Why Computer Algebra Systems Sometimes Can't Solve Simple Equations

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1 Introduction

Among the basic equations one might wish a computer to solve symbolically is the inverse of the power function, solving $y = z^w$ for z. (Note: $z^w \equiv \exp(w \ln z)$). While many special cases, easily solved, abound, the general question is fraught with implications: if this is so hard, how can we expect success in other ventures? Having solved this, we can naturally use it in a "composition" of solution methods for expressions of the form $y = f(z)^w$.

Can't we already do this? Is it not the case that the solution of $y = z^{a+bi}$ is trivially $z = y^{1/(a+bi))}$?

Not so. if this were the case, then a plot of the function $t(y) := y - (y^{1/(1+i))})^{1+i}$ would be indistinguishable from $t(y) \equiv 0$. For many values, $t(y)$ is (allowing for round-off error), zero. But if your computer system correctly computes with values in the complex plane, then, (to pick two complex points from a region described later), $t(-10000 + 4000i)$ is not zero, but about $-9981 + 3993i$ and $t(-0.01 + 0.002i)$ is about $5.34 - 1.06i$. These strange numbers are not the consequence of round-off error or some other numerical phenomena. The alleged solution is just not mathematically correct.

Computer algebra systems as usually programmed lack the expressive capability to return the exact and complete set of solutions, in general.

2 How to solve $y = z^w$ for z

We assume that y and w are complex-valued variables in general, and the solution sought for z is permitted to be complex as well. If we know specific values for y and w, we can simplify the question and answer it. For example, $9 = z^2$ has the solution set $\{-3,\ 3\}$. The equation $-9 = z^2$ has the solution set $\{-3i,\ 3i\}$. The equation $3 = z^{1/2}$ has the solution set $\{9\}$.

But $-3 = z^{1/2}$ has no solution for real or complex numbers, at least given the conventional meaning for this power function: if $z := r \cdot \exp(i\theta)$ then $z^{1/2}$ must be $\sqrt{r} \cdot \exp(i\theta/2)$ where the positive $\sqrt{r}$ of the positive value r is taken. There is no value for r and for $\pi < \theta \leq \pi$ to satisfy this equation. If you feel like arguing this point, read the footnote[1].

To some extent we can try to limit the scope of the answer even though we may not have specific values for w and y. One way of furthering this exploration is to inquire about the real and imaginary parts (or perhaps the argument and magnitude) of w and y. We've already seen situations above where the solution set has zero, one, or more distinct solutions, giving us some hint as to what to expect.

3 A systematic attempt

There are any number of ways of approaching this problem from a naive complex-variables direction. We've tried quite a few, and believe this is about as simple as it gets.

Let use define $y := s \cdot \exp(i\rho)$, $z := r \cdot \exp(i\theta)$ and $w := a + bi$. That's right, even though we have three complex variables, we don't use the same representation for w as for y and z. This is a matter of convenience; any alternative representation can be changed to this.

Note: s and r are nonnegative real, a and b are real, ρ and θ are in the half-open real interval $(-\pi, \pi]$. These are all conventional restrictions to make the representations of complex values canonical, and do not limit the "values" they can assume[2].

Then

$$z^w = \exp((a + bi) \cdot (\log r + i\theta))$$

$$= \exp(a \log r - b\theta + i(a\theta + b \log r))$$

[1] Even if you wish to specify that $()^{1/2}$ means a set of two values, then the equation still has no solution. If you think that a solution is $z = 9$, observe that $\{3\} \neq \{3,\ -3\}$. If you uniformly choose (somewhat perversely) the negative square root, then $3 = z^{1/2}$ has no solution. It appears that a solution would entail being able to magically distinguish the number 9 whose square-root is 3 from the number 9 whose square-root is -3.

[2] If $z = 0$ we will say that $r = 0$ and $\theta = 0$ for definiteness.

So

$$z^w = \exp(a \log r - b\theta) \cdot \exp(i\phi) \tag{1}$$

where $\phi = b \log r + a\theta$. We do not assume that ϕ is in $(-\pi, \pi]$. Note however, that the first factor in equation (1) is necessarily real because r is non-negative and a, b and θ are real.

