Multiple-valued complex functions and computer algebra

Helmer Aslaksen
Department of Mathematics
National University of Singapore
Singapore 0511
Republic of Singapore
aslaksen@math.nus.sg

1 Introduction

I recently taught a course on complex analysis. That forced me to think more carefully about branches. Being interested in computer algebra, it was only natural that I wanted to see how such programs dealt with these problems. I was also inspired by a paper by Stoutemyer ([3]).

While programs like Derive, Maple, Mathematica and Reduce are very powerful, they also have their fair share of problems. In particular, branches are somewhat of an Achilles' heel for them. As is well-known, the complex logarithm function is properly defined as a multiple-valued function. And since the general power and exponential functions are defined in terms of the logarithm function, they are also multiple valued. But for actual computations, we need to make them single valued, which we do by choosing a branch. In Section 2, we will consider some transformation rules for branches of multiple-valued complex functions in painstaking detail.

The purpose of this short article is not to do a comprehensive comparative study of different computer algebra system. (For an attempt at that, see [4].) My goal is simply to make the readers aware of some of the problems, and

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to encourage the readers to sit down and experiment with their favourite programs.

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2 Basic properties of branches multiple-valued complex functions

I will start with the following paradox due to the Danish mathematician Thomas Clausen ([1, 2]). It was published as an exercise in Crelle's journal in 1827.

Let n be an integer. Then

$$e^{1+2n\pi i} = e. \tag{1}$$

If we write

$$\left(e^{1+2n\pi i}\right)^{1+2n\pi i} = e^{1+2n\pi i} = e, \tag{2}$$

and

$$\left(e^{1+2n\pi i}\right)^{1+2n\pi i} = e^{1+4n\pi i-4n^2\pi^2} = ee^{-4n^2\pi^2}, \tag{3}$$

it follows that

$$e^{-4n^2\pi^2} = 1. \tag{4}$$

There are also a number of paradoxes involving square roots. Let me just give two.

$$1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1} = i\,i = -1, \tag{5}$$

and

$$\begin{aligned} \frac{1}{-1} &= \frac{-1}{1} \\ \frac{1}{\sqrt{-1}} &= \frac{\sqrt{-1}}{1} \\ \frac{1}{i} &= \frac{i}{1} \\ i^2 &= 1. \end{aligned} \tag{6}$$

In order to clarify such problems, we will take a fairly detailed look at some properties of elementary transcendental functions.

For $z = x + iy$, the complex exponential function is defined by

$$e^z = e^x(\cos y + i \sin y).$$

It satisfies the property $e^{(z+w)} = e^z e^w$, but does it satisfy the property $(e^z)^w = e^{(zw)}$? In order to answer this, we must look at the complex logarithm function.

We define the principal argument by $z = |z|e^{i\,\mathrm{Arg}(z)}$ and $\mathrm{Arg}(z) \in (-\pi, \pi]$. We do not define the principal argument of 0, and we will from now on assume that z is different from 0. Notice that we have defined the principal argument on the negative axis, too, but it is of course not continuous there. Extending the definition of the principal argument to the negative numbers gives us as a ready supply of counter examples.

We then define the principal logarithm $\mathrm{Log}(z)$ by $\mathrm{Log}(z) = \log|z| + i\,\mathrm{Arg}(z)$, where $\log|z|$ denotes the usual real logarithm of $|z|$. We clearly have $e^{\mathrm{Log}(z)} = z$, but do we have $\mathrm{Log}(e^z) = z$?

In order to study this, we will introduce the following terminology.

Definition 1 *Define the imaginary remainder* $\mathrm{Imr}(z)$ *and the imaginary quotient* $\mathrm{Imq}(z)$ *by*

$$\mathrm{Im}(z) = \mathrm{Imr}(z) + 2\pi\,\mathrm{Imq}(z),$$

where $\mathrm{Imr}(z) \in (-\pi, \pi]$ *and* $\mathrm{Imq}(z) \in \mathbb{Z}$.

