2D and 3D Measurements

VC.08 Transforming 3D Integrals
## Basics

B.1) 3D integrals

□**B.1.a)**

Here's a plot of the solid R whose top skin is the surface

$$z = 4 - x^2 - \frac{y^2}{4}$$

and whose bottom skin lies on the plane z = 1.

```
topskin = Plot3D[4 - x² - y²/4, {x, -1.8, 1.8}, {y, -3.5, 3.5},
  PlotRange → {0.9, 5}, ClipFill → None, DisplayFunction → Identity];
base =
 Graphics3D[Polygon[{{-2, -4, 1}, {2, -4, 1}, {2, 4, 1}, {-2, 4, 1}}]];

solid = Show[topskin, base, Boxed → False,
  BoxRatios → Automatic, AxesLabel → {"x", "y", "z"},
  ViewPoint → CMView, DisplayFunction → $DisplayFunction];
```
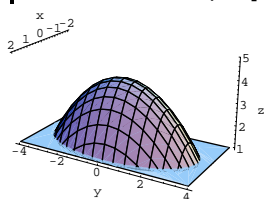

You could use a 2D integral to measure the volume of this solid. But just to get experience, use a 3D integral to measure the volume of this solid.

□**Answer:**

Just as

$$\iint_R dx\, dy$$

measures the area of a region R in two dimensions,

$$\iiint_R dx\, dy\, dz$$

measures the volume of a solid R in three dimensions.

You are free to choose any order you like for the variables so that

$$\iiint_R dx\, dy\, dz = \iiint_R dz\, dy\, dx = \iiint_R dz\, dx\, dy.$$

In the current situation, the top is

$$z = 4 - x^2 - \frac{y^2}{4}$$

and the bottom is

$$z = 1$$

With z so conveniently displayed, it's natural to integrate with respect to z first.

To set the limits for the first integral, fix x and y and enter the lowest and the highest values of z for this fixed x and y:

$$\int_{zlow[x,y]}^{zhigh[x,y]} dz.$$

```
Clear[zhigh, zlow, x, y, z]

{zlow[x_, y_], zhigh[x_, y_]} = {1, 4 - x² - y²/4};

firstintegral = ∫_{zlow[x,y]}^{zhigh[x,y]} dz
```
$$3 - x^2 - \frac{y^2}{4}$$

For the second integral, you've got your choice of whether to integrate with respect to y or x next. Go with x.

The main problem is deciding what limits to insert for xhigh[y] and xlow[y] in the second integral

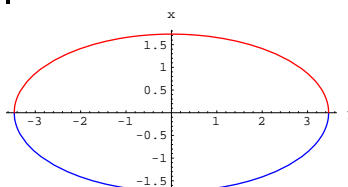$$\int_{xlow[y]}^{xhigh[y]} \int_{zlow[x,y]}^{zhigh[x,y]} dz\, dx.$$

The key to deciding how to set xhigh[y] and xlow[y] is to ask:

For a fixed y, what are the lowest and the highest values x can have no matter what z is?

The simplest way of answering this is to plot the shadow of this solid on xy-paper:

```
shadoweqn = Eliminate[{z == zhigh[x, y], z == zlow[x, y]}, z]
```
$$4\, x^2 == 12 - y^2$$
```
shadowcurves = Solve[shadoweqn, x]
```
$$\left\{\left\{x \to -\frac{1}{2}\sqrt{12-y^2}\right\}, \left\{x \to \frac{\sqrt{12-y^2}}{2}\right\}\right\}$$
```
Clear[xlow, xhigh]

xlow[y_] = -√(12 - y²)/2;

xhigh[y_] = √(12 - y²)/2;

shadowplot = Plot[{xlow[y], xhigh[y]}, {y, -2√3, 2√3},
  PlotStyle → {Blue, Red}, AxesLabel → {"y", "x"}];
```


The second integral $\int_{xlow[y]}^{xhigh[y]} \int_{zlow[x,y]}^{zhigh[x,y]} dz\, dx$ is:

```
secondintegral = ∫_{xlow[y]}^{xhigh[y]} firstintegral dx
```
$$\frac{1}{6}\,(12 - y^2)^{3/2}$$

For the third integral, you have no choice but to integrate with respect to y. The main problem is deciding what limits to insert for yhigh and ylow in the third integral

$$\int_{ylow}^{yhigh} \int_{xlow[y]}^{xhigh[y]} \int_{zlow[x,y]}^{zhigh[x,y]} dz\, dx\, dy.$$
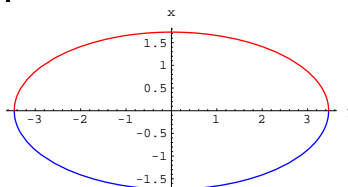
Note that ylow and yhigh are numbers, not functions.

The key to deciding how to set ylow and yhigh is to ask:

What are the lowest and the highest values y can have no matter what z and x are?

The simplest way of getting a hold on ylow and yhigh is to look at the shadow plot again:

```
Show[shadowplot];
```


In this problem, ylow and yhigh happen where xlow[y] and xhigh[y] meet:

```
ends = Solve[xhigh[y] == xlow[y]]
```
$$\{\{y \to -2\sqrt{3}\}, \{y \to 2\sqrt{3}\}\}$$
```
{ylow, yhigh} = {-2√3, 2√3}
```
$$\{-2\sqrt{3}, 2\sqrt{3}\}$$

The volume is measured by the third integral

$$\int_{ylow}^{yhigh} \int_{xlow[y]}^{xhigh[y]} \int_{zlow[x,y]}^{zhigh[x,y]} dz\, dx\, dy:$$

```
volume = ∫_{ylow}^{yhigh} secondintegral dy
```
$$9\,\pi$$

Or:
```
NIntegrate[secondintegral, {y, ylow, yhigh}]
```
28.2743

You can get this with a single 3D integral instruction for the volume measurement

$$\int_{ylow}^{yhigh} \int_{xlow[y]}^{xhigh[y]} \int_{zlow[x,y]}^{zhigh[x,y]} dz\, dx\, dy:$$

```
threeDintegral = ∫ylow^yhigh ∫xlow[y]^xhigh[y] ∫zlow[x,y]^zhigh[x,y] 1 dz dx dy
```

9π

Good.

However, doing this in one step does not relieve you of the experience of setting zhigh[x, y], zlow[x, y], xhigh[y], xlow[y], yhigh, and ylow. Tough break.

□**B.1.a.ii)**

Discuss the physical meaning of the three integrals that were calculated in part i):

$$\text{first integral} = \int_{zlow[x,y]}^{zhigh[x,y]} dz,$$

$$\text{second integral} = \int_{xlow[y]}^{xhigh[y]} \int_{zlow[x,y]}^{zhigh[x,y]} dz\, dx, \text{ and}$$

$$\text{third integral} = \int_{ylow}^{yhigh} \int_{xlow[y]}^{xhigh[y]} \int_{zlow[x,y]}^{zhigh[x,y]} dz\, dx\, dy.$$

□**Answer:**

The first integral,
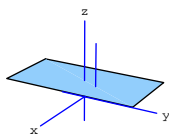
$$\int_{zlow[x,y]}^{zhigh[x,y]} dz,$$

fixes x and y and measures the length of the stick that runs from {x, y, zlow[x, y]} to {x, y, zhigh[x, y]}.

For example:

Feel free to reset {x0, y0} and rerun.

```
{x0, y0} = {0.5, 1};
stickinslice = Graphics3D[
```

```
   {Blue, Line[{{x0, y0, zlow[x0, y0]}, {x0, y0, zhigh[x0, y0]}}]}];
threedims = Axes3D[4.5];

Show[stickinslice, base, threedims, BoxRatios → Automatic,
  Boxed → False, ViewPoint → CMView];
```


The second integral

$$\int_{xlow[y]}^{xhigh[y]} \int_{zlow[x,y]}^{zhigh[x,y]} dz\, dx$$

holds y = y_0 fixed but releases x and measures the accumulated area of the slice of the solid swept out by the sticks as x runs from xlow[y] to xhigh[y]:

Feel free to reset y0 and rerun.

```
sticksinslice = Table[Graphics3D[
   {Blue, Line[{{x, y0, zlow[x, y0]}, {x, y0, zhigh[x, y0]}}]}],
   {x, xlow[y0], xhigh[y0], 1/50 (xhigh[y0] - xlow[y0])}];
Show[
  sticksinslice, base, threedims, Boxed → False, ViewPoint → CMView];
```
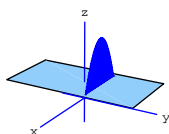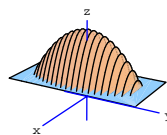

The third integral

$$\int_{ylow}^{yhigh} \int_{xlow[y]}^{xhigh[y]} \int_{zlow[x,y]}^{zhigh[x,y]} dz\, dx\, dy$$

releases y and measures the accumulated volume swept out by the slices as y runs from ylow to yhigh:

```
constituentslices = Table[Graphics3D[
   {Blue, Polygon[Table[{x, y, zhigh[x, y]}, {x, xlow[y] - 0.01,
     xhigh[y] + 0.01, 1/20 (xhigh[y] - xlow[y] + 0.02)}]]}],
  {y, ylow, yhigh, (yhigh - ylow)/20}];
accumulatedslices = Show[constituentslices, base, threedims,
  Boxed → False, ViewPoint → CMView];
```


The third integral measures the total volume.

□**B.1.a.iii) Triple integrals**

Calculate

$$\int\int\int_R z\, dx\, dy\, dz$$

where R is the solid plotted in part i)

□**Answer:**

If you are going to execute this, make sure that all the functions in part i) above are live.

The layout is essentially the same as the calculation of the volume of R done in part a.i) above.

```
Clear[f]
f[x_, y_, z_] = z;
firstintegral = ∫zlow[x,y]^zhigh[x,y] f[x, y, z] dz
```

$$-\frac{1}{2} + \frac{1}{2}\left(4 - x^2 - \frac{y^2}{4}\right)^2$$

```
secondintegral = ∫xlow[y]^xhigh[y] firstintegral dx
```

$$\frac{22\sqrt{12 - y^2}}{5} - \frac{17}{30} y^2 \sqrt{12 - y^2} + \frac{1}{60} y^4 \sqrt{12 - y^2}$$

```
thirdintegral = ∫ylow^yhigh secondintegral dy
```

18π

You can get this with one instruction for calculating

$$\int\int\int_R f[x, y, z]\, dx\, dy\, dz$$
$$= \int_{ylow}^{yhigh} \int_{xlow[y]}^{xhigh[y]} \int_{zlow[x,y]}^{zhigh[x,y]} f[x, y, z]\, dz\, dx\, dy:$$

```
onestep = ∫ylow^yhigh ∫xlow[y]^xhigh[y] ∫zlow[x,y]^zhigh[x,y] f[x, y, z] dz dx dy
```

18π

Good.

Doing this in one step does not relieve you of the burden of setting

zhigh[x, y], zlow[x, y],

xhigh[y], xlow[y],

yhigh, and ylow.

You set them up for calculating

$$\int\int\int_R f[x, y, z]\, dx\, dy\, dz$$

the same way you set them up for calculating the volume of R by calculating

$$\int\int\int_R dx\, dy\, dz$$

in part i).

$z^{(0,0,1)}[u, v, w] x^{(0,1,0)}[u, v, w] y^{(1,0,0)}[u, v, w] +$
$x^{(0,0,1)}[u, v, w] z^{(0,1,0)}[u, v, w] y^{(1,0,0)}[u, v, w] +$
$y^{(0,0,1)}[u, v, w] x^{(0,1,0)}[u, v, w] z^{(1,0,0)}[u, v, w] -$
$x^{(0,0,1)}[u, v, w] y^{(0,1,0)}[u, v, w] z^{(1,0,0)}[u, v, w]]$

Quite a pill to swallow, but who cares? The computer is the one that has to swallow it.

B.2) Transforming 3D integrals

□B.2.a)

In two dimensions, you know that
$$\iint_{R_{xy}} f[x, y] \, dx \, dy$$
$$= \iint_{R_{uv}} f[x[u, v], y[u, v]] A_{xy}[u, v] \, du \, dv.$$
where the area conversion factor $A_{xy}[u, v]$ is given by:

```
Clear[x, y, u, v, gradx, grady, Axy]
gradx[u_, v_] = {D[x[u, v], u], D[x[u, v], v]};
grady[u_, v_] = {D[y[u, v], u], D[y[u, v], v]};
Axy[u_, v_] = Abs[Det[{gradx[u, v], grady[u, v]}]]
```
$Abs[y^{(0,1)}[u, v] x^{(1,0)}[u, v] - x^{(0,1)}[u, v] y^{(1,0)}[u, v]]$

In three dimensions, what is the volume conversion factor
$$V_{xyz}[u, v, w]$$
that makes
$$\iiint_{R_{xyz}} f[x, y, z] \, dx \, dy \, dz$$

$$= \iiint_{R_{uvw}} f[x[u, v, w], y[u, v, w], z[u, v, w]] V_{xyz}[u, v, w] \, du \, dv \, dw$$

□Answer:

In mathematics, there is nothing like the power of analogy, and analogy works beautifully here.

In two dimensions the area conversion factor $A_{xy}[u, v]$ is given by:

```
Clear[x, y, u, v, gradx, grady, Axy]
gradx[u_, v_] = {D[x[u, v], u], D[x[u, v], v]};
grady[u_, v_] = {D[y[u, v], u], D[y[u, v], v]};
Axy[u_, v_] = Abs[Det[{gradx[u, v], grady[u, v]}]]
```
$Abs[y^{(0,1)}[u, v] x^{(1,0)}[u, v] - x^{(0,1)}[u, v] y^{(1,0)}[u, v]]$

Analogously, in three dimensions the volume conversion factor

$V_{xyz}[u, v, w]$ is given by:

```
Clear[x, y, z, u, v, gradx, grady, gradz, Vxyz]
gradx[u_, v_, w_] =
 {D[x[u, v, w], u], D[x[u, v, w], v], D[x[u, v, w], w]};
```

```
grady[u_, v_, w_] =
 {D[y[u, v, w], u], D[y[u, v, w], v], D[y[u, v, w], w]};
gradz[u_, v_, w_] =
 {D[z[u, v, w], u], D[z[u, v, w], v], D[z[u, v, w], w]};
Vxyz[u_, v_, w_] =
 Abs[Det[{gradx[u, v, w], grady[u, v, w], gradz[u, v, w]}]]
```
$Abs[z^{(0,0,1)}[u, v, w] y^{(0,1,0)}[u, v, w] x^{(1,0,0)}[u, v, w] -$
$y^{(0,0,1)}[u, v, w] z^{(0,1,0)}[u, v, w] x^{(1,0,0)}[u, v, w] -$
$z^{(0,0,1)}[u, v, w] x^{(0,1,0)}[u, v, w] y^{(1,0,0)}[u, v, w] +$
$x^{(0,0,1)}[u, v, w] z^{(0,1,0)}[u, v, w] y^{(1,0,0)}[u, v, w] +$
$y^{(0,0,1)}[u, v, w] x^{(0,1,0)}[u, v, w] z^{(1,0,0)}[u, v, w] -$
$x^{(0,0,1)}[u, v, w] y^{(0,1,0)}[u, v, w] z^{(1,0,0)}[u, v, w]]$

You just arrange the three gradients as (horizontal) rows of a matrix which most folks call the Jacobian matrix.

```
gradmatrix = {gradx[u, v, w], grady[u, v, w], gradz[u, v, w]}
```
$\{\{x^{(1,0,0)}[u, v, w], x^{(0,1,0)}[u, v, w], x^{(0,0,1)}[u, v, w]\},$
$\{y^{(1,0,0)}[u, v, w], y^{(0,1,0)}[u, v, w], y^{(0,0,1)}[u, v, w]\},$
$\{z^{(1,0,0)}[u, v, w], z^{(0,1,0)}[u, v, w], z^{(0,0,1)}[u, v, w]\}\}$