Our objective is for z^w, so expressed, to be equal to y:

$$y = s \cdot \exp(i\rho). \tag{2}$$

The magnitude and the argument (modulo 2π) of the two expressions must be equal, and so we are provided with two equations:

$$s = \exp(a \log r - b\theta)$$

or better, since r is non-negative, and a, b, s, and θ are real,

$$\log s = a \log r - b\theta \tag{3}$$

and also

$$\rho = b \log r + a\theta + 2n\pi \tag{4}$$

for some integer value n. Equations (3) and (4) are simultaneous linear equations for $\log r$ and θ. This set has a solution except when the determinant of the coefficient matrix is zero, the condition being $a^2 + b^2 = 0$. Since a and b are real, the only degenerate case is $a = b = 0$, which is discussed in the next section.

The solutions are

$$\log r = (a \log s + b\rho - 2nb\pi b)/(a^2 + b^2). \tag{5}$$

$$\theta = (-b \log s + a\rho - 2an\pi)/(a^2 + b^2). \tag{6}$$

Now what remains is for n to be chosen appropriately. Given a set of values for a, b, ρ, and s in equation (6) we can find some set of integer values for n, namely when

$$\frac{-\pi\ |w|^2 + b\ \log s - a\rho}{2a\pi} < n \leq \frac{\pi\ |w|^2 + b\ \log s - a\rho}{2a\pi}$$

(remember $w = a + ib$ so $|w|^2 = a^2 + b^2$) to constrain θ to be in $(-\pi,\ \pi]$. We then use those values to get corresponding values for r from equation (5). (When a is zero, use equation (5) for specifying the conditions on n. One of a or b is non-zero.)

It would be nice if this were the end of it. Unfortunately, it is not so simple.

4 Branch Cuts

The solutions of the previous section fall apart in various ways because of singularities and the necessity of defining a branch cut in the logarithm function. The branch cut is normally along the negative real axis, and the values along the cut are pasted to the "top" part. In more detail, let us consider the situation.

1. For $a = b = 0$, we are solving at a singular point, and the equation degenerates to $y = z^0$. The only solution is when $y = 1$, and then z is arbitrary.

2. If $b = 0$, $a \neq 0$ (the real exponent case), then the solution exists for the now simpler equations

$$r = \exp((\log s)/a)$$

and

$$\theta = \rho/a$$

Since $-\pi < \theta \leq \pi$, a solution can exist only when

$$-a\pi < \rho \leq a\pi. \tag{7}$$

Thus there is no solution for $y = z^a$ unless ρ (which is $\arg y$) abides by condition (7).

3. If $a = 0$ (but $b \neq 0$) we must avoid the division in equation (5) and go back to equations (3) and (4) giving us

$$\theta = -\log(s)/b$$

$$r = \exp(\rho/b)$$

The restrictions: since $-\pi < \theta \leq \pi$,

$$-|b|\pi < -\log(s) \leq |b|\pi$$

Since exp is monotonic, we impose

$$\exp(-|b|\pi) \leq s < \exp(|b|\pi). \tag{8}$$

Thus there is no solution for $y = z^{ib}$ unless s (which is $|y|$), abides by condition (8). For example, $\exp(\pi) = z^i$ has no solution, but $\exp(-\pi) = z^i$ has one. Geometrically, *the acceptable values of y appear in an annulus: the region between two concentric circles about the origin in the complex plane of radii $p = \exp(-|b|\pi)$ and $1/p$.* We would draw a figure, except this is not very hard to visualize.

4. If $a \neq 0$ and $b \neq 0$, consider, once again from (6) that $-\pi < \theta \leq \pi$ and therefore

$$-\pi(a^2 + b^2) < -b\ \log s + a\rho + 2an\pi \leq \pi(a^2 + b^2). \tag{9}$$

Consider the border curve of this region as determined by this equation:

$$-b \log s + a\rho = \pm\pi(a^2 + b^2 - 2an\pi).$$

The curve, in polar form is

$$s = K \exp((a/b) \cdot \rho) \tag{10}$$

for the constants

$$K = \exp(\pm\pi(a^2 + b^2 - 2an)/b).$$

Equation (10) defines a spiral starting on the real axis ($\rho = -\pi$) and ending on the real axis ($\rho = \pi$) after one revolution. The interior of the acceptable region includes one of these spirals (the one with the $-\pi$) but not the other. It is as though the annulus of the previous case were cut along the negative real axis and distorted. The two curves are joined in two places by segments of the negative real axis.

