Matrices, Geometry&*Mathematica*

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MGM.11 Function spaces and Root-Mean Square Approximation
BASICS

B.1) The root-mean-square distance between two functions f[t] and g[t] on [a,b] is $\sqrt{\int_a^b (f[t]-g[t])^2\,dt}$

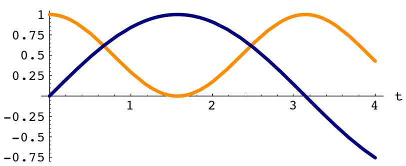
The Dot product f•g of two functions on [a,b] is $\int_a^b f[t]\,g[t]\,dt$.

Component of one function in the direction of another

B.1.a) Distance between two functions on an interval.

Here are two functions f[t] and g[t] on [a,b] with a = 0 and b = 2.5:

```
a = 0;
b = 4;
Clear[f, g, t];
f[t_] = Cos[t]^2;
g[t_] = Sin[t];
Plot[{f[t], g[t]}, {t, a, b},
  PlotStyle → {{Thickness[0.01], DarkOrange},
    {Thickness[0.01], NavyBlue}}, AxesLabel → {"t", ""}];
```



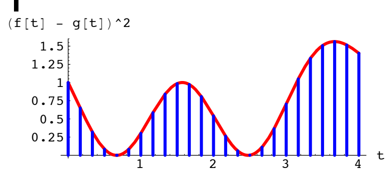
How do folks measure the root-mean-square distance between f[t] and g[t] on [a,b]?

Answer:

Folks plot $(f[t]-g[t])^2$ on [a,b] and shade between the curve and the t - axis:

```
rmsplot = Plot[(f[t] - g[t])^2,
    {t, a, b}, PlotStyle -> {{Thickness[0.01], Red}},
    AxesLabel -> {"t", "(f[t] - g[t])^2"},
```

```
    Epilog -> Table[{Blue, Thickness[0.01],
       Line[{{t, 0}, {t, (f[t] - g[t])^2}}]}, {t, a, b, (b - a) / 24}]];
```



They calculate the area measurement of the shaded region and then they say that the root-mean -square distance between f[t] and g[t] on [a,b] is the square root of the area measurement.

In short, the root mean square distance between f[t] and g[t] on [a,b] is

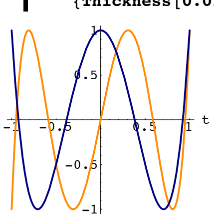
$$\sqrt{\int_a^b (f[t]-g[t])^2\,dt}$$

```
√NIntegrate[(f[t] - g[t])^2, {t, a, b}]
```

1.62424

Try it for two new functions f[t] and g[t] on a new interval [a,b]:

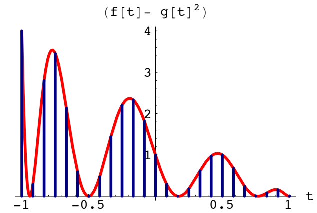
```
a = -1;
b = 1;
Clear[f, g, t];
f[t_] = 5 t - 20 t^3 + 16 t^5;
g[t_] = 1 - 8 t^2 + 8 t^4;
Plot[{f[t], g[t]}, {t, a, b},
 PlotStyle -> {{Thickness[0.01], DarkOrange},
   {Thickness[0.01], NavyBlue}}, AxesLabel -> {"t", ""}]
```



- Graphics -

```
rmsplot =
  Plot[(f[t] - g[t])^2, {t, a, b}, PlotStyle → {{Thickness[0.01], Red}},
```

```
  AxesLabel → {"t", "(f[t]- g[t]^2)"}, AspectRatio -> 1/GoldenRatio,
  Epilog → Table[{NavyBlue, Thickness[0.01],
     Line[{{t, 0}, {t, (f[t] - g[t])^2}}]}, {t, a, b, (b - a)/24}]];
```



The root mean square distance between f[t] and g[t] on [a,b] is

$$\sqrt{\int_a^b (f[t]-g[t])^2\,dt}:$$

```
√NIntegrate[(f[t] - g[t])^2, {t, a, b}]
```