Look at it in matrix form:

```
MatrixForm[gradmatrix]
```
$\begin{pmatrix} x^{(1,0,0)}[u, v, w] & x^{(0,1,0)}[u, v, w] & x^{(0,0,1)}[u, v, w] \\ y^{(1,0,0)}[u, v, w] & y^{(0,1,0)}[u, v, w] & y^{(0,0,1)}[u, v, w] \\ z^{(1,0,0)}[u, v, w] & z^{(0,1,0)}[u, v, w] & z^{(0,0,1)}[u, v, w] \end{pmatrix}$

Take the absolute value of its determinant:

```
Abs[Det[gradmatrix]]
```
$Abs[z^{(0,0,1)}[u, v, w] y^{(0,1,0)}[u, v, w] x^{(1,0,0)}[u, v, w] -$
$y^{(0,0,1)}[u, v, w] z^{(0,1,0)}[u, v, w] x^{(1,0,0)}[u, v, w] -$
$z^{(0,0,1)}[u, v, w] x^{(0,1,0)}[u, v, w] y^{(1,0,0)}[u, v, w] +$
$x^{(0,0,1)}[u, v, w] z^{(0,1,0)}[u, v, w] y^{(1,0,0)}[u, v, w] +$
$y^{(0,0,1)}[u, v, w] x^{(0,1,0)}[u, v, w] z^{(1,0,0)}[u, v, w] -$
$x^{(0,0,1)}[u, v, w] y^{(0,1,0)}[u, v, w] z^{(1,0,0)}[u, v, w]]$

This is the same as the volume conversion factor

$V_{xyz}[u, v, w]$:

```
Vxyz[u, v, w]
```
$Abs[z^{(0,0,1)}[u, v, w] y^{(0,1,0)}[u, v, w] x^{(1,0,0)}[u, v, w] -$
$y^{(0,0,1)}[u, v, w] z^{(0,1,0)}[u, v, w] x^{(1,0,0)}[u, v, w] -$

□B.2.b.i)

Check out the formula for the volume conversion factor
$$V_{xyz}[u, v, w]$$
$$= |Det[\{gradx[u, v, w], grady[u, v, w], gradz[u, v, w]\}]|$$
by using it to relate the volume of the ellipsoid
$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = r^2$$
to the known volume $\frac{4\pi r^3}{3}$ of the sphere of radius r
$$u^2 + v^2 + w^2 = r^2.$$

□Answer:

```
Clear[x, y, z, a, b, c, r]
ellipsoideqn = (x/a)^2 + (y/b)^2 + (z/c)^2 == r^2
```
$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} == r^2$

Go to uvw-space with:

```
Clear[u, v, w]
{x[u_, v_, w_], y[u_, v_, w_], z[u_, v_, w_]} = {a u, b v, c w};
```
Here is the uvw-equation of the original ellipsoid:

```
ellipsoideqn /. {x → x[u, v, w], y → y[u, v, w], z → z[u, v, w]}
```
$u^2 + v^2 + w^2 == r^2$

This tells you that if R_{xyz} is everything inside and on the ellipsoid
$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = r^2 ,$$
then R_{uvw} is everything inside and on the sphere
$$u^2 + v^2 + w^2 = r^2.$$

The volume of the sphere R_{uvw} is $\frac{4\pi r^3}{3}$.

The uvw-to-xyz-volume conversion factor is:

```
Clear[u, v, gradx, grady, gradz, Vxyz]
gradx[u_, v_, w_] =
 {D[x[u, v, w], u], D[x[u, v, w], v], D[x[u, v, w], w]};
grady[u_, v_, w_] =
 {D[y[u, v, w], u], D[y[u, v, w], v], D[y[u, v, w], w]};
gradz[u_, v_, w_] =
 {D[z[u, v, w], u], D[z[u, v, w], v], D[z[u, v, w], w]};
Vxyz[u_, v_, w_] =
 Abs[Det[{gradx[u, v, w], grady[u, v, w], gradz[u, v, w]}]]
```
$Abs[a b c]$

So the xyz-volume measurement of R_{xyz} is

a b c times the uvw-volume measurement of R_{uvw}.

In short, the volume measurement of the ellipsoid
$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = r^2$$
is

a b c $\frac{4\pi r^3}{3}$.

Makes sense when you think about it.

□B.2.b.ii)

Now get a bit more serious.
Calculate
$$\iiint_{R_{xyz}} e^{-x} Sin[y] \, dx \, dy \, dz$$
where R_{xyz} is the region inside and on the box bounded by the planes
$z = 2x - 1$ (bottom).
$z = 2x + 3$ (top),
$y = x$ (side),
$y = x + 4$ (side),
$y = -2x$ (side), and
$y = -2x + 5$ (side).

□Answer:

A direct calculation would be a revolting task. Even the task of plotting R_{xyz} wouldn't be pleasant. But after you transform the integral to a favorable uvw-space, then everything is fairly easy.

Look at the formulas for the boundary surfaces:

$z = 2x - 1$ (bottom), $z = 2x + 3$ (top),

$y = x$ (side), $y = x + 4$ (side),

$y = -2x$ (side), $y = -2x + 5$ (side),

and make the assignments:

```
Clear[u, v, w, x, y, z]
{u[x_, y_, z_], v[x_, y_, z_], w[x_, y_, z_]} = {z - 2 x, y - x, y + 2 x}
{-2 x + z, -x + y, 2 x + y}
```

Note:

$z = 2x - 1 \longleftrightarrow u[x, y, z] = -1,$

$z = 2x + 3 \longleftrightarrow u[x, y, z] = 3,$

$y = x \longleftrightarrow v[x, y, z] = 0,$

$y = x + 4 \longleftrightarrow v[x, y, z] = 4,$

$y = -2x \longleftrightarrow w[x, y, z] = 0,$ and

$y = -2x + 5 \longleftrightarrow w[x, y, z] = 5.$

In uvw-space, R_{uvw} is everything inside and on the box bounded by the planes

$u = -1, u = 3, v = 0, v = 4, w = 0,$ and $w = 5.$

Also

$$\iiint_{R_{xyz}} e^{-x} \, Sin[y] \, dx \, dy \, dz$$
$$= \iiint_{R_{uvw}} e^{-x[u,v,w]} \, Sin[y[u, v, w]] \, V_{xyz}[u, v, w] \, du \, dv \, dw$$
$$= \int_0^5 \int_0^4 \int_{-1}^3 e^{-x[u,v,w]} \, Sin[y[u, v, w]] \, V_{xyz}[u, v, w] \, du \, dv \, dw.$$

To calculate $x[u, v, w]$, $y[u, v, w]$, $z[u, v, w]$, and $V_{xyz}[u, v, w]$, use:

```
xyzsolved =
 Solve[{u == u[x, y, z], v == v[x, y, z], w == w[x, y, z]}, {x, y, z}]
```
$\{\{x \to \frac{1}{3} (-v + w), y \to \frac{1}{3} (2 v + w), z \to \frac{1}{3} (3 u - 2 v + 2 w)\}\}$

```
{x[u_, v_, w_], y[u_, v_, w_], z[u_, v_, w_]} = {x, y, z} /. xyzsolved[[1]]
```
$\{\frac{1}{3} (-v + w), \frac{1}{3} (2 v + w), \frac{1}{3} (3 u - 2 v + 2 w)\}$

Now turn everything over to Mathematica:

```
Clear[gradx, grady, gradz, Vxyz]
gradx[u_, v_, w_] =
 {D[x[u, v, w], u], D[x[u, v, w], v], D[x[u, v, w], w]};
grady[u_, v_, w_] =
 {D[y[u, v, w], u], D[y[u, v, w], v], D[y[u, v, w], w]};
gradz[u_, v_, w_] =
 {D[z[u, v, w], u], D[z[u, v, w], v], D[z[u, v, w], w]};
Vxyz[u_, v_, w_] =
 Abs[Det[{gradx[u, v, w], grady[u, v, w], gradz[u, v, w]}]]
```
$\frac{1}{3}$

So

$$\iiint_{R_{xyz}} e^{-x} \, Sin[y] \, dx \, dy \, dz$$
$$= \iiint_{R_{uvw}} e^{-x[u,v,w]} \, Sin[y[u, v, w]] \, V_{xyz}[u, v, w] \, du \, dv \, dw$$
$$= \int_0^5 \int_0^4 \int_{-1}^3 e^{-x[u,v,w]} \, Sin[y[u, v, w]] \, (\tfrac{1}{3}) \, du \, dv \, dw.$$

The first integral

$\int_{-1}^3 e^{-x[u,v,w]} \, Sin[y[u, v, w]] \, (\tfrac{1}{3}) \, du$:

```
firstintegral = ∫_{-1}^3 E^{-x[u,v,w]} Sin[y[u, v, w]] (1/3) du
```
$\frac{4}{3} E^{\frac{v-w}{3}} Sin[\frac{1}{3} (2 v + w)]$

The second integral

$\int_0^4 \int_{-1}^3 e^{-x[u,v,w]} \, Sin[y[u, v, w]] \, (\tfrac{1}{3}) \, du \, dv$ is

```
secondintegral = ∫_0^4 firstintegral dv
```

$\frac{4}{5} E^{-w/3} (2 Cos[\frac{w}{3}] - Sin[\frac{w}{3}]) - \frac{4}{5} E^{\frac{4-w}{3}} (2 Cos[\frac{8+w}{3}] - Sin[\frac{8+w}{3}])$

The third and final integral

$\int_0^5 \int_0^4 \int_{-1}^3 e^{-x[u,v,w]} \, Sin[y[u, v, w]] \, (\tfrac{1}{3}) \, du \, dv \, dw$

is, in decimals,

```
NIntegrate[secondintegral, {w, 0, 5}]
14.2704
```

You can get

$$\iiint_{R_{xyz}} e^{-x} \, Sin[y] \, dx \, dy \, dz$$
$$= \int_0^5 \int_0^4 \int_{-1}^3 e^{-x[u,v,w]} \, Sin[y[u, v, w]] \, (\tfrac{1}{3}) \, du \, dv \, dw$$

in one step:

```
quickway = NIntegrate[1/3 E^{-x[u,v,w]} Sin[y[u, v, w]], {w, 0, 5},
 {v, 0, 4}, {u, -1, 3}]
14.2704
```

Went through that one like a hot knife going through butter.

□**B.2.b.iii)**

The solid R_{xyz} consists of everything under the surface

$z = 0.2 \, e^{-y}$

and over the region inside the ellipse

$x^2 + (\frac{y}{3})^2 = 1$

in the xy-plane.

Here's a look:

```
Clear[x, y, z, r, t, s]
{x[r_, t_], y[r_, t_]} = {r Cos[t], 3 r Sin[t]};
z[r_, t_] = 0.2 E^{-y[r,t]};
{{rlow, rhigh}, {tlow, thigh}} = {{0, 1}, {0, 2 π}};
topskin = ParametricPlot3D[{x[r, t], y[r, t], z[r, t]},
 {r, rlow, rhigh}, {t, tlow, thigh}, DisplayFunction → Identity];
baseskin = ParametricPlot3D[
 {x[r, t], y[r, t], 0}, {r, rlow, rhigh}, {t, tlow, thigh},
 PlotPoints → {2, Automatic}, DisplayFunction → Identity];
sideskin = ParametricPlot3D[
 {x[rhigh, t], y[rhigh, t], s z[rhigh, t]}, {s, 0, 1}, {t, tlow, thigh},
```

```
 PlotPoints → {2, Automatic}, DisplayFunction → Identity];

Show[topskin, baseskin, sideskin, BoxRatios → Automatic,
 PlotRange → All, AxesLabel → {"x", "y", "z"}, ViewPoint → CMView,
 DisplayFunction → $DisplayFunction];
```
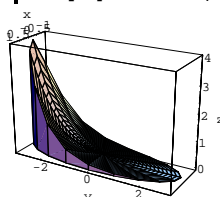

Calculate

$$\iiint_{R_{xyz}} e^{-y} \, dx \, dy \, dz$$

by transforming to an easy 3D integral.

□**Answer:**

Look at the plotting functions above:

```
{x[r, t], y[r, t]}
{r Cos[t], 3 r Sin[t]}
z[r, t]
0.2 E^{-3 r Sin[t]}
```

Go to rst-space with:

```
Clear[x, y, z, r, s, t]
{x[r_, s_, t_], y[r_, s_, t_], z[r_, s_, t_]} =
 {r Cos[t], 3 r Sin[t], 0.2 s E^{3 r Sin[t]}}
{r Cos[t], 3 r Sin[t], 0.2 E^{3 r Sin[t]} s}
```

As you run

r from rlow to rhigh,

t from tlow to thigh, and

s from 0 to 1,

$\{x[r, s, t], y[r, s, t], z[r, s, t]\}$ sweeps out all of R_{xyz}; so R_{rst} is the box

$rlow \leq r \leq rhigh, 0 \leq s \leq 1, tlow \leq t \leq thigh,$

and

$$\iiint_{R_{xyz}} e^{-y} \, dx \, dy \, dz$$
$$= \iiint_{R_{rst}} e^{-y[r,s,t]} \, V_{xyz}[r, s, t] \, dr \, ds \, dt$$
$$= \int_{tlow}^{thigh} \int_0^1 \int_{rlow}^{rhigh} e^{-y[r,s,t]} \, V_{xyz}[r, s, t] \, dr \, ds \, dt.$$

```
Clear[gradx, grady, gradz, Vxyz]
gradx[r_, s_, t_] =
 {D[x[r, s, t], r], D[x[r, s, t], s], D[x[r, s, t], t]};
grady[r_, s_, t_] =
 {D[y[r, s, t], r], D[y[r, s, t], s], D[y[r, s, t], t]};
gradz[r_, s_, t_] =
 {D[z[r, s, t], r], D[z[r, s, t], s], D[z[r, s, t], t]};
Vxyz[r_, s_, t_] =
 TrigReduce[Det[{gradx[r, s, t], grady[r, s, t], gradz[r, s, t]}]]
```
$-0.6 \, E^{3 \, r \, Sin[t]} \, r$

This is negative no matter what r, s, and t are in this box. Reset it:

```
Clear[Vxyz]
Vxyz[r_, s_, t_] =
 -TrigReduce[Det[{gradx[r, s, t], grady[r, s, t], gradz[r, s, t]}]]
```
$0.6 \, E^{3 \, r \, Sin[t]} \, r$

So

$$\iiint_{R_{xyz}} e^{-y} \, dx \, dy \, dz$$
$$= \iiint_{R_{rst}} e^{-y[r,s,t]} \, V_{xyz}[r, s, t] \, dr \, ds \, dt$$
$$= \int_{tlow}^{thigh} \int_0^1 \int_{rlow}^{rhigh} e^{-y[r,s,t]} \, V_{xyz}[r, s, t] \, dr \, ds \, dt$$

is given by:

```
∫_tlow^thigh ∫_0^1 ∫_rlow^rhigh E^-y[r,s,t] Vxyz[r, s, t] dr ds dt
```
1.88496

Nice answer.

But if you hadn't gone to rst-space to calculate the integral, getting this nice answer wouldn't have been nearly so simple.

B.3) Mass and density

□**B.3.a)**

Here x, y, and z are measured in centimeters.
The top of a solid R is the surface
$$z = 32 - x^2 - y^2$$
and the bottom is the surface
$$z = x^2 + y^2 - 40.$$
The density of the solid varies from point to point, and is given by
$$density[x, y, z] = 0.3 \, (x^2 + y^2 + z^2) \, \left(\frac{grams}{centimeter^3}\right)$$
at {x, y, z}.
It is fluffy near {0, 0, 0} and heavily dense far away from {0, 0, 0}.
Measure the total mass (in grams) of this solid.
If you dump it in a lake, will it sink or will it float?