5 An example of the complex case

If we look at a particular instance of our equation, namely

$$y = z^{i+1}$$

we can easily but rather cavalierly solve it as

$$z = y^{1/(i+1)} = y^{1/2-i/2}.$$

For this case, the two spirals are governed by $K = \exp(\pm\pi(2 - 2n))$. For $n = 0$, the values of K are $\{535.492, 0.00186744\}$. The outer logarithmic spiral hits the negative real axis at about $-12,392$ and the inner spiral hits the negative real axis at about -0.0432. The acceptable values for y are between the spirals and the line connecting them on the negative real axis (this line is joined upward to the figure).

Is $n = 0$ acceptable? By equation (6), using $a = 1$ and $b = 1$, we require $\theta = (-\log s + \rho)/2$ to be in $(-\pi, \pi]$ when ρ is in $(-\pi, \pi]$. This is no problem, since for every value of ρ there is a satisfactorily corresponding s. There may be other n possible, in which case there is a pair of spirals such as illustrated below. The solution exists between them. We give two figures to show the general shape; first, two spirals, and then a close-up of them near the origin.

6 Conclusion

This paper proceeds from common elementary functions of exp and log as used in all complex variables text (we deliberately omit classical references). The editor advises me that there will be further discussion of the computational issues in other papers in this issue of the SIGSAM Bulletin.

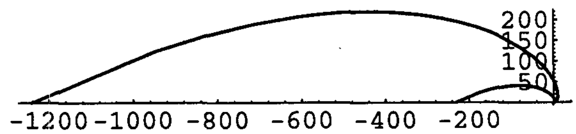

Figure 1: Two spirals.

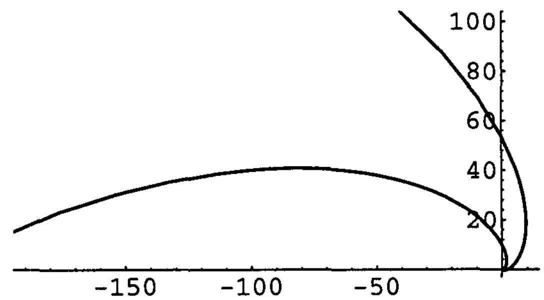

Figure 2: A closer view of spirals.

We show that in the case of solving equations, computer algebra systems probably do not provide the information one would like to have. We would expect as much of the following information as possible: a description of the inverse function's domain; the number of distinct solutions; a formula to numerically evaluate each inverse.

If particular cases simplify, and the result can be expressed as lists of equations, this seems most plausible to be useful.

If one cannot make determinations as to whether $w = 0$, or if its real part is 0, or whether various conditions hold on y, then some alternatives must be considered.

1. We could ask the user to resolve these questions, or inquire of some assumption "knowledge base" in the system to try to determine the answers. Some CAS designers (e.g. Maple) are opposed to halting the computation to ask the user. Others (e.g. Macsyma) are not so shy. A reference to a set of "assumptions" is possible in either of these systems.

2. We could give a symbolic `If [ ...]` answer along the lines of the construction above. This is rarely useful unless (a) it collapses substantially as a consequence of arithmetic and/or logical simplification, and (b) the CAS is able to continue computation with "conditional" expressions, including, for example, adding and multiplying them, inverting them (!), etc. This is a challenge, but in some sense inevitable if the answer to the questions of domain are not and cannot yet be "known".

3. A variation of the above is to carry along with every expression "provisos" which restrict the correctness of the result. This means that the computer algebra system must maintain what it believes to be the "most appropriate" answer through computation, but with the additional computation of side-conditions, such that the end result is correct only if the side-conditions are satisfied.

4. We could assume that if one cannot prove (say) $a = 0$, then it is definitely the case that $a \neq 0$. Although some CAS use this "closed world assumption" (that all true things are known, and anything that is not known or provable is false), the consequences are potentially dreadful.

5. We could defer execution of the program that evaluates the test conditions until the time that it is in fact needed to proceed with a computation. At that time our computer program would insist that any information that is indeed needed is provided. This so-called lazy evaluation is rarely used in computer algebra systems with the exception of some series computations in which terms are computed "as needed".

6. One could refuse to solve the equation as given.

We recommend some version of 2 or 3, as being almost inevitable in cases where "symbolic" parameters must be used, although in some circumstances, approach 5 is workable. We continue to work on this approach in our own thinking about computation. Most existing computer algebra systems seem to use some version of 1 or 4.

7 Acknowledgments

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