Notice that $\mathrm{Imq}(z) = \lceil (\mathrm{Im}(z) + \pi)/2\pi) \rceil$, where $\lceil\ \rceil$ is the ceiling function.

We can now prove the following.

Theorem 2 *We have*

$$\mathrm{Log}(e^z) = \mathrm{Re}(z) + i\,\mathrm{Imr}(z).$$

In particular,

$$\mathrm{Log}(e^z) = z \quad \textit{if and only if} \quad \mathrm{Im}(z) \in (-\pi, \pi].$$

Proof: We have

$$\mathrm{Log}(e^z) = \log|e^z| + i\,\mathrm{Arg}(e^z) = x + i\,\mathrm{Arg}(e^{i\,\mathrm{Im}(z)}) =$$
$$x + i\,\mathrm{Arg}(e^{i(\mathrm{Imr}(z)+2\pi\,\mathrm{Imq}(z))}) = x + i\,\mathrm{Arg}(e^{i\,\mathrm{Imr}(z)}) == x + i\,\mathrm{Imr}(z). \quad \square$$

We will next study whether the complex logarithm satisfies the property $\mathrm{Log}(zw) = \mathrm{Log}(z) + \mathrm{Log}(w)$. To this end, we must first study $\mathrm{Arg}(uv)$. It is easy to see that

$$\mathrm{Arg}(zw) = \mathrm{Arg}(z) + \mathrm{Arg}(w) + d2\pi, \qquad \text{where } d \text{ is } 0 \text{ or } \pm 1.$$

We now make the following definitions.

Definition 3 *Define the principal product excess* $\mathrm{ppe}(u, v)$ *of two complex numbers by*

$$\mathrm{ppe}(z, w) = (\mathrm{Arg}(zw) - \mathrm{Arg}(z) - \mathrm{Arg}(w))/(2\pi).$$

Definition 4 *Define the complex sign* $\mathrm{csgn}(z)$ *of a complex number by*

$$\mathrm{csgn}(z) = \begin{cases} 1, & \text{if } \mathrm{Re}(z) > 0 \text{ or } (\mathrm{Re}(z) = 0 \text{ and } \mathrm{Im}(z) > 0) \\ 0, & \text{if } z = 0 \\ -1, & \text{if } \mathrm{Re}(z) < 0 \text{ or } (\mathrm{Re}(z) = 0 \text{ and } \mathrm{Im}(z) < 0). \end{cases}$$

We can now define the right (left) half-plane as the set of points where $\mathrm{csgn}(z)$ *is positive (negative).*

The next lemma is immediate.

Lemma 5

1. $\mathrm{ppe}(z, w)$ *is always* 0 *or* ± 1.
2. *If either* z *or* w *is positive, then* $\mathrm{ppe}(z, w) = 0$.
3. *If both* z *and* w *lie in the right half-plane, then* $\mathrm{ppe}(z, w) = 0$.
4. *If both* z *and* w *lie in the left half-plane, then* $\mathrm{ppe}(z, w) \neq 0$.
5. $\mathrm{ppe}(z, z) = 0$ *if and only if* z *lies in the right half-plane.*

We can now prove the following.

Theorem 6 *We have*

$$\operatorname{Log}(zw) = \operatorname{Log}(z) + \operatorname{Log}(w) + 2\pi i \operatorname{ppe}(z, w).$$

In particular $\operatorname{Log}(z^2) = 2\operatorname{Log}(z)$ *if and only if* z *lies in the right half-plane.*

Proof:

$$\begin{aligned}\operatorname{Log}(zw) &= \log|zw| + i \operatorname{Arg}(zw) \\ &= \log|z| + \log|w| + i(\operatorname{Arg}(z) + \operatorname{Arg}(w) + 2\pi \operatorname{ppe}(z, w)) \\ &= \operatorname{Log}(z) + \operatorname{Log}(w) + 2\pi i \operatorname{ppe}(z, w). \quad \square\end{aligned}$$

We have the following result for quotients.