□**Answer:**

The density function is what you integrate to calculate mass.

As a result, the mass of a solid R whose density is density[x, y, z] $\frac{grams}{centimeter^3}$ at a point {x, y, z} is measured by
$$\iiint_R density[x, y, z] \, dx \, dy \, dz.$$

```
Clear[x, y, z, density]
density[x_, y_, z_] = 0.03 (x^2 + y^2 + z^2);
```

The top of R is the surface
$$z = 32 - x^2 - y^2$$
and the bottom is the surface
$$z = x^2 + y^2 - 40.$$
With z so conveniently displayed, it is natural to integrate with respect to z first, so calculate
$$\iiint_R density[x, y, z] \, dx \, dy \, dz$$
$$= \iiint_R density[x, y, z] \, dz \, dy \, dx$$

To set the limits for the first integral,
$$\int_\square^\square density[x, y, z] \, dz,$$
fix x and y and enter the lowest and the highest values of z for this fixed x and y. The top skin of R is the surface
$$z = 32 - x^2 - y^2$$
and the bottom is the surface
$$z = x^2 + y^2 - 40.$$

```
Clear[zhigh, zlow]
zhigh[x_, y_] = 32 - x^2 - y^2;
zlow[x_, y_] = x^2 + y^2 - 40;
firstintegral = ∫_zlow[x,y]^zhigh[x,y] density[x, y, z] dz
```
$967.68 - 76.56 \, x^2 + 2.1 \, x^4 - 0.02 \, x^6 - 76.56 \, y^2 + 4.2 \, x^2 \, y^2 - 0.06 \, x^4 \, y^2 + 2.1 \, y^4 - 0.06 \, x^2 \, y^4 - 0.02 \, y^6$

For the second integral, you have our choice whether to integrate with respect to y or to x.
Go with y.
The main problem is deciding what limits to insert for yhigh[x] and ylow[x] in the second integral
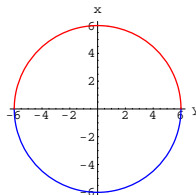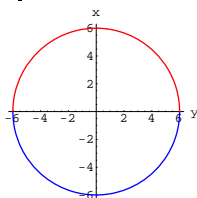$$\int_{ylow[x]}^{yhigh[x]} \int_{zlow[x,y]}^{zhigh[x,y]} density[x, y, z] \, dz \, dy.$$
The key to deciding how to set yhigh[x] and ylow[x] is to ask:
For a fixed x, what are the lowest and the highest values y can have no matter what z is?
The simplest way of answering this is to plot the shadow of this solid on xy-paper with the y-axis running on the horizontal:

```
shadoweqn = Eliminate[{z == zhigh[x, y], z == zlow[x, y]}, z]
```
$x^2 == 36 - y^2$
```
shadowcurves = Solve[shadoweqn, y]
```

$\left\{\left\{y \to -\sqrt{36 - x^2}\right\}, \left\{y \to \sqrt{36 - x^2}\right\}\right\}$
```
Clear[ylow, yhigh]
{ylow[x_], yhigh[x_]} = {-√(36 - x^2), √(36 - x^2)}
```
$\left\{-\sqrt{36 - x^2}, \sqrt{36 - x^2}\right\}$
```
shadowplot = Plot[{ylow[x], yhigh[x]}, {x, -6, 6},
 PlotStyle → {Blue, Red}, AxesLabel → {"y", "x"}];
```


The second integral
$$\int_{ylow[x]}^{yhigh[x]} \int_{zlow[x,y]}^{zhigh[x,y]} density[x, y, z] \, dz \, dy$$
is:

```
secondintegral = ∫_ylow[x]^yhigh[x] Expand[firstintegral] dy
```
$919.954 \, \sqrt{36 - x^2} - 70.6469 \, x^2 \, \sqrt{36 - x^2} + 1.91086 \, x^4 \, \sqrt{36 - x^2} - 0.0182857 \, x^6 \, \sqrt{36 - x^2}$

For the third integral, you have no choice but to integrate with respect to x.
The main problem is in deciding what limits to insert for xhigh and xlow in the third integral
$$\int_{xlow}^{xhigh} \int_{ylow[x]}^{yhigh[x]} \int_{zlow[x,y]}^{zhigh[x,y]} density[x, y, z] \, dz \, dy \, dx.$$
   Note that xlow and xhigh are numbers, not functions.
The key to deciding how to set xlow and xhigh is to ask:
What are the lowest and the highest values x can have no matter what z and y are?

The simplest way of getting a hold on xlow and xhigh is to look at the shadow plot again:

```
Show[shadowplot];
```


In this one, xlow and xhigh appear where ylow[x] and yhigh[x] meet:

```
ends = Solve[yhigh[x] == ylow[x]]
```

$\{\{x \to -6\}, \{x \to 6\}\}$

That gives:

```
{xlow, xhigh} = {-6, 6};
```

The mass in grams is measured by the third integral

$$\int_{xlow}^{xhigh} \int_{ylow[x]}^{yhigh[x]} \int_{zlow[x,y]}^{zhigh[x,y]} \text{density[x, y, z]} \, dz \, dy \, dx:$$

```
mass = NIntegrate[secondintegral, {x, xlow, xhigh}]
```
29803.4

To see whether this solid sinks or floats, calculate its volume:

```
volume = ∫xlow^xhigh ∫ylow[x]^yhigh[x] ∫zlow[x,y]^zhigh[x,y] 1 dz dy dx
```
1296π

Compare:

```
{N[volume], mass}
```
{4071.5, 29803.4}

One centimeter^3 of water weighs one gram.

So this solid outweighs the volume of water it displaces.

When you dump it in the lake, it sinks fast.

B.4) Average value of a function

B.4.a.i) One variable

Given a function f[x] and an interval $a \le x \le b$, folks like to say that the average value of f[x] and this interval is given by

$$\frac{1}{(b-a)} \int_a^b f[x] \, dx.$$

Why do they like to say this?

□**Answer:**

They like to say this because it works.

To see it in action, go with

f[x] = Sin[x]

and look at an experimental average of random values of f[x] for

$a = 0 \le x \le b = \pi$:

```
Clear[f, x, k]
f[x_] = Sin[x];
{a, b} = {0, π};
pointcount = 200;
experimentalaverage = (∑_{k=1}^{pointcount} N[f[Random[Real, {N[a], N[b]}]]]) / pointcount
```
0.628107

And the theoretical average value:

```
theoreticalaverage = N[(∫a^b f[x] dx) / (b - a)]
```
0.63662

Compare:

```
{theoreticalaverage "= theoretical average",
  experimentalaverage "= experimental average"}
```
{0.63662 = theoretical average, 0.628107 = experimental average}

See what happens when you use more points:

```
pointcount = 400;
newexperimentalaverage = (∑_{k=1}^{pointcount} N[f[Random[Real, {N[a], N[b]}]]]) / pointcount;
{theoreticalaverage "= theoretical average",
  newexperimentalaverage "= experimental average"}
```
{0.63662 = theoretical average, 0.656004 = experimental average}

Uh-huh.

Reset the point count and rerun.

□B.4.a.ii)

Wait a minute! This is supposed to be a mathematics course. Where do you get off saying that something is correct because it works?

□**Answer:**

Good point.

You can explain why it works another way by looking at plots like this:

```
Clear[f, x]
f[x_] = Sin[x];
{a, b} = {0, π};
fplot = Plot[f[x], {x, a, b},
   PlotStyle → {{Thickness[0.01], Red}}, DisplayFunction → Identity];

average = N[(∫a^b f[x] dx) / (b - a)];
averagefplot = Graphics[
   {Blue, Polygon[{{a, 0}, {b, 0}, {b, average}, {a, average}}]}];

Show[averagefplot, fplot, Axes → True,
  AxesLabel → {"x", "y"}, PlotRange → All, AspectRatio → Automatic,
  DisplayFunction → $DisplayFunction];
```
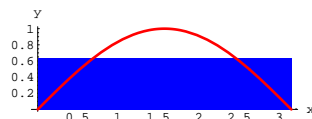

The height of the rectangle is $\frac{1}{(b-a)} \int_a^b f[x] \, dx$.

The width of the rectangle is $(b - a)$.

The area inside the rectangle measures out to

$$\frac{\int x \, dx}{(b-a)} (b - a) = \int_a^b f[x] \, dx$$

in square units.

This tells you that the area of the flat rectangle measures the (signed) area under the curve y = f[x] and over the x-axis and for $a \le x \le b$.

This is why folks say that the average value of f[x] for $a \le x \le b$ is the height of the rectangle,

$$\frac{1}{(b-a)} \int_a^b f[x] \, dx.$$

Try it again for

f[x] = $2.3 \, e^{-0.5 x}$ for $-1 = a \le x \le b = 2$:

```
Clear[f, x]
f[x_] = 2.3 E^-0.5 x;
{a, b} = {-1, 2};
fplot = Plot[f[x], {x, a, b},
   PlotStyle → {{Thickness[0.01], Red}}, DisplayFunction → Identity];

average = (∫a^b f[x] dx) / (b - a);
averagefplot = Graphics[
   {Blue, Polygon[{{a, 0}, {b, 0}, {b, average}, {a, average}}]}];

Show[averagefplot, fplot, Axes → True,
  AxesLabel → {"x", "y"}, PlotRange → All, AspectRatio → Automatic,
  DisplayFunction → $DisplayFunction];
```
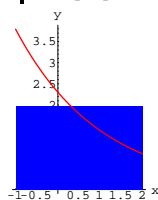

And experimentally a couple of times:

```
pointcount = 50;
experimentalaverage = (Σ_{k=1}^{pointcount} N[f[Random[Real, {N[a], N[b]}]]]) / pointcount;
{average "= theoretical average",
 experimentalaverage "= experimental average"}
```
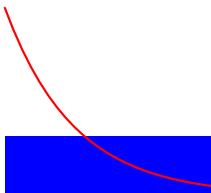{1.96396 = theoretical average, 2.02415 = experimental average}

```
newpointcount = 300;
newexperimentalaverage = (Σ_{k=1}^{pointcount} N[f[Random[Real, {N[a], N[b]}]]]) / pointcount;
{average "= theoretical average",
 newexperimentalaverage "= experimental average"}
```
{1.96396 = theoretical average, 1.91133 = experimental average}

Usually not far off.

```
Clear[f, x]
f[x_] = E^-x;
{a, b} = {-1, 2};
fplot = Plot[f[x], {x, a, b},
  PlotStyle → {{Thickness[0.01], Red}}, DisplayFunction → Identity];
average = (∫_a^b f[x] dx) / (b - a);
averagefplot = Graphics[
  {Blue, Polygon[{{a, 0}, {b, 0}, {b, average}, {a, average}}]}];

Show[averagefplot, fplot, AxesLabel → {"x", "y"},
 AspectRatio → Automatic, DisplayFunction → $DisplayFunction];
```


And experimentally several times:

```
theoreticalaverage = N[(∫_a^b f[x] dx) / (b - a)];
pointcount = 200;
experimentalaverage = (Σ_{k=1}^{pointcount} N[f[Random[Real, {N[a], N[b]}]]]) / pointcount;
{theoreticalaverage, experimentalaverage}
```
{0.860982, 0.992238}

See what happens when you use more points:

```
pointcount = 200;
experimentalaverage = (Σ_{k=1}^{pointcount} N[f[Random[Real, {N[a], N[b]}]]]) / pointcount;
{theoreticalaverage, experimentalaverage}
```
{0.860982, 0.834493}

```
pointcount = 200;
experimentalaverage = (Σ_{k=1}^{pointcount} N[f[Random[Real, {N[a], N[b]}]]]) / pointcount;
{theoreticalaverage, experimentalaverage}
```
{0.860982, 0.843936}

```
pointcount = 200;
experimentalaverage = (Σ_{k=1}^{pointcount} N[f[Random[Real, {N[a], N[b]}]]]) / pointcount;
{theoreticalaverage, experimentalaverage}
```
{0.860982, 0.777761}

Usually not far off.

□B.4.b) 2D averages.

Given a function f[x, y] and region R in xy-paper, folks like to say that the average value of f[x, y] on R is given by
$$\frac{\int\int_R f[x,y]\,dx\,dy}{\int\int_R dx\,dy}$$
Why do they like to say this?

□**Answer:**

Look at these:

```
{{a, b}, {c, d}} = {{0, 2}, {-1, 4}};
Clear[f, x, y]
f[x_, y_] = Sin[2 x + y] + 1;
```

```
fplot = Plot3D[f[x, y], {x, a, b},
  {y, c, d}, Axes → True, AxesLabel → {"x", "y", "z"},
  ViewPoint → CMView, DisplayFunction → Identity];
theoreticalaverage = N[(∫_a^b ∫_c^d f[x, y] dy dx) / (∫_a^b ∫_c^d 1 dy dx)];
averagefplot = Plot3D[theoreticalaverage, {x, a, b}, {y, c, d},
  PlotPoints → {2, 2}, Axes → True, AxesLabel → {"x", "y", "z"},
  ViewPoint → CMView, DisplayFunction → Identity];

Show[GraphicsArray[{fplot, averagefplot}],
 DisplayFunction → $DisplayFunction];
```
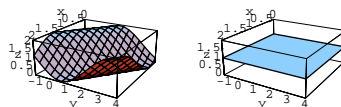

The volume measurements of the solids between the xy-plane and the plotted surfaces are the same.

Reason:

$$\text{volume left} = \int\int_R f[x, y]\,dx\,dy$$
$$\text{volume right} = \text{base times height}$$
$$= \int\int_R dx\,dy \left(\frac{\int\int_R f[x,y]\,dx\,dy}{\int\int_R dx\,dy}\right)$$
$$= \int\int_R f[x, y]\,dx\,dy.$$

Both solids have the same base, so the height of the solid on the right must be the average value of f[x, y] for {x, y} in R.

See what happens experimentally:

```
pointcount = 300;
experimentalaverage = 1/pointcount (Σ_{k=1}^{pointcount} N[

    f[Random[Real, {N[a], N[b]}], Random[Real, {N[c], N[d]}]]]);

{theoreticalaverage "= theoretical average",
 experimentalaverage "= experimental average"}
```
{0.961822 = theoretical average, 0.985445 = experimental average}

Again:

```
pointcount = 500;
experimentalaverage = 1/pointcount (Σ_{k=1}^{pointcount} N[

    f[Random[Real, {N[a], N[b]}], Random[Real, {N[c], N[d]}]]]);

{theoreticalaverage "= theoretical average",
 experimentalaverage "= experimental average"}
```
{0.961822 = theoretical average, 0.955158 = experimental average}

Lookin' good and feeling good.

□B.4.c) 3D averages.

Given a function f[x, y, z] and a solid R in xyz-space, folks like to say that the average value of f[x, y, z] on R is given by
$$\int\int\int_R f[x, y, z]\,dx\,dy\,dz \,/\, \int\int\int_R dx\,dy\,dz$$
Why do they like to say this?

□**Answer:**

This time, a plot is not possible.