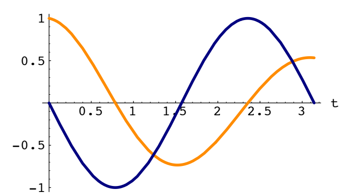
1.405

B.1.b) The dot product f•g of two functions f[t] and g[t] on an interval [a,b]

Here are two functions f[t] and g[t] on [a,b] with a = 0 and b = Pi:

```
a = 0;
b = π;
Clear[f, g, t];
f[t_] = e^(-0.2 t) Cos[2 t];
g[t_] = -Sin[2 t];
Plot[{f[t], g[t]}, {t, a, b},
  PlotStyle → {{Thickness[0.01], DarkOrange},
    {Thickness[0.01], NavyBlue}}, AxesLabel → {"t", ""}];
```

π

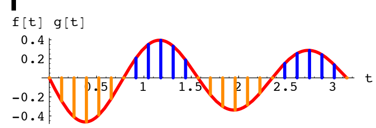


How do folks measure calculate dot product f•g of f[t] and g[t] on [a,b]?

Answer:

Folks plot f[t] g[t] on [a,b] and shade between the curve and the t - axis:

```
dotplot =
  Plot[f[t] g[t], {t, a, b}, PlotStyle -> {{Thickness[0.01], Red}},
   AxesLabel -> {"t", "f[t] g[t]"}, Epilog ->
    Table[{If[f[t] g[t] > 0, Blue, DarkOrange], Thickness[0.01],
      Line[{{t, 0}, {t, f[t] g[t]}}]}, {t, a, b, (b - a) / 24}]];
```



They put f•g equal the signed area measurement of the shaded region (subtracting the orange from the blue).

This is the same as

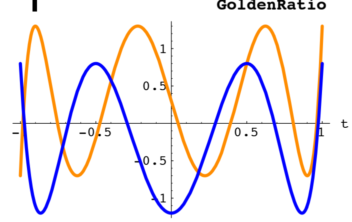
$$f\bullet g = \int_a^b f[t]\,g[t]\,dt:$$

```
NIntegrate[f[t] g[t] , {t, a, b}]
```

-0.0581686

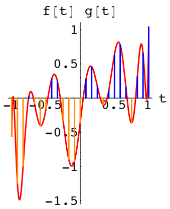
Try it for two new functions f[t] and g[t] on a new interval [a,b]:

```
a = -1;
b = 1;
Clear[f, g, t];
f[t_] = 0.3 - 7 t + 56 t^3 - 112 t^5 + 64 t^7;
g[t_] = -1.2 + 18 t^2 - 48 t^4 + 32 t^6;
Plot[{f[t], g[t]}, {t, a, b}, PlotStyle ->
   {{Thickness[0.01], DarkOrange}, {Thickness[0.01], Blue}},
  AspectRatio -> 1/GoldenRatio, AxesLabel -> {"t", ""}];
```




```
dotplot =
  Plot[f[t] g[t], {t, a, b}, PlotStyle -> {{Thickness[0.01], Red}},
```

```
AxesLabel -> {"t", "f[t] g[t]"}, Epilog ->
  Table[{If[f[t] g[t] > 0, Blue, DarkOrange], Thickness[0.01],
    Line[{{t, 0}, {t, f[t] g[t]}}]}, {t, a, b, (b - a) / 24}]];
```



$f\bullet g = \int_a^b f[t]\,g[t]\,dt$:

```
NIntegrate[f[t] g[t] , {t, a, b}]
```

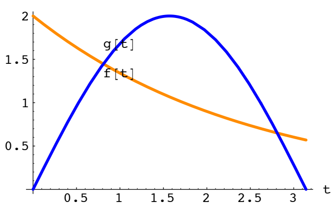
-0.137143

These two functions are close to being perpendicular.

□B.1.c.i) The component of one function in the direction of another on an interval

Here are two functions f[t] and g[t] on [a,b] with $a = 0$ and $b = \pi$:

```
a = 0;
b = π;
Clear[f, g, t];
f[t_] = 2 e^(-0.4 t);
g[t_] = 2 Sin[t];
Plot[{f[t], g[t]}, {t, a, b}, PlotStyle →
   {{Thickness[0.01], DarkOrange}, {Thickness[0.01], Blue}},
  AxesLabel → {"t", ""}, Epilog →
   {Text["f[t]", {a + 1, f[a + 1]}], Text["g[t]", {a + 1, g[a + 1]}]}];
```



How do folks come up with the component of f[t] in the direction of g[t] on the plotted interval?