Theorem 7 *We have*

$$\operatorname{Arg}(1/z) = \begin{cases} -\operatorname{Arg}(z), & \text{if } z \text{ is not negative} \\ -\operatorname{Arg}(z) + 2\pi, & \text{if } z \text{ is negative.} \end{cases}$$

Hence

$$\operatorname{Log}(1/z) = \begin{cases} -\operatorname{Log}(z), & \text{if } z \text{ is not negative} \\ -\operatorname{Log}(z) + 2\pi i, & \text{if } z \text{ is negative.} \end{cases}$$

Proof: We have $1/z = \overline{z}/|z|^2$, but $\operatorname{Arg}(\overline{z}) = -\operatorname{Arg} z$ unless z is negative, in which case both $\operatorname{Arg}(z)$ and $\operatorname{Arg}(1/z)$ are equal to π. $\square$

We are now ready to define the complex power and exponential functions.

Definition 8 *We define the complex power and exponential functions by*

$$z^a = e^{\operatorname{Log}(z)a}, \quad \text{and} \quad a^z = e^{\operatorname{Log}(a)z} \quad \text{for } a \neq e.$$

We are now ready to consider whether $(e^z)^w$ equals e^{zw}. The key issue is that $(e^z)^w$ involves the exponential function with base e^z and not just e. So while e^{zw} is a genuine single-valued function, we need to choose a branch in order to make $(e^z)^w$ single valued.

Theorem 9 *We have*

$$(e^z)^w = e^{zw} e^{-w 2\pi i \operatorname{Imq}(z)}.$$

Proof:

$$(e^z)^w = e^{\mathrm{Log}(e^z)w} = e^{(\mathrm{Re}(z)+i\,\mathrm{Imr}(z))w} = e^{(z-i2\pi\,\mathrm{Imq}(z))w} = e^{zw}e^{-wi2\pi\,\mathrm{Imq}(z)} \quad \Box$$

Using Theorem 9, we can easily resolve Clausen's paradox. In equation (3) we said that

$$(e^{1+2n\pi i})^{1+2n\pi i} = e^{1+4n\pi i-4n^2\pi^2}.$$

This should be replaced by

$$(e^{1+2n\pi i})^{1+2n\pi i} = e^{1+4n\pi i-4n^2\pi^2}e^{-(1+2n\pi i)2\pi i\,\mathrm{Imq}(1+2n\pi i)} =$$
$$e^{1-4n^2\pi^2}e^{-(1+2n\pi i)2\pi i n} = e^{1-4n^2\pi^2}e^{-2n\pi i+4n^2\pi^2} = e,$$

which agrees with equation (2).

We can also prove the following corollary.

Corollary 10 *We have*

$$(e^z)^{1/2} = (-1)^{\mathrm{Imq}(z)}e^{z/2}.$$

In particular,

$$(e^z)^{1/2} = e^{z/2} \qquad \text{if and only if } \mathrm{Im}(z) \in ((4n-1)\pi, (4n+1)\pi],\ n \in \mathbb{Z}.$$

We also have the following immediate generalization.

Theorem 11 *We have*

$$(a^z)^w = a^{zw}e^{-w2\pi i\,\mathrm{Imq}(z\,\mathrm{Log}(a))}.$$

We can now ask similar questions about the power function.

Theorem 12 *We have*

$$(zw)^a = z^a w^a e^{a2\pi i\,\mathrm{ppe}(z,w)}.$$

Proof:

$$(zw)^a = e^{a\,\mathrm{Log}(zw)} = e^{a(\mathrm{Log}(z)+\mathrm{Log}(w)+2\pi i\,\mathrm{ppe}(z,w))} = z^a w^a e^{a2\pi i\,\mathrm{ppe}(z,w)}. \quad \Box$$

We will derive some consequences of Theorem 12.

Theorem 13 *We have*

$$\sqrt{zw} = (-1)^{\mathrm{ppe}(z,w)}\sqrt{z}\sqrt{w}.$$

In particular,

$$\sqrt{z^2} = \mathrm{csgn}(z)z,$$

so

$$\sqrt{z^2} = z \quad \textit{if and only if } z \textit{ lies in the right half-plane.}$$

We will finish this section with the following theorem.