Try an experiment with R as the box

$$0 \le x \le 2,\ -1 \le y \le 4,\ \text{and}\ -2 \le z \le 3:$$

```
{{xlow, xhigh}, {ylow, yhigh}, {zlow, zhigh}} =
 {{0, 2}, {-1, 4}, {-2, 3}};
Clear[f, x, y, z]
f[x_, y_, z_] = Sin[2 x + y - z] + 1;
```

$$\text{theoreticalaverage} = N\left[\frac{\int_{xlow}^{xhigh}\int_{ylow}^{yhigh}\int_{zlow}^{zhigh} f[x, y, z]\, dz\, dy\, dx}{\int_{xlow}^{xhigh}\int_{ylow}^{yhigh}\int_{zlow}^{zhigh} 1\, dz\, dy\, dx}\right];$$

```
pointcount = 100;
```

$$\text{experimentalaverage} = \frac{1}{\text{pointcount}}$$

$$\left(\sum_{k=1}^{\text{pointcount}} N[f[\text{Random}[\text{Real}, \{N[xlow], N[xhigh]\}], \text{Random}[\text{Real},}\right.$$

$$\left.\{N[ylow], N[yhigh]\}], \text{Random}[\text{Real}, \{N[zlow], N[zhigh]\}]]]\right);$$

```
{theoreticalaverage "= theoretical average",
 experimentalaverage "= experimentalaverage"}
```
{1.00368 = theoretical average, 1.10411 = experimentalaverage}

```
pointcount = 200;
```

$$\text{newexperimentalaverage} = \frac{1}{\text{pointcount}}$$

$$\left(\sum_{k=1}^{\text{pointcount}} N[f[\text{Random}[\text{Real}, \{N[xlow], N[xhigh]\}], \text{Random}[\text{Real},}\right.$$

$$\left.\{N[ylow], N[yhigh]\}], \text{Random}[\text{Real}, \{N[zlow], N[zhigh]\}]]]\right);$$

```
{theoreticalaverage "= theoretical average",
 newexperimentalaverage "= experimentalaverage"}
```
{1.00368 = theoretical average, 1.09911 = experimentalaverage}

And look at this:

```
{theoreticalaverage "= theoretical average",
 1
 ─ (experimentalaverage + newexperimentalaverage)
 2
 "= averageexperimentalaverage"}
```
{1.00368 = theoretical average, 1.10161 = averageexperimentalaverage}

Nice.

Another way to look at this is to note that when you go with

$$\text{average} = \iiint_R f[x, y, z]\, dx\, dy\, dz \,/ \iiint_R dx\, dy\, dz,$$

then you get

$$\iiint_R f[x, y, z]\, dx\, dy\, dz = \iiint_R \text{average}\, dx\, dy\, dz.$$

VC.08 Transforming 3D Integrals Tutorials

T.1) Cylinders, spheres, and tubes:
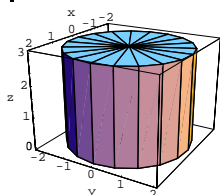
Plotting them and integrating on them
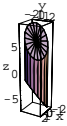
□T.1.a.i) Cylinders

Here's a cylinder:

```
Clear[x, y, z, r, s, t]
{x[r_, s_, t_], y[r_, s_, t_], z[r_, s_, t_]} = {r Cos[t], r Sin[t], s};
{{rlow, rhigh}, {slow, shigh}, {tlow, thigh}} =
 {{0, 2}, {0, 3}, {0, 2 π}};
sideplot =
 ParametricPlot3D[{x[rhigh, s, t], y[rhigh, s, t], z[rhigh, s, t]},
  {s, slow, shigh}, {t, tlow, thigh},
  PlotPoints → {2, Automatic}, DisplayFunction → Identity];
topplot =
 ParametricPlot3D[{x[r, shigh, t], y[r, shigh, t], z[r, shigh, t]},
  {r, rlow, rhigh}, {t, tlow, thigh},
  PlotPoints → {2, Automatic}, DisplayFunction → Identity];
baseplot =
 ParametricPlot3D[{x[r, slow, t], y[r, slow, t], z[r, slow, t]},
  {r, rlow, rhigh}, {t, tlow, thigh},
  PlotPoints → {2, Automatic}, DisplayFunction → Identity];

Show[sideplot, topplot, baseplot,
 AxesLabel → {"x", "y", "z"}, ViewPoint → CMView, BoxRatios → Automatic,
 DisplayFunction → $DisplayFunction];
```


Measure the volume contained in this cylinder.

You don't need calculus for this one.

The base of this cylinder is a circle of radius 2; its area measures out to:

```
basearea = 2² π
```
4 π

The height of the cylinder is:

```
height = shigh
```
3

The volume of the cylinder measures out to:

```
volume = basearea height
```
12 π

Now try it using calculus:

```
Clear[gradx, grady, gradz, Vxyz]
gradx[r_, s_, t_] =
 {D[x[r, s, t], r], D[x[r, s, t], s], D[x[r, s, t], t]};
grady[r_, s_, t_] =
 {D[y[r, s, t], r], D[y[r, s, t], s], D[y[r, s, t], t]};
gradz[r_, s_, t_] =
 {D[z[r, s, t], r], D[z[r, s, t], s], D[z[r, s, t], t]};
Vxyz[r_, s_, t_] =
 TrigExpand[Det[{gradx[r, s, t], grady[r, s, t], gradz[r, s, t]}]]
```
−r

When you look at the plotting instructions, you see that the points

$$\{x[r, s, t],\ y[r, s, t],\ z[r, s, t]\}$$

with rlow ≤ r ≤ rhigh, slow ≤ s ≤ shigh, and zlow ≤ z ≤ zhigh describe the cylinder and everything inside it. This tells you that the volume of this cylinder measures out to

$$\int_{tlow}^{thigh}\int_{slow}^{shigh}\int_{rlow}^{rhigh}(-V_{xyz}[r, s, t])\, dr\, ds\, dt\text{:}$$

```
volume = -∫_tlow^thigh ∫_slow^shigh ∫_rlow^rhigh Vxyz[r, s, t] dr ds dt
```
12 π

Same as you got without calculus.

□T.1.a.ii)

This time, go with a cylinder whose top is on the plane:

```
Clear[top, x, y]
top[x_, y_] = -x - y + 6
```
6 − x − y

And whose bottom is on the plane:

```
Clear[bottom, x, y]
bottom[x_, y_] = x - y - 4
```
−4 + x − y

Put:

```
Clear[x, y, z, s]
{slow, shigh} = {0, 1};
z[x_, y_, s_] = bottom[x, y] + s (top[x, y] - bottom[x, y])
```
−4 + s (10 − 2 x) + x − y

And note that:

```
{z[x, y, slow] == bottom[x, y], z[x, y, shigh] == top[x, y]}
```
{True, True}

This gives you an easy way of plotting the cockeyed cylinder
→ whose top skin runs with the plane z = top[x, y],
→ whose bottom skin runs with the plane z = bottom[x, y] and
→ whose sides run with circles of radius 2 perpendicular to the z-axis and centered on the z-axis:

```
Clear[x, y, newz, r, s, t]
{{rlow, rhigh}, {tlow, thigh}} = {{0, 2}, {0, 2 π}};
{x[r_, s_, t_], y[r_, s_, t_]} = {r Cos[t], r Sin[t]};
newz[r_, s_, t_] = z[x[r, s, t], y[r, s, t], s];
sideplot =
 ParametricPlot3D[{x[rhigh, s, t], y[rhigh, s, t], newz[rhigh, s, t]},
  {s, slow, shigh}, {t, tlow, thigh},
  PlotPoints → {2, Automatic}, DisplayFunction → Identity];
topplot =
 ParametricPlot3D[{x[r, shigh, t], y[r, shigh, t], newz[r, shigh, t]},
  {r, rlow, rhigh}, {t, tlow, thigh},
  PlotPoints → {2, Automatic}, DisplayFunction → Identity];
baseplot =
 ParametricPlot3D[{x[r, slow, t], y[r, slow, t], newz[r, slow, t]},
  {r, rlow, rhigh}, {t, tlow, thigh},
  PlotPoints → {2, Automatic}, DisplayFunction → Identity];
```

```
Show[sideplot, topplot, baseplot,
 AxesLabel → {"x", "y", "z"}, ViewPoint → CMView, BoxRatios → Automatic,
 DisplayFunction → $DisplayFunction];
```


Measure the volume contained in this weirdo cylinder.

□ **Answer:**

You do want to use calculus for this one.

```
Clear[gradx, grady, gradz, Vxyz]
gradx[r_, s_, t_] =
 {D[x[r, s, t], r], D[x[r, s, t], s], D[x[r, s, t], t]};
grady[r_, s_, t_] =
 {D[y[r, s, t], r], D[y[r, s, t], s], D[y[r, s, t], t]};
gradnewz[r_, s_, t_] =
 {D[newz[r, s, t], r], D[newz[r, s, t], s], D[newz[r, s, t], t]};
Vxyz[r_, s_, t_] = Abs[
 TrigExpand[Det[{gradx[r, s, t], grady[r, s, t], gradnewz[r, s, t]}]]]
```
Abs[-10 r + 2 r² Cos[t]]

The volume inside this weirdo measures out (in cubic units) to
$\int_{tlow}^{thigh} \int_{slow}^{shigh} \int_{rlow}^{rhigh} V_{xyz}[r, s, t] \, dr \, ds \, dt$:

```
volume = NIntegrate[Vxyz[r, s, t], {t, tlow, thigh}, {s, slow, shigh},
 {r, rlow, rhigh}]
```
125.664

Looking bad, but feeling good.

□ **T.1.b) Spheres**

One way to see how to plot the sphere
$x^2 + y^2 + z^2 = r^2$
is to slice it with the plane
$z = c$ where $-r \le c \le r$,
and then to think about what you get.
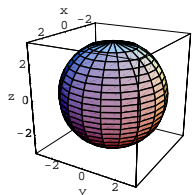What you get is the circle
$x^2 + y^2 = r^2 - c^2$
plotted in the plane $z = c$.
Now, when you go with a given radius r, you can plot the sphere
$x^2 + y^2 + z^2 = r^2$.
Take a look in the case that r = 3:

```
Clear[c, t]
radius = 3;
{clow, chigh} = {-radius, radius};
{tlow, thigh} = {0, 2π};
sphereplot =
 ParametricPlot3D[{√(radius² - c²) Cos[t], √(radius² - c²) Sin[t], c},
  {c, clow, chigh}, {t, tlow, thigh}, ViewPoint → CMView,
  Axes → True, AxesLabel → {"x", "y", "z"}];
```


Agree that R is the region consisting of everything on and inside the sphere
$x^2 + y^2 + z^2 = 9$
as plotted above, and calculate the integral
$\iiint_R x^2 \, dx \, dy \, dz.$

□ **Answer:**

Look at the plotting code:
ParametricPlot3D [{Sqrt[radius² − c²] Cos[t],
      Sqrt[radius² − c²] Sin[t], c},
      {c, clow, chigh}, {t, tlow, thigh}].
And go with
$\{x[c, r, t], y[c, r, t], z[c, r, t]\} = \{r \, Cos[t], r \, Sin[t], c\};$
with
$0 \le r \le Sqrt[radius^2 - c^2], -radius \le c \le radius,$
and $0 \le t \le 2\pi.$
This gives you
$$\iiint_R x^2 \, dx \, dy \, dz$$
$$= \iiint_{R_{crt}} x[c, r, t]^2 \, V_{xyz}[c, r, t] \, dc \, dr \, dt$$
$$= \iiint_{R_{crt}} x[c, r, t]^2 \, V_{xyz}[c, r, t] \, dr \, dc \, dt$$
$$= \int_0^{2\pi} \int_{radius}^{radius} \int_0^{\sqrt{radius^2 - c^2}} x[c, r, t]^2 \, V_{xyz}[c, r, t] \, dr \, dc \, dt.$$
You gotta integrate with respect to r before you integrate with respect to c because the limit of integration with respect to r changes as c changes.

Now calculate the volume conversion factor:

```
Clear[x, y, z, c, r, t]
{x[c_, r_, t_], y[c_, r_, t_], z[c_, r_, t_]} = {r Cos[t], r Sin[t], c};
Clear[gradx, grady, gradz, Vxyz]
gradx[c_, r_, t_] =
 {D[x[c, r, t], c], D[x[c, r, t], r], D[x[c, r, t], t]};
grady[c_, r_, t_] =
 {D[y[c, r, t], c], D[y[c, r, t], r], D[y[c, r, t], t]};
gradz[c_, r_, t_] =
 {D[z[c, r, t], c], D[z[c, r, t], r], D[z[c, r, t], t]};
Vxyz[c_, r_, t_] =
 TrigExpand[Det[{gradx[c, r, t], grady[c, r, t], gradz[c, r, t]}]]
```
r

Calculate
$$\iiint_R x^2 \, dx \, dy \, dz$$
$$= \int_0^{2\pi} \int_{radius}^{radius} \int_0^{\sqrt{radius^2 - c^2}} x[c, r, t]^2 \, V_{xyz}[c, r, t] \, dr \, dc \, dt.$$

```
∫₀²π ∫₋ᵣₐ𝒹ᵢᵤₛ^radius ∫₀^√(radius²-c²)
      x[c, r, t]² Vxyz[c, r, t] dr dc dt
```
$\frac{324\pi}{5}$

Done.

T.1.c) Tubes

Remember the main unit normal and the binormal from the lesson on perpendicularity:

```
Clear[P, x, y, z, t, unittan, mainunitnormal, binormal]
x[t_] = t;
y[t_] = Sin[t];
z[t_] = Cos[t];
P[t_] = {x[t], y[t], z[t]};
{tlow, thigh} = {0, 2};
curve = ParametricPlot3D[
 Evaluate[P[t]], {t, tlow, thigh}, DisplayFunction → Identity];
scaler = 0.5;
unittan[t_] = P'[t] / √(P'[t] . P'[t]);
unittans = Table[Arrow[unittan[t], Tail → P[t],
 VectorColor → Blue, ScaleFactor → scaler], {t, tlow, thigh, 0.5}];
mainunitnormal[t_] = D[unittan[t], t] / √(Expand[D[unittan[t], t] . D[unittan[t], t]]);
mainnormalvectors = Table[Arrow[mainunitnormal[t], Tail → P[t],
 VectorColor → Red, ScaleFactor → scaler], {t, tlow, thigh, 0.5}];
binormal[t_] = unittan[t]×mainunitnormal[t];
binormalvectors = Table[Arrow[binormal[t], Tail → P[t],
 VectorColor → Red, ScaleFactor → scaler], {t, tlow, thigh, 0.5}];

everything = Show[curve, unittans, mainnormalvectors, binormalvectors,
 ViewPoint → CMView, PlotRange → All, BoxRatios → Automatic,
 AxesLabel → {"x", "y", "z"}, DisplayFunction → $DisplayFunction];
```
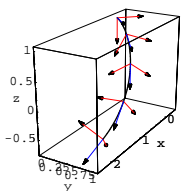
Lots of folks like to call what you see above by the name "moving frame".

Remember how much fun it was to make a tube consisting of all circles of a fixed radius centered on the curve and lying in planes perpendicular to the curve:

```
radius = 0.2;
{slow, shigh} = {0, 2 π};
tubeplot = ParametricPlot3D[Evaluate[
  P[t] + radius Cos[s] mainunitnormal[t] + radius Sin[s] binormal[t]],
  {t, tlow, thigh}, {s, slow, shigh}, ViewPoint → CMView,
  BoxRatios → Automatic, AxesLabel → {"x", "y", "z"}];
```
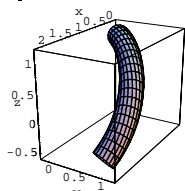

Put flat caps on each end of this tube, and then measure the volume of the resulting container.

□Answer:

To plot the skin of the container, you plotted

$P[t] +$ radius $\cos[s]$ mainunitnormal$[t] +$ radius $\sin[s]$ binormal$[t]$

where

radius = 0.2, slow ≤ s ≤ shigh, and tlow ≤ t ≤ thigh.

To describe the whole container and everything inside it, you go with

$\{x[r, s, t], y[r, s, t], z[r, s, t]\} =$
$P[t] + r \cos[s]$ mainunitnormal$[t] + r \sin[s]$ binormal$[t]$

with $0 \le r \le 0.2$, slow ≤ s ≤ shigh, and tlow ≤ t ≤ thigh.

