x] - D[m[x, y], y]
0
```

What condition on
D[f[x, y], {x, 2}] + D[f[x, y], {y, 2}]
$= \partial^2 \frac{f[x,y]}{\partial x^2} + \partial^2 \frac{f[x,y]}{\partial y^2}$

```
D[f[x, y], {x, 2}] + D[f[x, y], {y, 2}]
f^(0,2)[x, y] + f^(2,0)[x, y]
```

guarantees that divField[x, y] = 0 at each point {x, y}?
The Laplacian pops up again.

G.8.b.iv)

Run the following cells and use the result of each to determine which of the following functions has a gradient field suitable for modeling sourceless and sinkless water flow:

```
Clear[x, y, f]
f[x_, y_] = Sin[x] Sinh[y];
D[f[x, y], {x, 2}] + D[f[x, y], {y, 2}]
0
```

```
Clear[x, y, f]
f[x_, y_] = Cos[x] Sin[y];
D[f[x, y], {x, 2}] + D[f[x, y], {y, 2}]
-2 Cos[x] Sin[y]
```

```
Clear[x, y, f]
f[x_, y_] = x^2 + y^2;
D[f[x, y], {x, 2}] + D[f[x, y], {y, 2}]
4
```

```
Clear[x, y, f]
f[x_, y_] = x^2 - y^2;
D[f[x, y], {x, 2}] + D[f[x, y], {y, 2}]
0
```

G.9) Is parallel flow always irrotational?

Some of the ideas for this problem came from Gilbert Strang's book
Calculus, Wellesley-Cambridge Press, 1991.
For a book printed on paper, this one is not bad at all.

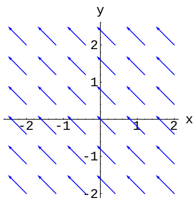
□G.9.a)

Here is a 2D vector field consisting of equal parallel vectors:

```
Clear[Field1, m1, n1, x, y]
{m1[x_, y_], n1[x_, y_]} = 0.5 {-1, 1};
Field1[x_, y_] = {m1[x, y], n1[x, y]}
{-0.5, 0.5}
```

Take a look:

```
field1plot = Table[Arrow[Field1[x, y], Tail → {x, y},
    VectorColor → Blue], {x, -2, 2, 4/5}, {y, -2, 2, 4/5}];

Show[field1plot, Axes → True, AxesLabel → {"x", "y"}];
```



A calming, steady flow.
Check its rotation:

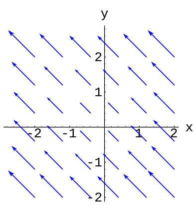
```
D[n1[x, y], x] - D[m1[x, y], y]
0
```

Just as the plot suggests, this field has no swirls at all.
Here's another vector field flowing in the same direction.

```
Clear[Field2, m2, n2, x, y]
{m2[x_, y_], n2[x_, y_]} = 0.2 (√(x² + y²) + 1) {-1, 1};
Field2[x_, y_] = {m2[x, y], n2[x, y]}
{-0.2 (1 + √(x² + y²)), 0.2 (1 + √(x² + y²))}
```

Take a look:

```
field2plot = Table[Arrow[Field2[x, y], Tail → {x, y},
    VectorColor → Blue], {x, -2, 2, 4/5}, {y, -2, 2, 4/5}];

Show[field2plot, Axes → True, AxesLabel → {"x", "y"}];
```



Another calming, steady parallel flow.
The farther {x, y} is from {0, 0}, the faster the flow of Field2[x, y].
Both Field1[x, y] and Field2[x, y] represent parallel flows.
Field1[x, y] represents flow at a steady speed, but the flow represented by Field2[x, y] is of variable speed.
Is this Field2[x, y] also rotation-free?
If not, plot some sample points at which Field2[x, y] is swirling in the counterclockwise way.

□G.9.b.i)

One way to make a non-constant parallel flow in the direction of {−1, 1} that is free of all rotation is to take any nonnegative function g[x] and put:

```
Clear[Field, m, n, g, x, y]
{m[x_, y_], n[x_, y_]} = g[x - y] {-1, 1};
Field[x_, y_] = {m[x, y], n[x, y]}
{-g[x - y], g[x - y]}
```

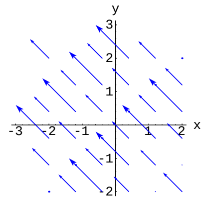
Check rotField[x, y]:

```
D[n[x, y], x] - D[m[x, y], y]
0
```

Here's what you get when you go with

$g[x] = \mathrm{Sin}[x]^2$:

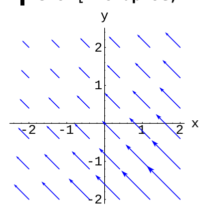
```
Clear[Field, m, n, g, x, y]
g[x_] = Sin[x]^2;
{m[x_, y_], n[x_, y_]} = g[x - y] {-1, 1};
Field[x_, y_] = {m[x, y], n[x, y]};
fieldplot = Table[Arrow[Field[x, y], Tail → {x, y}, VectorColor → Blue],
    {x, -2, 2, 4/5}, {y, -2, 2, 4/5}];

Show[fieldplot, Axes → True, AxesLabel → {"x", "y"}];
```



This flow is guaranteed to be free of rotation at all points because
rotField[x, y] = 0 at all points {x, y},
and because there are no singularities.
Here's what you get when you go with

$g[x] = 0.4\, e^{0.2\, x}$:

```
Clear[Field, m, n, g, x, y]
g[x_] = 0.4 E^(0.2 x);
{m[x_, y_], n[x_, y_]} = g[x - y] {-1, 1};
Field[x_, y_] = {m[x, y], n[x, y]};
fieldplot = Table[Arrow[Field[x, y], Tail → {x, y}, VectorColor → Blue],
    {x, -2, 2, 4/5}, {y, -2, 2, 4/5}];

Show[fieldplot, Axes → True, AxesLabel → {"x", "y"}];
```



This flow is guaranteed to be free of rotation at all points because
rotField[x, y] = 0 at all points {x, y},
and there are no singularities.
Given a and b with a ≠ 0, your job is to say how to set c in terms of a and b so that when you go with a nonconstant nonnegative function g[x], then the vector field

Field[x, y] = {m[x, y], n[x, y]} = g[x + c y] {a, b}

gives you rotation-free, nonconstant, parallel flow on the direction of {a, b}.
Show off your work with a plot or two.

□G.9.b.ii)

Given a and b with a = 0 and b ≠ 0, is it possible to set c in terms of b so that when you go with a nonconstant, nonnegative function g[x], the vector field

Field[x, y] = {m[x, y], n[x, y]}
= g[x + c y] {a, b}
= g[x + c y] {0, b}

gives you rotation-free, nonconstant, parallel flow on the direction of {a, b}?
Why not?

□Tip:

Saying g[x] is nonconstant tells you that there are points at which g'[x] ≠ 0.

□G.9.c)

Given definite numbers a, b, c, and d with {a, b} ≠ {0, 0}, agree or disagree with the following statement and explain why:
No matter what nonconstant function g[x] you go with, when you make a nonconstant, parallel flow in the direction of {a, b} by setting

Field[x, y] = {m[x, y], n[x, y]} = g[d x + c y] {a, b},

then either

→ Field[x, y] is not free of sources or sinks,

or

→ Field[x, y] is not rotation free.

G.10) Spin fields

If you did the problem on spin fields in the last lesson, you can skip the first part of this problem.

Start with a vector field

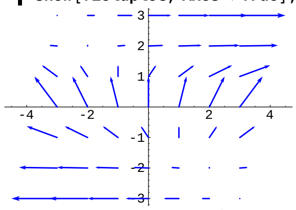
Field[x, y] = {m[x, y], n[x, y]}

and make what some folks call the spin field

spinField[x, y] = {−n[x, y], m[x, y]}.

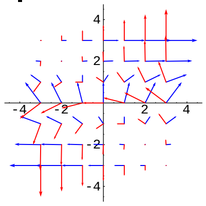
Here's a vector field:

```
Clear[x, y, m, n, Field, spinField]
{m[x_, y_], n[x_, y_]} = {(x + y)/4, E^(-y^2)};
Field[x_, y_] = {m[x, y], n[x, y]};
spinField[x_, y_] = {-n[x, y], m[x, y]};
fieldplot = Table[Arrow[Field[x, y],
   Tail → {x, y}, VectorColor → Blue], {x, -3, 3}, {y, -3, 3}];

Show[fieldplot, Axes → True];
```



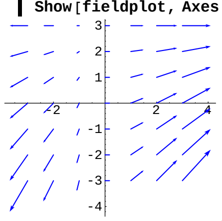
Here's the same vector field plotted together with its spin field:

```
spinfieldplot = Table[Arrow[spinField[x, y],
   Tail → {x, y}, VectorColor → Red], {x, -3, 3}, {y, -3, 3}];

Show[fieldplot, spinfieldplot, Axes → True];
```



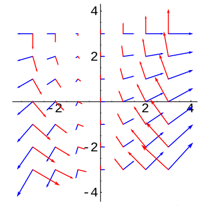
Again:

```
Clear[x, y, m, n, Field, spinField]
{m[x_, y_], n[x_, y_]} = {0.3 x + 0.2, 0.2 x (1 - y/3)};
Field[x_, y_] = {m[x, y], n[x, y]};
spinField[x_, y_] = {-n[x, y], m[x, y]};
fieldplot = Table[Arrow[Field[x, y], Tail → {x, y}, VectorColor → Blue],
  {x, -3, 3, 1}, {y, -3, 3, 1}];

Show[fieldplot, Axes → True];
```



Here's the same vector field plotted together with its spin field:

```
spinfieldplot = Table[Arrow[spinField[x, y],
   Tail → {x, y}, VectorColor → Red], {x, -3, 3, 1}, {y, -3, 3, 1}];

Show[fieldplot, spinfieldplot, Axes → True];
```



Run some more of these to get a feeling for the relationship between a vector field and its spin field.

□G.10.a)

Try to describe the geometric relationship between a vector field and its spin field.

□Tip:

Think rotation.

□G.10.b)

Here's a cleared vector field and its spin field:

```
Clear[x, y, m, mm, n, nn, Field, spinField]
Field[x_, y_] = {m[x, y], n[x, y]};
{mm[x_, y_], nn[x_, y_]} = {-n[x, y], m[x, y]};
spinField[x_, y_] = {mm[x, y], nn[x, y]};
{Field[x, y], spinField[x, y]}
{{m[x, y], n[x, y]}, {-n[x, y], m[x, y]}}
```

Here is divField[x, y]:

```
Clear[divField]
divField[x_, y_] = D[m[x, y], x] + D[n[x, y], y]
n^(0,1)[x, y] + m^(1,0)[x, y]
```

Here is rotSpinField[x, y]:

```
Clear[rotSpinField]
rotSpinField[x_, y_] = D[nn[x, y], x] - D[mm[x, y], y]
n^(0,1)[x, y] + m^(1,0)[x, y]
```

Here is rotField[x, y]:

```
Clear[rotField]
rotField[x_, y_] = D[n[x, y], x] - D[m[x, y], y]
-m^(0,1)[x, y] + n^(1,0)[x, y]
```

Here is divSpinField[x, y]:

```
Clear[divSpinField]
divSpinField[x_, y_] = D[mm[x, y], x] + D[nn[x, y], y]
m^(0,1)[x, y] - n^(1,0)[x, y]
```

Here they are together:

```
{{divField[x, y], rotSpinField[x, y]},
 {rotField[x, y], divSpinField[x, y]}}
{{n^(0,1)[x, y] + m^(1,0)[x, y], n^(0,1)[x, y] + m^(1,0)[x, y]},
 {-m^(0,1)[x, y] + n^(1,0)[x, y], m^(0,1)[x, y] - n^(1,0)[x, y]}}
```

Examine what you see above, and then answer the questions below:

→ If {x, y} is a source of spinField[x, y], then which way (clockwise or counterclockwise) is Field[x, y] swirling at {x, y}?

→ If {x, y} is a sink of spinField[x, y], then which way (clockwise or counterclockwise) is Field[x, y] swirling at {x, y}?

→ If {x, y} is a source of (Field[x, y]), then which way (clockwise or counterclockwise) is spinField[x, y] swirling at {x, y}?

→ If {x, y} is a sink of Field[x, y], then which way (clockwise or counterclockwise) is spinField[x, y] swirling at {x, y}

→ If Field[x, y] is a gradient field, then how do you know that spinField[x, y] has no sources or sinks?

→ If spinField[x, y] is a gradient field, then how do you know that Field[x, y] has no sources or sinks?

□G.10.c.i)

Here's a cleared vector field and its spin field:

```
Clear[x, y, m, mm, n, nn, Field, spinField]
Field[x_, y_] = {m[x, y], n[x, y]};
{mm[x_, y_], nn[x_, y_]} = {-n[x, y], m[x, y]};
spinField[x_, y_] = {mm[x, y], nn[x, y]};

{Field[x, y], spinField[x, y]}
{{m[x, y], n[x, y]}, {-n[x, y], m[x, y]}}
```

Here is divField[x, y]:

```
Clear[divField]
divField[x_, y_] = D[m[x, y], x] + D[n[x, y], y]
n^(0,1)[x, y] + m^(1,0)[x, y]
```

Here is rotSpinField[x, y]:

```
Clear[rotSpinField]
rotSpinField[x_, y_] = D[nn[x, y], x] - D[mm[x, y], y]
```

$n^{(0,1)}[x, y] + m^{(1,0)}[x, y]$

Suppose you know that
Field[x, y] = gradf[x, y]
for a function f[x, y], and
spinField[x, y] = gradg[x, y]
for another function g[x, y].
Also suppose you know that neither Field[x, y] nor spinField[x, y] has any singularities.
Why is it totally impossible to have a point $\{x_0, y_0\}$ such that
$f[x_0, y_0] > f[x, y]$ for $\{x, y\} \neq \{x_0, y_0\}$?

□ **Tip:**

First explain why the fact that spinField[x, y] is a gradient field guarantees that

divField[x, y] = 0 at all points {x, y},

so that the netflow of Field[x, y] across any given closed curve must be 0.

Then rule out the possibility of finding a point $\{x_0, y_0\}$ such that

$f[x_0, y_0] > f[x, y]$ for $\{x, y\} \neq \{x_0, y_0\}$

by centering a small circle C at $\{x_0, y_0\}$, and asking yourself what the net flow of the gradf[x, y] across C must be.

Remember gradf[x, y] points in the direction of greatest initial increase at {x, y}.

□ G.10.c.ii)

Suppose you know that
Field[x, y] = gradf[x, y]
for a function f[x, y] and
spinField[x, y] = gradg[x, y]
for another function g[x, y].
Try to explain why it must be that

$$\frac{\partial^2 f[x,y]}{\partial x^2} + \frac{\partial^2 f[x,y]}{\partial y^2} = 0$$

and

$$\frac{\partial^2 g[x,y]}{\partial x^2} + \frac{\partial^2 g[x,y]}{\partial y^2} = 0$$

at all points {x, y} other than singularities.

□ **Tip:**

```
Clear[x, y, m, mm, n, nn, f, g, gradf, gradg, Field, spinField]
Field[x_, y_] = {D[f[x, y], x], D[f[x, y], y]};
{m[x_, y_], n[x_, y_]} = Field[x, y]
```

$\{f^{(1,0)}[x, y], f^{(0,1)}[x, y]\}$

```
spinField[x_, y_] = {D[g[x, y], x], D[g[x, y], y]};
{mm[x_, y_], nn[x_, y_]} = spinField[x, y]
```

$\{g^{(1,0)}[x, y], g^{(0,1)}[x, y]\}$

Saying that spinField[x, y] is a gradient field tells you that

rotSpinField[x, y] = 0

at all points {x, y} other than singularities. If you did the part above, you'll be able to explain why this means that divField[x, y] = 0 at all points {x, y} other than singularities:

Now check out divField[x, y] = D[m[x, y], x] + D[n[x, y], y]:

```
D[m[x, y], x] + D[n[x, y], y]
```

$f^{(0,2)}[x, y] + f^{(2,0)}[x, y]$

Now you take over.