Theorem 14 *We have*

$$\sqrt{1/z} = \begin{cases} 1/\sqrt{z}, & \textit{if } z \textit{ is not negative} \\ -1/\sqrt{z}, & \textit{if } z \textit{ is negative.} \end{cases}$$

In particular, if z is real, then

$$\sqrt{1/z} = \mathrm{sgn}(z)/\sqrt{z},$$

Proof: If z is not negative, we have

$$\sqrt{1/z} = e^{\mathrm{Log}(1/z)/2} = e^{-\mathrm{Log}(z)/2} = 1/\sqrt{z},$$

while if z is negative, we have

$$\sqrt{1/z} = e^{\mathrm{Log}(1/z)/2} = e^{(-\mathrm{Log}(z)+2\pi i)/2} = -e^{-\mathrm{Log}(z)/2} = -1/\sqrt{z}.$$

The last two results resolve the two square root paradoxes given at the beginning of this section.

3 Computer tests

Computer algebra systems are in general much better at reducing the difference between two equivalent expressions to 0, than simplifying an expression to a specific form. I therefore suggest that the readers experiments with the following eight tests (adapted from [3]) using their favourite computer algebra system.

Notice that some programs simplify expressions automatically, while others only do so when you use an explicit `simplify` command. Sometimes you can control the behaviour by using a special option to the `simplify` command, or a different command such as `PowerExpand`. In some programs you can explicitly restrict the domain of a variable, use statements like `on expandlogs` or program your own transformation rules to change the behaviour.

Test 1

(a) $\sqrt{zw} - \sqrt{z}\sqrt{w}$ *should not simplify when* z *and* w *are complex.*

(b) $\sqrt{zw} - \sqrt{z}\sqrt{w}$ *should simplify to* 0 *when* z *and* w *are both positive.*

Test 2

(a) $\sqrt{z^2}$ *should not simplify, or simplify to* $\operatorname{csgn}(z)z$ *when* z *is complex.*

(b) $\sqrt{z^2}$ *should not simplify, or simplify to* $\operatorname{sgn}(z)z = |z|$ *when* z *is real.*

(c) $\sqrt{z^2}$ *should simplify to* z *when* z *is positive.*

Test 3

(a) $\sqrt{1/z} - 1/\sqrt{z}$ *should not simplify when* z *is complex.*

(b) $\sqrt{1/z} - 1/\sqrt{z}$ *should not simplify, or simplify to* $(\operatorname{sgn}(z) - 1)/\sqrt{z}$ *when* z *is real.*

(c) $\sqrt{1/z} - 1/\sqrt{z}$ *should simplify to* 0 *when* z *is positive.*

Test 4

(a) $\sqrt{e^z} - e^{z/2}$ *should not simplify when* z *is complex.*

(b) $\sqrt{e^z} - e^{z/2}$ *should simplify to* 0 *when* z *is real.*

Test 5

(a) $\operatorname{Log}(zw) - \operatorname{Log}(z) - \operatorname{Log}(w)$ *should not simplify when* z *and* w *are complex.*

(b) $\log(zw) - \log(z) - \log(w)$ *should simplify to* 0 *when* z *and* w *are both positive.*

Test 6

(a) $\mathrm{Log}(z^2) - 2\,\mathrm{Log}(z)$ *should not simplify when* z *is complex.*

(b) $\log(z^2) - 2\log(z)$ *should simplify to* 0 *when* z *is positive.*

Test 7

(a) $\mathrm{Log}(1/z) + \mathrm{Log}(z)$ *should not simplify when* z *is complex.*

(b) $\log(1/z) + \log(z)$ *should simplify to* 0 *when* z *is positive.*

Test 8

(a) $\mathrm{Log}(e^z)$ *should not simplify when* z *is complex.*

(b) $\log(e^z)$ *should simplify to* z *when* z *is real.*

References

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