Call the container and everything inside it R_{xyz} and measure its volume by calculating

$$\iiint_{R_{xyz}} dx\, dy\, dz$$
$$= \iiint_{R_{rst}} V_{xyz}[r, s, t]\, dr\, ds\, dt$$
$$= \int_{tlow}^{thigh} \int_{slow}^{shigh} \int_0^{0.2} V_{xyz}[r, s, t]\, dr\, ds\, dt.$$

Now if Mathematica can hack this, you're out of here.

```
Clear[x, y, z, r, s, t]
{x[r_, s_, t_], y[r_, s_, t_], z[r_, s_, t_]} = Chop[
  TrigExpand[P[t] + r Cos[s] mainunitnormal[t] + r Sin[s] binormal[t]]];
Clear[gradx, grady, gradz, Vxyz]
gradx[r_, s_, t_] =
 {D[x[r, s, t], r], D[x[r, s, t], s], D[x[r, s, t], t]};
grady[r_, s_, t_] =
 {D[y[r, s, t], r], D[y[r, s, t], s], D[y[r, s, t], t]};
gradz[r_, s_, t_] =
 {D[z[r, s, t], r], D[z[r, s, t], s], D[z[r, s, t], t]};
Vxyz[r_, s_, t_] = Simplify[TrigExpand[
  TrigReduce[Det[{gradx[r, s, t], grady[r, s, t], gradz[r, s, t]}]]]]
```
$\frac{r\,(2 - r\,Cos[s])}{\sqrt{2}}$

Haw.

volume of container $= \int_{tlow}^{thigh} \int_{slow}^{shigh} \int_0^{0.2} V_{xyz}[r, s, t]\, dr\, ds\, dt$

in cubic units is measured by:

```
volume = NIntegrate[Abs[Vxyz[r, s, t]], {t, tlow, thigh},
  {s, slow, shigh}, {r, 0, 0.2}]
0.355431
```

Done.

T.2) Integrating on solids bounded by sets of surfaces

□T.2.a)

A solid R is given as all points that are simultaneously between:
→ the planes $x + y + z = -2$, and $x + y + z = 7$,
→ the planes $x - y - z = -4$, and $x - y - z = 8$, and
→ the surfaces $x^3 - y + 2z = -2$, and $x^3 - y + 2z = 6$.
Measure the volume of this solid.

□Answer:

Going in blind is the thing to do.

Look again at the description of the surface:

The solid R was given as all points that are simultaneously between:

→ the planes $x + y + z = -2$, and $x + y + z = 7$,

→ the planes $x - y - z = -4$, and $x - y - z = 8$, and

→ the surfaces $x^3 - y + 2z = -2$, and $x^3 - y + 2z = 6$.

Put:

```
Clear[x, y, z, u, v, w]
{u[x_, y_, z_], ulow, uhigh} = {x + y + z, -2, 7}
{v[x_, y_, z_], vlow, vhigh} = {x - y - z, -4, 8}
{w[x_, y_, z_], wlow, whigh} = {x^3 - y + 2 z, -3, 6}
{x + y + z, -2, 7}
{x - y - z, -4, 8}
{x^3 - y + 2 z, -3, 6}
```

R_{uvw} is the rectangular box

ulow ≤ u ≤ uhigh, vlow ≤ v ≤ vhigh, and

wlow ≤ w ≤ whigh.

So:

$$\text{volume} = \iiint_R dx\, dy\, dz$$
$$= \iiint_{R_{uvw}} V_{xyz}[u, v, w]\, du\, dv\, dw$$

$$= \int_{wlow}^{whigh} \int_{vlow}^{vhigh} \int_{ulow}^{uhigh} V_{xyz}[u, v, w]\, du\, dv\, dw.$$

Calculate x[u, v, w], y[u, v, w], and z[u, v, w]:

```
xyzsolved =
 Solve[{u == u[x, y, z], v == v[x, y, z], w == w[x, y, z]}, {x, y, z}]
```
$\{\{y \to \frac{1}{24} (8 u + u^3 - 8 v + 3 u^2 v + 3 u v^2 + v^3 - 8 w),$
$z \to \frac{1}{24} (4 u - u^3 - 4 v - 3 u^2 v - 3 u v^2 - v^3 + 8 w), x \to \frac{u + v}{2}\}\}$
```
{x[u_, v_, w_], y[u_, v_, w_], z[u_, v_, w_]} = {x, y, z} /. xyzsolved[[1]]
```
$\{\frac{u + v}{2}, \frac{1}{24} (8 u + u^3 - 8 v + 3 u^2 v + 3 u v^2 + v^3 - 8 w),$
$\frac{1}{24} (4 u - u^3 - 4 v - 3 u^2 v - 3 u v^2 - v^3 + 8 w)\}$

Calculate the volume conversion factor:

```
Clear[gradx, grady, gradz, Vxyz]
gradx[u_, v_, w_] =
 {D[x[u, v, w], u], D[x[u, v, w], v], D[x[u, v, w], w]};
grady[u_, v_, w_] =
 {D[y[u, v, w], u], D[y[u, v, w], v], D[y[u, v, w], w]};
gradz[u_, v_, w_] =
 {D[z[u, v, w], u], D[z[u, v, w], v], D[z[u, v, w], w]};
Vxyz[u_, v_, w_] =
 Abs[Det[{gradx[u, v, w], grady[u, v, w], gradz[u, v, w]}]]
```
$\frac{1}{6}$

So:

$$\text{volume} = \iiint_R dx\, dy\, dz$$
$$= \int_{wlow}^{whigh} \int_{vlow}^{vhigh} \int_{ulow}^{uhigh} V_{xyz}[u, v, w]\, du\, dv\, dw.$$

is measured (in cubic units) by:

```
volume = ∫_wlow^whigh ∫_vlow^vhigh ∫_ulow^uhigh Vxyz[u, v, w] du dv dw
162
```

Really easy.
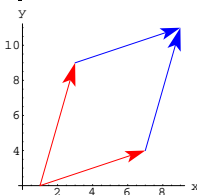
T.3) Parallelepipeds

Here are two vectors in two dimensions plotted with their tails at the same base point:

```
vector1 = {2, 7};
vector2 = {6, 2};
basepoint = {1, 2};
Show[Arrow[vector1, Tail → basepoint, VectorColor → Red],
  Arrow[vector2, Tail → basepoint, VectorColor → Red], Axes → True,
  AxesLabel → {"x", "y"}];
```
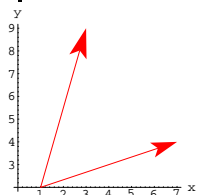

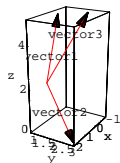Here is the parallelogram the two vectors and the basepoint determine:

```
Show[Arrow[vector1, Tail → basepoint, VectorColor → Red],
  Arrow[vector2, Tail → basepoint, VectorColor → Red],
  Arrow[vector1, Tail → basepoint + vector2, VectorColor → Blue],
  Arrow[vector2, Tail → basepoint + vector1, VectorColor → Blue],
  Axes → True, AxesLabel → {"x", "y"}];
```


Here are three vectors in 3D plotted with their tails at the same base point:

```
vector1 = {-1, 0, 3};
vector2 = {1, 2, -2};
vector3 = {-2, 1, 3};
basepoint = {1, 1, 2};
labels = {Graphics3D[Text["vector1", basepoint + 0.4 vector1]],
```

```
  Graphics3D[Text["vector2", basepoint + 0.5 vector2]],
  Graphics3D[Text["vector3", basepoint + 0.7 vector3]]};
threevectors =
  Show[Arrow[vector1, Tail → basepoint, VectorColor → Red],
   Arrow[vector2, Tail → basepoint, VectorColor → Red],
   Arrow[vector3, Tail → basepoint, VectorColor → Red], labels,
   Axes → True, AxesLabel → {"x", "y", "z"}, ViewPoint → CMView];
```


Here is the parallelepiped the three vectors and the basepoint determine:

```
"parallelepiped" is just jargon that fancy folks use to name a box all
of whose faces are parallelograms lying in pairs of parallel planes.
parallelepiped =
  Show[Arrow[vector1, Tail → basepoint, VectorColor → Red],
   Arrow[vector2, Tail → basepoint, VectorColor → Red],
   Arrow[vector3, Tail → basepoint, VectorColor → Red], Arrow[vector2,
    Tail → basepoint + vector1, VectorColor → Blue], Arrow[vector3,
    Tail → basepoint + vector1, VectorColor → Blue], Arrow[vector1,
    Tail → basepoint + vector2, VectorColor → Blue], Arrow[vector3,
    Tail → basepoint + vector2, VectorColor → Blue], Arrow[vector1,
    Tail → basepoint + vector3, VectorColor → Blue], Arrow[vector2,
    Tail → basepoint + vector3, VectorColor → Blue], Arrow[vector3,
    Tail → basepoint + vector1 + vector2, VectorColor → Blue], Arrow[
    vector1, Tail → basepoint + vector2 + vector3, VectorColor → Blue],
   Arrow[vector2, Tail → basepoint + vector1 + vector3,
    VectorColor → Blue], Arrow[vector3,
    Tail → basepoint + vector1 + vector2, VectorColor → Blue],
   labels, Axes → True, AxesLabel → {"x", "y", "z"}, ViewPoint → CMView];
```
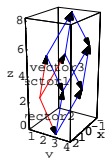

□ **T.3.a.i)**

Agree that R is the region consisting of everything on and inside the parallelepiped, and measure the volume of R by transforming the integral

$$\int\int\int_R d\mathrm{x}\,d\mathrm{y}\,d\mathrm{z}.$$

□ **Answer:**

The parallelepiped has six planar sides; they come in parallel pairs.

A normal vector for one of the pairs is the cross product

vector1 × vector2;

a normal vector for another of the pairs is the cross product

vector1 × vector3

and a normal vector for the last pair is the cross product

vector2 × vector3:

```
Clear[normal]
{normal[1], normal[2], normal[3]} =
  {vector1 × vector2, vector1 × vector3, vector2 × vector3}
{{-6, 1, -2}, {-3, -3, -1}, {8, 1, 5}}
```

Three of these six planes go through the basepoint.

The other three planes go through the tip of

basepoint + vector1 + vector2 + vector3.

Here are their equations:

```
Clear[k]
Table[
```

```
  {Expand[normal[k] . ({x, y, z} - basepoint)] == 0, Expand[normal[k] .
     ({x, y, z} - (basepoint + vector1 + vector2 + vector3))] == 0},
  {k, 1, 3}]
{{9 - 6 x + y - 2 z == 0, 2 - 6 x + y - 2 z == 0},
 {8 - 3 x - 3 y - z == 0, 15 - 3 x - 3 y - z == 0},
 {-19 + 8 x + y + 5 z == 0, -26 + 8 x + y + 5 z == 0}}
```

Introduce uvw-space:

```
Clear[u, v, w]
{u[x_, y_, z_], ulow, uhigh} = {-6 x + y - 2 z, -9, -2}
{v[x_, y_, z_], vlow, vhigh} = {-3 x - 3 y - z, -15, -8}
{w[x_, y_, z_], wlow, whigh} = {8 x + y + 5 z, 19, 26}
{-6 x + y - 2 z, -9, -2}
{-3 x - 3 y - z, -15, -8}
{8 x + y + 5 z, 19, 26}
```

R$_{uvw}$ is the rectangular box

ulow ≤ u ≤ uhigh, vlow ≤ v ≤ vhigh, and

wlow ≤ w ≤ whigh.

So:

$$\text{volume} = \int\int\int_R d\mathrm{x}\,d\mathrm{y}\,d\mathrm{z}$$
$$= \int\int\int_{R_{uvw}} V_{xyz}[u, v, w]\,du\,dv\,dw$$
$$= \int_{wlow}^{whigh}\int_{vlow}^{vhigh}\int_{ulow}^{uhigh} V_{xyz}[u, v, w]\,du\,dv\,dw.$$

Calculate x[u, v, w], y[u, v, w], and z[u, v, w]:

```
xyzsolved =
  Solve[{u == u[x, y, z], v == v[x, y, z], w == w[x, y, z]}, {x, y, z}]
{{x → 1/7 (-2 u - v - w), y → 1/7 (u - 2 v), z → 1/7 (3 u + 2 v + 3 w)}}
```

Extract x[u, v, w], y[u, v, w], and z[u, v, w]:

```
{x[u_, v_, w_], y[u_, v_, w_], z[u_, v_, w_]} =
  Expand[{x, y, z} /. xyzsolved[[1]]]
{-2u/7 - v/7 - w/7, u/7 - 2v/7, 3u/7 + 2v/7 + 3w/7}
```

Calculate the volume conversion factor:

```
Clear[gradx, grady, gradz, Vxyz]
gradx[u_, v_, w_] =
  {D[x[u, v, w], u], D[x[u, v, w], v], D[x[u, v, w], w]};
```

```
grady[u_, v_, w_] =
  {D[y[u, v, w], u], D[y[u, v, w], v], D[y[u, v, w], w]};
gradz[u_, v_, w_] =
  {D[z[u, v, w], u], D[z[u, v, w], v], D[z[u, v, w], w]};
Vxyz[u_, v_, w_] =
  Abs[Det[{gradx[u, v, w], grady[u, v, w], gradz[u, v, w]}]]
```
$\frac{1}{49}$

So:

$$\text{volume} = \iiint_R d\text{x}\,d\text{y}\,d\text{z}$$
$$= \int_{\text{wlow}}^{\text{whigh}} \int_{\text{vlow}}^{\text{vhigh}} \int_{\text{ulow}}^{\text{uhigh}} V_{xyz}[u, v, w]\,d u\,d v\,d w.$$

is measured by:

```
volume = ∫_wlow^whigh ∫_vlow^vhigh ∫_ulow^uhigh Vxyz[u, v, w] du dv dw
```
7

Done.

□**T.3.a.ii)**

    To run this successfully, all the cells in part a.i)
              above should be active.

Stay with the same three vectors
     vector1, vector2, and vector3
as in the last part.
Here they are together with the parallelepiped they determine:

```
Show[parallelepiped];
```
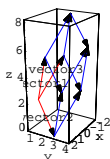

The volume of this parallelepiped measures out to:

```
volume = ∫_wlow^whigh ∫_vlow^vhigh ∫_ulow^uhigh Vxyz[u, v, w] du dv dw
```
7

Now look at this calculation of vector1 . (vector2×vector3):

```
boxproduct = Abs[vector1 . vector2 × vector3]
```
7

Compare:

```
volume == boxproduct
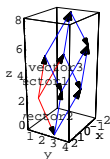```
True

Do you think this was an accident?

□**Answer:**

Gimmee a break! This is getting old.

---
**In mathematics, there are no accidents.**

---

To see why this is not an accident, look at the parallelpided again:

```
Show[parallelepiped];
```


Look at the face F determined by vector2 and vector3 and remember that

     area of F = ||vector2×vector3||.

Second, letting a be the angle between vector1 and vector2×vector3, you have

     |vector1 . vector2×vector3|

     = ||vector1|| ||vector2×vector3|| |Cos[a]|

     = ||vector1|| (area of F) |Cos[a]|

     = ||vector1|| |Cos[a]| (area of F).

Because vector2×vector3 is perpendicular to F,

     ||vector1|| |Cos[a]|

measures the perpendicular height of the parallelepiped as measured from F.

The upshot:

     |vector1 . vector2×vector3|

     = (area of a face) (perpendicular height)

     = volume of the parallelepiped.

□**T.3.a.iii)**

Now everyone knows that you can calculate the volume of the parallelepiped determined by
     vector1, vector2, and vector3
simply by calculating
     vector1 . (vector2×vector3).
Was all that jazz on transformation of the 3D integral in part i) a waste of time?

□**Answer:**

    To run this successfully, all the cells in part a.i)
              above should be active.

If all you wanted to do was to measure the volume of the

parallelepided, the transformation turns out to be a waste of time.

On the other hand if you had wanted to calculate an integral like

$$\iiint_R \text{x}^2\,d\text{x}\,d\text{y}\,d\text{z}$$

where R is everything inside and on the parallelepiped, then your time

on setting up the transformation would not have been wasted.

To wit:

$$\iiint_R \text{x}^2\,d\text{x}\,d\text{y}\,d\text{z}$$
$$= \iiint_{R_{uvw}} \text{x}[u, v, w]^2\, V_{xyz}[u, v, w]\,d u\,d v\,d w$$

$$= \int_{\text{wlow}}^{\text{whigh}} \int_{\text{vlow}}^{\text{vhigh}} \int_{\text{ulow}}^{\text{uhigh}} \text{x}[u, v, w]^2\, V_{xyz}[u, v, w]\,d u\,d v\,d w.$$

is given by:

```
∫_wlow^whigh ∫_vlow^vhigh ∫_ulow^uhigh x[u, v, w]² Vxyz[u, v, w] du dv dw
```
$\frac{7}{2}$

Done in two shakes of a lamb's tail.

T.4) Switching the order of integration

□**T.4.a.)**

Here is a calculation of the triple integral
$$\int_0^1 \int_0^2 \int_0^{2-x} \text{z}^4\,d\text{z}\,d\text{x}\,d\text{y}$$

```
Clear[x, y, z]
∫_0^1 ∫_0^2 ∫_0^{2-x} z⁴ dz dx dy
```
$\frac{32}{15}$

Rewrite the integral in the order
$$\int_\square^\square \int_\square^\square \int_\square^\square \text{z}^4\,d\text{y}\,d\text{x}\,d\text{z},$$
filling in the □'s with the appropriate limits.
Confirm your answer by calculating it.

□**Answer:**

To do this, it's a good idea to pin down the region R in three

dimensions such that

$$\iiint_R \text{z}^4\,d\text{x}\,d\text{y}\,d\text{z} = \int_0^1 \int_0^2 \int_0^{2-x} \text{z}^4\,d\text{z}\,d\text{x}\,d\text{y}.$$
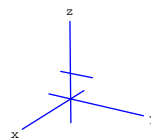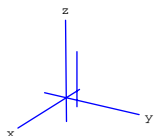
Try to take a look:

The first integral

$$\int_0^{2-x} \text{z}^4\,d\text{z}$$

fixes x and y and integrates over the stick that runs from

{x, y, 0} to {x, y, 4 − x}:

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```
{x0, y0} = {0.4, 0.6};
stickinslice =
 Graphics3D[{Blue, Line[{{x0, y0, 0}, {x0, y0, 2 - x0}}]}];
threedims = Axes3D[2.3];

Show[stickinslice, threedims, Boxed → False, PlotRange → All,
 ViewPoint → CMView];
```
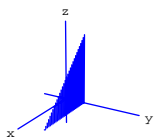
The second integral

$$\int_0^2 \int_0^{2-x} z^4 \, dz \, dx$$

holds y fixed, but releases x and integrates over the slice of the solid

swept out by the sticks as x runs from 0 to 2:

```
Clear[x]
sticksinslice = Table[Graphics3D[

   {Blue, Line[{{x, y0, 0}, {x, y0, 2 - x}}]}], {x, 0, 2, 1/15}];
slice = Show[sticksinslice, threedims, Boxed → False,
   PlotRange → All, ViewPoint → CMView];
```
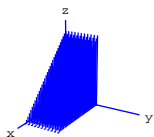
The limits on the third integral

$$\int_0^1 \int_0^2 \int_0^{2-x} z^4 \, dz \, dx \, dy.$$

release y and integrate over the accumulated slices as y runs from 0 to

1.

So R looks something like this:

```
Clear[y]
allsticks = Table[Graphics3D[{Blue, Line[{{x, y, 0}, {x, y, 2 - x}}]}],

   {x, 0, 2, 1/15}, {y, 0, 1, 1/10}];
Rplot = Show[allsticks, threedims, Boxed → False, PlotRange → All,
   ViewPoint → CMView];
```

R is the region whose top is the plane z = 2 − x, whose bottom is the

plane z = 0, and whose sides are the planes y = 0, y = 1, and x = 0.

To write the integral in the order

$$\int_\square^\square \int_\square^\square \int_\square^\square z^4 \, dy \, dx \, dz,$$

you've got to describe the same region, but first starting with sticks

parallel to the y-axis. This means x and z are fixed and the stick runs

from y = 0 to y = 1:

This gives you the first integral

$$\int_0^1 z^4 \, dy$$

which integates over a stick:

```
{x0, z0} = {0.5, 1};
newstickinslice = Graphics3D[{Blue, Line[{{x0, 0, z0}, {x0, 1, z0}}]}];
Show[newstickinslice, threedims, Boxed → False, PlotRange → All,
   ViewPoint → CMView];
```

Remember:

R is the region whose top is the plane z = 2 − x, whose bottom is the

plane z = 0, and whose sides are the planes y = 0, y = 1, and x = 0.

The second integral

$$\int_\square^\square \int_0^1 z^2 \, dy \, dx$$

holds z fixed but releases x and integrates over the slice composed of

the sticks as x runs from x = 0 to the plane z = 2 − x. The second

integral is

$$\int_0^{2-z} \int_0^1 z^2 \, dy \, dx$$

which integrates over a slice:

```
Clear[x]
newsticksinslice = Table[Graphics3D[

   {Blue, Line[{{x, 0, z0}, {x, 1, z0}}]}], {x, 0, 2 - z0, (2 - z0)/10}];
newslice = Show[newsticksinslice, threedims, Boxed → False,
   PlotRange → All, ViewPoint → CMView];
```

The third integral

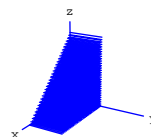$$\int_\square^\square \int_0^{2-z} \int_0^1 z^2 \, dy \, dx \, dz$$

releases z and integrates over the accumulated slices as z runs from as

small as it can be to as big as it can be. Remembering that R is the

region whose top is the plane z = 2 − x, whose bottom is the plane

z = 0, and whose sides are the planes y = 0, y = 1, and x = 0, you can

see that the smallest z is 0 and the biggest z is 2 (which happens when

x = 0).

The third integral is

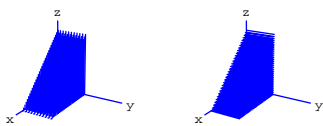$$\int_0^2 \int_0^{2-z} \int_0^1 z^2 \, dy \, dx \, dz$$

which integrates over R:

```
allnewsticks = Table[Graphics3D[{Blue, Line[{{x, 0, z}, {x, 1, z}}]}],

   {z, 0, 2, 1/15}, {x, 0, 2 - z, 1/15}];
newRplot = Show[allnewsticks, threedims, Boxed → False,
   PlotRange → All, ViewPoint → CMView];
```

Compare this to the original R plot:

```
Show[GraphicsArray[{Rplot, newRplot}]];
```

The same solid region but described with different sticks; that's why

$$\int_0^2 \int_0^{2-z} \int_0^1 z^4 \, dy \, dx \, dz = \int_0^1 \int_0^2 \int_0^{2-x} z^4 \, dz \, dx \, dy:$$

```
∫₀² ∫₀²⁻ᶻ ∫₀¹ z⁴ dy dx dz
```

$\frac{32}{15}$

```
∫₀¹ ∫₀² ∫₀²⁻ˣ z⁴ dz dx dy
```

$\frac{32}{15}$

Copacetic.

VC.08 Transforming 3D Integrals
Give it a Try!

Experience with the starred problems will be useful for understanding developments later in the course.

G.1) 3D integrals*

☐G.1.a)

Calculate
$$\iiint_R x^2 \, y^2 \, z^2 \, dx \, dy \, dz$$
where R is the box with
$$0 \le x \le 2, \ 1 \le y \le 3, \text{ and } -2 \le z \le 2.$$

☐G.1.b)

Calculate
$$\iiint_{R_{xyz}} \left(\left(\tfrac{x}{3}\right)^2 + \left(\tfrac{y}{2}\right)^2 \right) dx \, dy \, dz$$

where R_{xyz} is the region consisting of everything inside and on the elliptical cylinder with sides
$$\left(\tfrac{x}{3}\right)^2 + \left(\tfrac{y}{2}\right)^2 = 1,$$
and bottom z = 0, and top z = 4.

☐Tip:

Go with

$x[r, s, t] = 3 \, r \, Cos[s]$

$y[r, s, t] = 2 \, r \, Sin[s]$

$z[r, s, t] = t.$

☐G.1.c)

Calculate
$$\iiint_R x^2 \, y^2 \, z^2 \, dx \, dy \, dz,$$
where R is the solid whose top skin is the surface
$$z = 8 - x^2 - y^2,$$
and whose bottom skin is the surface
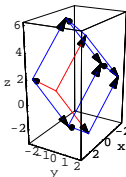$$z = x^2 + y^2.$$

☐G.1.d)

Calculate
$$\iiint_R x^2 \, y^2 \, z^2 \, dx \, dy \, dz,$$
where R is the solid inside the parallelpiped plotted below:

```
vector1 = {3, 0, 2};
vector2 = {0, 3, -3};
vector3 = {-3, 1, 4};
basepoint = {0, -2, 0};
Show[Arrow[vector1, Tail → basepoint, VectorColor → Red],
 Arrow[vector2, Tail → basepoint, VectorColor → Red],
 Arrow[vector3, Tail → basepoint, VectorColor → Red], Arrow[vector2,
  Tail → basepoint + vector1, VectorColor → Blue], Arrow[vector3,
  Tail → basepoint + vector1, VectorColor → Blue], Arrow[vector1,
```

```
  Tail → basepoint + vector2, VectorColor → Blue], Arrow[vector3,
  Tail → basepoint + vector2, VectorColor → Blue], Arrow[vector1,
  Tail → basepoint + vector3, VectorColor → Blue], Arrow[vector2,
  Tail → basepoint + vector3, VectorColor → Blue], Arrow[vector3,
  Tail → basepoint + vector1 + vector2, VectorColor → Blue], Arrow[
  vector1, Tail → basepoint + vector2 + vector3, VectorColor → Blue],
 Arrow[vector2, Tail → basepoint + vector1 + vector3,
  VectorColor → Blue], Arrow[vector3,
  Tail → basepoint + vector1 + vector2, VectorColor → Blue],
 Axes → True, AxesLabel → {"x", "y", "z"}, ViewPoint → CMView];
```
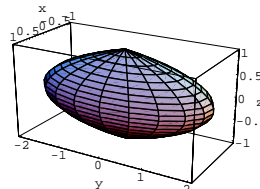

G.2) Volume*

☐G.2.a.i)

Here's the plot of the outside skin of a solid:

```
Clear[s, t]
skin =
 ParametricPlot3D[{(1 - s²) Cos[t], 2 (1 - s²) Sin[t], s}, {s, -1, 1},
  {t, 0, 2 π}, ViewPoint → CMView, AxesLabel → {"x", "y", "z"}];
```
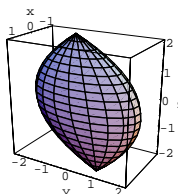

Measure the volume enclosed by this skin.

☐G.2.a.ii)

Here's the plot of the outside skin of a related solid:

```
Clear[s, t]
skin = ParametricPlot3D[
  {-s, -s, 2 s} + {(1 - s²) Cos[t], 2 (1 - s²) Sin[t], 0}, {s, -1, 1},
  {t, 0, 2 π}, ViewPoint → CMView, AxesLabel → {"x", "y", "z"}];
```


Measure the volume enclosed by this skin.

☐G.2.b)

A solid R is given as all points that are simultaneously between:
→ the planes x + y + z = −2, and x + y + z = 5,
→ the planes x + y − z = −3, and x + y − z = 7, and
→ the surfaces $x - y + 2 z^3 = 0$, and $x - y + 2 z^3 = 5$.
Measure the volume of this solid.

G.3) Drilling and slicing spheres

One way to see how to plot the sphere
$$x^2 + y^2 + z^2 = \text{radius}^2$$
is to slice it with the plane
$$z = c, \text{ where } -\text{radius} \le c \le \text{radius},$$
and then to think about what you get.
What you get is the circle
$$x^2 + y^2 = \text{radius}^2 - c^2$$
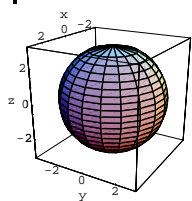plotted in the plane z = c.
Now when you go with a given radius, you can plot the sphere
$$x^2 + y^2 + z^2 = \text{radius}^2.$$
Take a look in the case that radius = 3:

```
Clear[c, t]
radius = 3;
{clow, chigh} = {-radius, radius};
```

```
{tlow, thigh} = {0, 2 π};
ParametricPlot3D[{√(radius² - c²) Cos[t], √(radius² - c²) Sin[t], c},
 {c, clow, chigh}, {t, tlow, thigh}, ViewPoint → CMView,
 Axes → True, BoxRatios → Automatic, AxesLabel → {"x", "y", "z"}];
```


□G.3.a.i)

Use a 3D integral to measure the volume of this sphere.

□G.3.a.ii)

Use a 3D integral to come up with a formula for a function volume[radius] that measures the volume of a sphere of a given radius.

□G.3.b.i)

Here's a sphere of radius 2 shown in true scale with an axis running from top to bottom:

```
Clear[c, t]
radius = 2;
{clow, chigh} = {-radius, radius};
{tlow, thigh} = {0, 2 π};
sphereplot =
 ParametricPlot3D[{√(radius² - c²) Cos[t], √(radius² - c²) Sin[t], c},
  {c, clow, chigh}, {t, tlow, thigh}, DisplayFunction → Identity];
Show[sphereplot, Graphics3D[
  {Thickness[0.01], Blue, Line[{{0, 0, -2.5}, {0, 0, 2.5}}]}],
 ViewPoint → CMView, Axes → True, BoxRatios → Automatic,
 AxesLabel → {"x", "y", "z"}, DisplayFunction → $DisplayFunction];
```
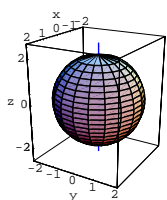

Plot what you get after you take a circular drill of radius 1 and bore a straight circular hole through the sphere, centering the drill on the plotted axis.

□G.3.b.ii)

Go back to the set-up in part b.i) immediately above, and imagine that the sphere is the skin of a solid ball made of redwood. Measure the volume of the redwood that remains after you take a circular drill of radius 1 and bore a straight circular hole through the sphere, centering the drill on the plotted axis.

□G.3.c)

Here's a sphere whose top has been sliced off:

```
Clear[c, t]
radius = 2;
{clow, chigh} = {-radius, 1};
{tlow, thigh} = {0, 2 π};
sphereplot =
 ParametricPlot3D[{√(radius² - c²) Cos[t], √(radius² - c²) Sin[t], c},
  {c, clow, chigh}, {t, tlow, thigh}, ViewPoint → CMView,
  Axes → True, BoxRatios → Automatic, AxesLabel → {"x", "y", "z"}];
```
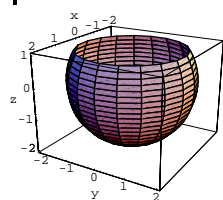

Think of this as a container for your favorite cold liquid refreshment and measure the volume of liquid this container can hold.

□G.3.d)

A cubic foot of water weighs 62.4 pounds. A cubic foot of seasoned balsa wood weighs 7.3 pounds.

If you carve a perfect sphere 2 feet in diameter from a block of balsa wood, paint it with waterproof paint, let the paint dry, throw it into a pond, and wait until it stops bobbing, then how deeply will it be submerged?

□Tip:

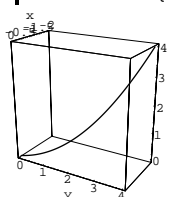The floating balsa sphere will displace the volume of water that weighs as much as the balsa sphere.

One way to communicate your answer is with a plot.

G.4) Tubes, horns, and squashed doughnuts*

□G.4.a.i)

Here's a base curve:

```
Clear[s, P]
P[s_] = {-s, 2 s, s²};
curve = ParametricPlot3D[Evaluate[P[s]], {s, 0, 2},
 AxesLabel → {"x", "y", "z"}, ViewPoint → CMView];
```


Plot the tube whose skin consists of each circle of a radius 1.4 centered on the curve and lying in the plane that cuts the curve perpendicularly at the center of the circle.

□G.4.a.ii)

Imagine that flat caps have been attached to the open ends of the tube in part a.i) above, and measure the volume of the resulting container.

□G.4.b)

Look at this horn:

```
Clear[s, P]
P[s_] = {-s, 2 s, s²};
unittan[s_] = P'[s] / √(P'[s] . P'[s]);
mainunitnormal[s_] = N[unittan'[s] / √(Expand[unittan'[s] . unittan'[s]])];
Clear[binormal]
binormal[s_] = unittan[s] × mainunitnormal[s];
Clear[hornskin, r]
r[s_] = 0.25 s;
hornskin[s_, t_] = Together[
  P[s] + r[s] Cos[t] mainunitnormal[s] + r[s] Sin[t] binormal[s]];

ParametricPlot3D[Evaluate[hornskin[s, t]], {s, 0, 2}, {t, 0, 2 π},
 AxesLabel → {"x", "y", "z"}, ViewPoint → CMView];
```
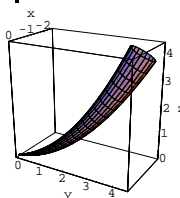

Imagine that a flat cap has been attached to the open end, and measure the volume of the resulting container.

□G.4.c) Squashed doughnuts

Here is a squashed doughnut:

```
Clear[P, s, unittan, mainunitnormal, binormal]
P[s_] := {6 Cos[s], 6 Sin[s], 0};
unittan[s_] = ─────────P'[s]─────────;
              √TrigExpand[P'[s].P'[s]]
mainunitnormal[s_] = N[──────────unittan'[s]──────────────];
                       √TrigExpand[unittan'[s].unittan'[s]]
binormal[s_] = N[TrigExpand[unittan[s]×mainunitnormal[s]]];
Clear[donutskin, rad, t]
rad[s] = Sin[s]² + 0.5;
donutskin[s_, t_] =
 P[s] + rad[s] Cos[t] mainunitnormal[s] + rad[s] Sin[t] binormal[s];

ParametricPlot3D[Evaluate[donutskin[s, t]], {s, 0, 2 π}, {t, 0, 2 π},
 PlotPoints → {30, Automatic}, AxesLabel → {"x", "y", "z"},
 ViewPoint → CMView];
```
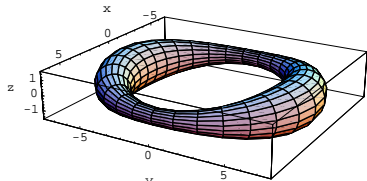

Try to explain why the plot came out the way it did, and then measure
the volume inside this squashed treat.

G.5) Measuring when you can't see

□G.5.a.i)

Here is a modest curve in three dimensions:

```
Clear[curveplotter, t]
{tlow, thigh} = {0, 4};
curveplotter[t_] = {3 Cos[2 t], 3 t Sin[t], 3 t};
curve = ParametricPlot3D[Evaluate[curveplotter[t]],
  {t, tlow, thigh}, DisplayFunction → Identity];

Show[
 curve, ViewPoint → CMView, Axes → True, AxesLabel → {"x", "y", "z"},
 PlotRange → All, DisplayFunction → $DisplayFunction];
```
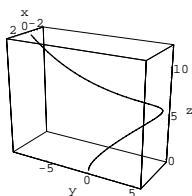

Here is the same curve shown with a couple of sample circles parallel
to the xy-plane and centered on the curve:

```
Clear[s]
circle1 =
 ParametricPlot3D[Evaluate[curveplotter[1] + 2 {Cos[s], Sin[s], 0}],
  {s, 0, 2 π}, DisplayFunction → Identity];
circle2 =
 ParametricPlot3D[Evaluate[curveplotter[2] + 2 {Cos[s], Sin[s], 0}],
  {s, 0, 2 π}, DisplayFunction → Identity];

Show[curve, circle1, circle2, ViewPoint → CMView,
 Axes → True, AxesLabel → {"x", "y", "z"}, PlotRange → All,
 DisplayFunction → $DisplayFunction];
```
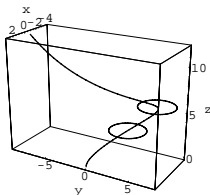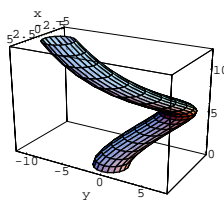

Here's what you get when you make the tube consisting of all such
circles:

```
Clear[tubeplotter, s, t]
tubeplotter[s_, t_] = curveplotter[t] + 2 {Cos[s], Sin[s], 0};
tube = ParametricPlot3D[Evaluate[tubeplotter[s, t]],
  {t, tlow, thigh}, {s, 0, 2 π}, DisplayFunction → Identity];

tubeplot = Show[tube, curve, ViewPoint → CMView,
 Axes → True, AxesLabel → {"x", "y", "z"}, PlotRange → All,
 DisplayFunction → $DisplayFunction];
```


Imagine that the ends of this tube are covered with flat plates, and
measure the volume of the resulting container.

□G.5.a.ii)

In part i), the core curve was plotted by
    curveplotter[t] = {3 Cos[2 t], 3 t Sin[t], 3 t}
with 0 ≤ t ≤ 4.
This time, suppose the core curve is given by
    curveplotter[t] = {f[t], g[t], 3 t}
with 0 ≤ t ≤ 4, where f[t] and g[t] are unspecified functions. When
you center circles of radius 2 on the curve so that they are all parallel
to the xy-plane as above, you can't plot the tube because you don't
have concrete formulas for f[t] and g[t]. Nevertheless, you can
imagine that flat plates are put on the open end and you can measure
the volume of the resulting container.
Do it.

□G.5.b)

A solid R is given as all points that are simultaneously between:
→ the planes x + y + 2 z = −1, and x + y + 2 z = 5,
→ the planes x − y − 2 z = −3, and x − y − 2 z = 6, and
→ the surfaces Sinh[x] − y + 2 z = −1, and Sinh[x] − y + 2 z = 2.
When you try to measure the volume of this solid, you go with:

```
Clear[x, y, z, u, v, w]
{u[x_, y_, z_], ulow, uhigh} = {x + y + 2 z, -1, 5};
{v[x_, y_, z_], vlow, vhigh} = {x - y - 2 z, -3, 6};
{w[x_, y_, z_], wlow, whigh} = {Sinh[x] - y + 2 z, -1, 2};
```

Then you observe that R_{uvw} is the rectangular box
    ulow ≤ u ≤ uhigh,
    vlow ≤ v ≤ vhigh, and
    wlow ≤ w ≤ whigh.
So:
$$\text{volume} = \iiint_R d\text{x}\,d\text{y}\,d\text{z}$$
$$= \iiint_{R_{uvw}} V_{xyz}[u, v, w]\,d\text{u}\,d\text{v}\,d\text{w}$$
$$= \int_{wlow}^{whigh} \int_{vlow}^{vhigh} \int_{ulow}^{uhigh} V_{xyz}[u, v, w]\,d\text{u}\,d\text{v}\,d\text{w}.$$

```
xyzsolved =
 Solve[{u == u[x, y, z], v == v[x, y, z], w == w[x, y, z]}, {x, y, z}]
```

$\{\{y \to \frac{1}{4} (u - v - 2 w + 2 Sinh[\frac{u+v}{2}]), z \to \frac{1}{8} (u - v + 2 w - 2 Sinh[\frac{u+v}{2}]),$
$x \to \frac{u+v}{2}\}\}$

Instead of finding x[u, v, w], y[u, v, w], and z[u, v, w] in order to
calculate the volume conversion factor
    $V_{xyz}[u, v, w]$
for converting uvw-volume measurements into xyz-volume
measurements, you can calculate
    $V_{uvw}[x, y, z]$
which converts xyz-volume measurements into uvw-volume
measurements:

```
Clear[gradu, gradv, gradw]
gradu[x_, y_, z_] = {D[u[x, y, z], x], D[u[x, y, z], y], D[u[x, y, z], z]};
gradv[x_, y_, z_] = {D[v[x, y, z], x], D[v[x, y, z], y], D[v[x, y, z], z]};
gradw[x_, y_, z_] = {D[w[x, y, z], x], D[w[x, y, z], y], D[w[x, y, z], z]};
Vuvw[x_, y_, z_] =
 Abs[Det[{gradu[x, y, z], gradv[x, y, z], gradw[x, y, z]}]]
```
8

Look at this and use just a smidgen of common sense to figure out
what
    $V_{xyz}[u, v, w]$
has got to be. Once you have
    $V_{xyz}[u, v, w]$
nailed down, use it to help to measure the volume of the solid
described at the beginning.

G.6) Average values, centroids, and centers of mass

□G.6.a.i)

Agree that avf$_R$ stands for the average value of f[x, y, z] as {x, y, z} varies through a specified solid region R in xyz-space. The formula for avf$_R$ is

$$\text{avf}_R = \frac{\int\int\int_R f[x,y,z]\, dx\, dy\, dz}{\int\int\int_R dx\, dy\, dz}.$$

How do you know that avf$_R$ is a number that falls between the minimum value of f[x, y, z] on R and the maximum value of f[x, y, z] on R?

□G.6.a.ii)

Suppose f[x, y, z] is not constant on R.
Why do you expect that there are points
$\quad$ {x$_1$, y$_1$, z$_1$} and {x$_2$, y$_2$, z$_2$}
in R with
$\quad$ f[x$_1$, y$_1$, z$_1$] < avf$_R$ and
$\quad$ f[x$_2$, y$_2$, z$_2$] > avf$_R$?

□G.6.b)

Calculate the average value of f[x, y, z] = $x^2 + y^2 + z^2$ on the trapezoidal box
$\quad 3 \le x \le 7,$
$\quad 5 \le y \le 9,$ and
$\quad 0 \le z \le 6 + 6x + 6y.$

□G.6.c) Centroids

You get the centroid
$\quad$ {x$_c$, y$_c$, z$_c$}
of a solid R by setting
$\quad$ x$_c$ = the average value of f[x, y, z] = x on R,
$\quad$ y$_c$ = the average value of g[x, y, z] = y on R, and
$\quad$ z = the average value of h[x, y, z] = z on R.
Calculate the centroid of the trapezoidal box
$\quad 3 \le x \le 7,$
$\quad 5 \le y \le 9,$ and
$\quad 0 \le z \le 6 + 6x + 6y.$

□G.6.d.i) Weighted averages

Take a nonnegative function p[x, y, z] and think of it this way:
The bigger p[x, y, z] is the more weight the point {x, y, z} has.
The weighted average value of a function f[x, y, z] with respect to p[x, y, z] on a region R in three dimensions is the number

$$\text{avfp}_R = \frac{\int\int\int_R f[x,y,z]\, p[x,y,z]\, dx\, dy\, dz}{\int\int\int_R p[x,y,z]\, dx\, dy\, dz}.$$

□G.6.d.ii)

Given a weight function p[x, y, z], how do you know that avfp$_R$ is a number that falls between the minimum value of f[x, y, z] on R and the maximum value of f[x, y, z] on R?

□G.6.d.iii)

Take p[x, y, z] = e^{-x-y-z}.
Calculate the weighted average of f[x, y, z] = $x^2 + y^2 + z^2$ with respect to p[x, y, z] of the trapezoidal box
$\quad 0 \le x \le 1,$
$\quad 0 \le y \le 2,$ and
$\quad 0 \le z \le 1 + 2x + y.$

□G.6.e) Center of mass

A solid R has density p[x, y, z] $\frac{\text{grams}}{\text{cm}^3}$ at a point {x, y, z}.
You get the center of mass
$\quad$ {x$_\mu$, y$_\mu$, z$_\mu$}
of a solid R by setting
$\quad$ x$_\mu$ = the average value of f[x, y, z] = x with respect to p[x, y, z] on R,
$\quad$ y$_\mu$ = the average value of g[x, y, z] = y with respect to p[x, y, z] on R, and
$\quad$ z$_\mu$ = the average value of h[x, y, z] = z with respect to p[x, y, z] on R.
The top of a solid is the domed surface
$\quad$ z = $8 - x^2 - y^2$,
and the bottom is the paraboloid
$\quad$ z = $x^2 + y^2$.
The density p[x, y, z] at a point {x, y, z} in the solid is proportional to the distance of {x, y, z} from the plane z = 0, so for z ≠ 0,
p[x, y, z] = c z, where c is a positive constant of proportionality.
Calculate the center of mass {x$_\mu$, y$_\mu$, z$_\mu$} of the solid.
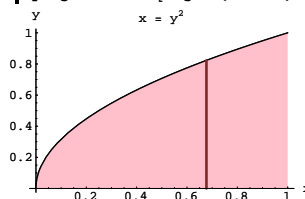
G.7) Switching the order of integration

□G.7.a.i)

Here's a nice little region in the xy-plane.

```
Clear[x, y, region, pregion, vline, x0];
Needs["Graphics`FilledPlot`"];
region = FilledPlot[√x, {x, 0, 1},
    AspectRatio → 1 / GoldenRatio, Fills → Pink, AxesLabel → {"x", "y"},
    PlotLabel → "x = y² ", DisplayFunction → Identity];
Show[region, DisplayFunction → $DisplayFunction];
```
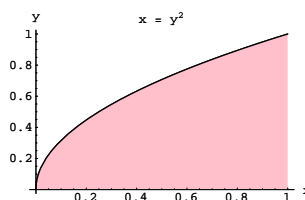

You want to calculate $\int\int_R (x^2 + y^3)\, dA$. Here's a view of the region with a randomly placed vertical line.

```
x0 = Random[];
vline =
  Graphics[{Thickness[0.01], Brown, Line[{{x0, 0}, {x0, √x0 }}]}];
pregion = Show[region, vline, DisplayFunction → $DisplayFunction];
```
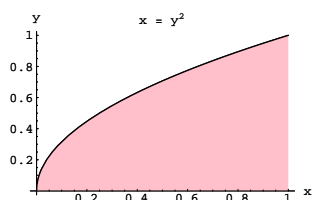

Use what you see to fill in the boxes so that
$$\int\int_R (x^2 + y^3)\, dA = \int_\square^\square \int_\square^\square (x^2 + y^3)\, dy\, dx.$$
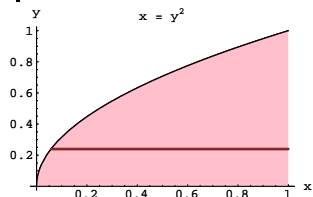
□G.7.a.ii)

Here's that nice little region in the xy-plane again.

```
Clear[x, y, region, pregion, hline, y0];
region = FilledPlot[√x, {x, 0, 1},
    AspectRatio → 1 / GoldenRatio, Fills → Pink, AxesLabel → {"x", "y"},
    PlotLabel → "x = y² ", DisplayFunction → Identity];
Show[region, DisplayFunction → $DisplayFunction];
```

You want to calculate $\int\int_R (x^2 + y^3)\, d\mskip2mu A$. Here's a view of the region with a randomly placed horizontal line.

```
y0 = Random[];
hline = Graphics[{Thickness[0.01], Brown, Line[{{y0^2, y0}, {1, y0}}]}];
pregion = Show[region, hline, DisplayFunction → $DisplayFunction];
```
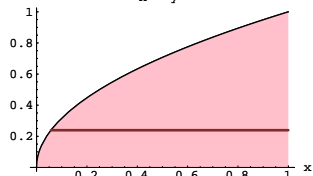

Use what you see to fill in the boxes so that
$$\int\int_R (x^2 + y^3)\, d\mskip2mu A = \int_\square^\square \int_\square^\square (x^2 + y^3)\, dx\, dy.$$

□G.7.a.iii)

In calculating the limits for the integral over the region here,

```
Show[pregion];
```


you could start with inequalities $y^2 \le x \le 1$ and $0 \le y \le 1$ from the picture. That would give you the limits you needed just above here.

If you want the limits for the integral in the reverse order, you can start from these inequalities and get the corresponding inequalities for the other direction. Do it.

□G.7.b)

Calculate the double integral
$$\int_0^1 \int_0^{1-x} \operatorname{Sin}[\pi(x + y)]\, dy\, dx.$$
Then rewrite the integral in the order
$$\int_\square^\square \int_\square^\square \operatorname{Sin}[\pi(x + y)]\, dx\, dy,$$
filling in the □'s with the appropriate limits.
Confirm your answer by calculating both integrals.

□G.7.c)

Calculate the triple integral
$$\int_0^1 \int_0^{1-x} \int_0^{x+y} \operatorname{Sin}[\pi(x + y + z)]\, dz\, dy\, dx.$$
Then rewrite the integral in the order
$$\int_\square^\square \int_\square^\square \int_\square^\square \operatorname{Sin}[\pi(x + y + z)]\, dy\, dx\, dz,$$
filling in the □'s with the appropriate limits.
Confirm your answer by calculating both integrals.

G.8) Linear equations and volume

□G.8.a.i) Tripping in 3D

Go with:

```
Clear[x, y, z, u, v, w]
{{au, bu, cu}, {av, bv, cv}, {aw, bw, cw}} =
  {{1, 2, 0.3}, {2, -3, 1}, {3, -1, 3}};
{u[x_, y_, z_], v[x_, y_, z_], w[x_, y_, z_]} =
  {au x + bu y + cu z, av x + bv y + cv z, aw x + bw y + cw z}
{x + 2 y + 0.3 z, 2 x - 3 y + z, 3 x - y + 3 z}
```

Make three vectors:

```
{directionx, directiony, directionz} =
  {{au, av, aw}, {bu, bv, bw}, {cu, cv, cw}}
{{1, 2, 3}, {2, -3, -1}, {0.3, 1, 3}}
```

Note the relationship of these vectors to the variables x, y, and z, and look at these plots of
    x directionx with its tail at {0, 0, 0},
    y directiony with its tail at the tip of x directionx, and
    z directionz with its tail at the tip of x directionx + y directiony,
and the point
    {u[x, y, z], v[x, y, z], w[x, y, z]}
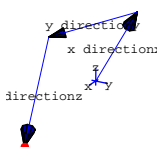for x = 1.2, y = 0.9 and z = 0.7:

```
{x, y, z} = {1.2, 0.9, 0.7};
uvwpoint = Graphics3D[
  {Red, PointSize[0.05], Point[{u[x, y, z], v[x, y, z], w[x, y, z]}]}];
trip = {Arrow[x directionx,
    Tail → {0, 0, 0}, VectorColor → Blue], Arrow[y directiony,
    Tail → x directionx, VectorColor → Blue], Arrow[z directionz,
    Tail → x directionx + y directiony, VectorColor → Blue]};
labels = {Graphics3D[Text["x directionx", (x directionx)/2]], Graphics3D[
    Text["y directiony", x directionx + (y directiony)/2]], Graphics3D[
    Text["z directionz", x directionx + y directiony + (z directionz)/2]]};
threedims = Axes3D[1.0];

Show[uvwpoint, labels, trip, threedims, PlotRange → All,
  Boxed → False, ViewPoint → CMView];
```
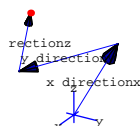

Here's the same thing for {x, y, z} = {2.1, 1.6, −3.1}:

```
{x, y, z} = {2.1, 1.6, -3.1};
uvwpoint = Graphics3D[
  {Red, PointSize[0.05], Point[{u[x, y, z], v[x, y, z], w[x, y, z]}]}];
trip = {Arrow[x directionx,
    Tail → {0, 0, 0}, VectorColor → Blue], Arrow[y directiony,
    Tail → x directionx, VectorColor → Blue], Arrow[z directionz,
```

```
    Tail → x directionx + y directiony, VectorColor → Blue]};
labels = {Graphics3D[Text["x directionx", (x directionx)/2]], Graphics3D[
    Text["y directiony", x directionx + (y directiony)/2]], Graphics3D[
    Text["z directionz", x directionx + y directiony + (z directionz)/2]]};
threedims = Axes3D[1.0];

Show[uvwpoint, labels, trip, threedims, PlotRange → All,
  Boxed → False, ViewPoint → CMView];
```


Read the code, and play with your own choices of x, y, and z until you get the feel of what's happening.
Once you feel that you have an idea of what's happening, explain the physical process of going with an xyz-point {x, y, z} and the vectors,
    directionx, directiony, and directionz,
to take a trip starting at {0, 0, 0} and ending at the uvw-point
    {u[x, y, z], v[x, y, z], w[x, y, z]}.

□Tip:

What you saw was the geometry of the trip.

Here's the algebra:

```
Clear[x, y, z, u, v, w, au, bu, cu, av, bv, cv, aw, bw, cw]
{u[x_, y_, z_], v[x_, y_, z_], w[x_, y_, z_]} =
  {au x + bu y + cu z, av x + bv y + cv z, aw x + bw y + cw z};
{directionx, directiony, directionz} =
  {{au, av, aw}, {bu, bv, bw}, {cu, cv, cw}};
{u[x, y, z], v[x, y, z], w[x, y, z]} ==
  x directionx + y directiony + z directionz
True
```

□G.8.a.ii) Tripping in 3D when directionx, directiony, and directionz

are not coplanar

Continue to go with the set-up in part i):

```
Clear[x, y, z, u, v, w]
{{au, bu, cu}, {av, bv, cv}, {aw, bw, cw}} =
  {{1, 2, 0.3}, {2, -3, 1}, {3, -1, 3}};
{u[x_, y_, z_], v[x_, y_, z_], w[x_, y_, z_]} =
  {au x + bu y + cu z, av x + bv y + cv z, aw x + bw y + cw z}
{x + 2 y + 0.3 z, 2 x - 3 y + z, 3 x - y + 3 z}
```

And the vectors you tripped on in part i):

```
{directionx, directiony, directionz} =
  {{au, av, aw}, {bu, bv, bw}, {cu, cv, cw}}
{{1, 2, 3}, {2, -3, -1}, {0.3, 1, 3}}
```

Note that if you are given three numbers u, v, and w, Mathematica has no trouble solving the linear system

$u = u[x, y, z] = au\,x + bu\,y + cu\,z,$
$v = v[x, y, z] = av\,x + bv\,y + cv\,z,$ and
$w = w[x, y, z] = aw\,x + bw\,y + cw\,z$

for x, y, and z:

```
{u, v, w} = {0.3, -6.8, 18.43};
Solve[{u == au x + bu y + cu z, v == av x + bv y + cv z, w == aw x + bw y + cw z},
  {x, y, z}]
{{x → -7.88966, y → 1.89513, z → 14.6647}}
```

To see why Mathematica has no problem delivering the solutions above, calculate

$|directionx . (directiony \times directionz)|$:

```
Abs[directionx . directiony × directionz]
11.9
```

This is not 0; so the parallelepiped determined by the three vectors

directionx = {au, av, aw},
directiony = {bu, bv, bw}, and
directionz = {cu, cv, cw}

has positive volume (see the Tutorial problem on parallelepipeds).
The upshot:
The vectors directionx, directiony, and directionz cannot be coplanar.
How does this fact help you explain why it is that no matter what

three numbers {u, v, w} you go with Mathematica will be able to solve the linear system

$u = u[x, y, z] = au\,x + bu\,y + cu\,z,$
$v = v[x, y, z] = av\,x + bv\,y + cv\,z,$ and
$w = w[x, y, z] = aw\,x + bw\,y + cw\,z$

for x, y, and z?

□G.8.a.iii)

Here's a new set-up:

```
Clear[x, y, z, u, v, w]
{{au, bu, cu}, {av, bv, cv}} = {{1, 2, 1}, {2, -3, 1}};
{aw, bw, cw} = {au, bu, cu} + {av, bv, cv};
{u[x_, y_, z_], v[x_, y_, z_], w[x_, y_, z_]} =
  {au x + bu y + cu z, av x + bv y + cv z, aw x + bw y + cw z}
{x + 2 y + z, 2 x - 3 y + z, 3 x - y + 2 z}
```

And the three vectors:

```
{directionx, directiony, directionz} =
  {{au, av, aw}, {bu, bv, bw}, {cu, cv, cw}}
{{1, 2, 3}, {2, -3, -1}, {1, 1, 2}}
```

If you are given three numbers u, v, and w, Mathematica can run into big trouble solving the linear system

$u = u[x, y, z] = au\,x + bu\,y + cu\,z,$
$v = v[x, y, z] = av\,x + bv\,y + cv\,z,$ and
$w = w[x, y, z] = aw\,x + bw\,y + cw\,z$

for x, y, and z:

```
{u, v, w} = {1, 1, 1};
Solve[{u == au x + bu y + cu z, v == av x + bv y + cv z, w == aw x + bw y + cw z},
  {x, y, z}]
{}
```

To see why Mathematica has a big problem delivering the solutions above, calculate

$|directionx . (directiony \times directionz)|$:

```
Abs[directionx . directiony × directionz]
0
```

This is 0; so the parallelepiped determined by the three vectors

directionx = {au, av, aw},
directiony = {bu, bv, bw}, and

directionz = {cu, cv, cw}

has zero volume.
The upshot:
The vectors directionx, directiony, and directionz are coplanar.
How does this help you explain why when you go with the numbers
    au, bu, cu, av, bv, cv, aw, bw, and cw
used here, then for all but very special choices of u, v, and w, it will be impossible to solve

$u = u[x, y, z] = au\,x + bu\,y + cu\,z,$
$v = v[x, y, z] = av\,x + bv\,y + cv\,z,$ and
$w = w[x, y, z] = aw\,x + bw\,y + cw\,z$

for x, y, and z?

□G.8.a.iv)

Run with everything cleared:

```
Clear[x, y, z, u, v, w, au, av, aw, bu, bv, bw, cu, cv, cw]
{u[x_, y_, z_], v[x_, y_, z_], w[x_, y_, z_]} =
  {au x + bu y + cu z, av x + bv y + cv z, aw x + bw y + cw z}
{au x + bu y + cu z, av x + bv y + cv z, aw x + bw y + cw z}
```

And the three vectors:

```
{directionx, directiony, directionz} =
  {{au, av, aw}, {bu, bv, bw}, {cu, cv, cw}}
{{au, av, aw}, {bu, bv, bw}, {cu, cv, cw}}
```

Make a matrix:

```
coefficientmatix = {{au, bu, cu}, {av, bv, cv}, {aw, bw, cw}};
MatrixForm[coefficientmatix]
```
$$\begin{pmatrix} au & bu & cu \\ av & bv & cv \\ aw & bw & cw \end{pmatrix}$$

Take its determinant:

```
detmatrix = Det[coefficientmatix]
-aw bv cu + av bw cu + aw bu cv - au bw cv - av bu cw + au bv cw
```

Look at:

$|directionx . (directiony \times directionz)|$:

```
vol = Expand[directionx . directiony × directionz]
-aw bv cu + av bw cu + aw bu cv - au bw cv - av bu cw + au bv cw
```

And look at:

```
detmatrix == vol
True
```

Explain this:
When you go with numbers
    au, bu, cu, av, bv, cv, aw, bw, and cw
and the determinant of the resulting coefficient matrix:

```
MatrixForm[coefficientmatix]
```
$$\begin{pmatrix} au & bu & cu \\ av & bv & cv \\ aw & bw & cw \end{pmatrix}$$

is not 0, then you are always guaranteed that given any numbers u, v, and w it will be possible to solve the system

$u = u[x, y, z] = au\,x + bu\,y + cu\,z,$
$v = v[x, y, z] = av\,x + bv\,y + cv\,z,$ and
$w = w[x, y, z] = aw\,x + bw\,y + cw\,z$

for x, y, and z.
And explain this:
When you go with numbers
    au, bu, cu, av, bv, cv, aw, bw, and cw
and the determinant of the resulting coefficient matrix

```
MatrixForm[coefficientmatix]
```
$$\begin{pmatrix} au & bu & cu \\ av & bv & cv \\ aw & bw & cw \end{pmatrix}$$

is 0, then for all but very special choices of u, v, and w it will be impossible to solve

$u = u[x, y, z] = au\,x + bu\,y + cu\,z,$
$v = v[x, y, z] = av\,x + bv\,y + cv\,z,$ and
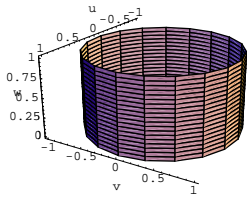$w = w[x, y, z] = aw\,x + bw\,y + cw\,z$

for x, y, and z.

G.9) The volume conversion factor $V_{xyz}[u, v, w]$

□G.9.a)

Here's a cylinder plotted in uvw-space:

```
Clear[s, t]
{{slow, shigh}, {tlow, thigh}} = {{0, 1}, {0, 2 π}};
uvwplot = ParametricPlot3D[{Cos[t], Sin[t], s}, {s, slow, shigh},
  {t, tlow, thigh}, BoxRatios → Automatic, Boxed → False,
  ViewPoint → CMView, AxesLabel → {"u", "v", "w"}];
```
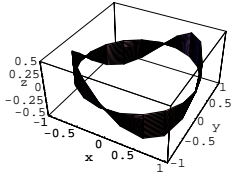

Go from uvw-space to xyz-space with:

```
Clear[x, y, z]
{x[u_, v_, w_], y[u_, v_, w_], z[u_, v_, w_]} = {u, v, u v w}
{u, v, u v w}
```

Here's how that cylinder plots out in xyz-space:

```
ParametricPlot3D[Evaluate[
  {x[Cos[t], Sin[t], s], y[Cos[t], Sin[t], s], z[Cos[t], Sin[t], s]}],
  {s, slow, shigh}, {t, tlow, thigh}, AxesLabel → {"x", "y", "z"}];
```


Damaged goods.

Look at the volume conversion factor $V_{xyz}[u, v, w]$:

```
Clear[gradx, grady, gradz, Vxyz]
gradx[u_, v_, w_] =
  {D[x[u, v, w], u], D[x[u, v, w], v], D[x[u, v, w], w]};
grady[u_, v_, w_] =
  {D[y[u, v, w], u], D[y[u, v, w], v], D[y[u, v, w], w]};
gradz[u_, v_, w_] =
  {D[z[u, v, w], u], D[z[u, v, w], v], D[z[u, v, w], w]};
Vxyz[u_, v_, w_] =
  Abs[Det[{gradx[u, v, w], grady[u, v, w], gradz[u, v, w]}]]
```

```
Abs[u v]
```

Use the volume conversion factor to help to explain why the xyz-plot
of the original uvw-cylinder turned out the way it did.

□G.9.b)

Take any cleared functions f[u] and g[u, v].
Someone said that if you start in uvw-space with a solid R_{uvw} and go
to xyz-space via
    x[u, v, w] = u,
    y[u, v, w] = v + f[u], and
    z[u, v, w] = w + g[u, v],
then the xyz-plot, R_{xyz}, of R_{uvw} is likely to be a different shape from
the original shape of R_{uvw}, but the xyz-volume measurement of R_{xyz} is
likely to be the same as the uvw-volume measurement of R_{uvw}.
What's your opinion?

□Tip:

Start with:

```
Clear[x, y, z, u, v, w, f, g]
x[u_, v_, w_] = u;
y[u_, v_, w_] = v + f[u];
z[u_, v_, w_] = w + g[u, v];
```