□Answer:

They do it just the way you do it with vectors in kD:

$$\text{comp}_g\, f[t] = \frac{f\bullet g}{g\bullet g}\; g[t]$$

where

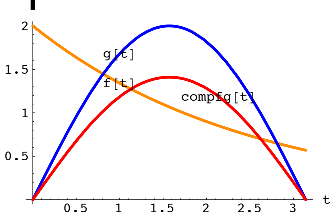
$$f\bullet g = \int_a^b f[t]\,g[t]\,dt \text{ and } g\bullet g = \int_a^b g[t]\,g[t]\,dt.$$

See its formula:

```
f·g = NIntegrate[f[t] g[t], {t, a, b}];
g·g = NIntegrate[g[t] g[t], {t, a, b}];
Clear[compfg];
     compfg[t_]  =  (f·g)/(g·g)  g[t]
```

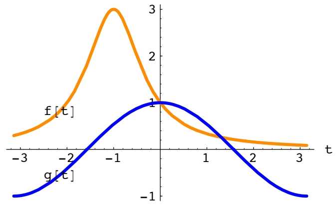
1.41001 Sin[t]

```
compplot  = Plot[{f[t], g[t], compfg[t]},
   {t, a, b}, PlotStyle -> {{Thickness[0.01], DarkOrange},
     {Thickness[0.01], Blue}, {Thickness[0.01], Red}},
   AxesLabel -> {"t", ""}, Epilog -> {Text["f[t]", {a + 1, f[a + 1]}],
     Text["g[t]", {a + 1, g[a + 1]}],
          Text["compfg[t]", {b - 1, compfg[b - 1]}]}];
```



Try it for two new functions f[t] and g[t] on a new interval [a,b]

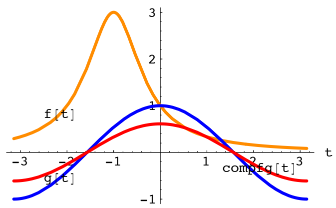
```
a = -π;
b = π;
Clear[f, g, t];
f[t_] = 3 / (1 + 2 (t + 1)^2);
g[t_] = Cos[t];
Plot[{f[t], g[t]}, {t, a, b}, PlotStyle →
   {{Thickness[0.01], DarkOrange}, {Thickness[0.01], Blue}},
  AxesLabel → {"t", ""}, Epilog →
   {Text["f[t]", {a + 1, f[a + 1]}], Text["g[t]", {a + 1, g[a + 1]}]}];
```




```
f·g = NIntegrate[f[t] g[t], {t, a, b}];
g·g = NIntegrate[g[t] g[t], {t, a, b}];
Clear[compfg];
     compfg[t_]  =  (f·g)/(g·g)  g[t]
```

0.611397 Cos[t]

```
compplot  = Plot[{f[t], g[t], compfg[t]},
   {t, a, b}, PlotStyle -> {{Thickness[0.01], DarkOrange},
     {Thickness[0.01], Blue}, {Thickness[0.01], Red}},
   AxesLabel -> {"t", ""}, Epilog -> {Text["f[t]", {a + 1, f[a + 1]}],
     Text["g[t]", {a + 1, g[a + 1]}],
          Text["compfg[t]", {b - 1, compfg[b - 1]}]}];
```



The component of f[x] in the direction of g[t] is a multiple of g[t].

□B.1.c.ii) Explanation of where the formula $\text{comp}_g\, f[t] = \frac{f\bullet g}{g\bullet g}\, g[t]$ comes from

Explain where the formula

$$\text{compfg}[t] = \frac{f\bullet g}{g\bullet g}\; g[t]$$

comes from.

□Answer:

The root mean square distance between f[t] and c g[t] on the interval [a,b] is

$$\text{rmssquared}[c] = \int_a^b (f[t] - c\, g[t])^2\, dt$$

Differentiate with respect to c:

$$\begin{aligned}\text{rmssquared}'[c] &= \int_a^b \partial_c (f[t] - c\, g[t])^2\, dt\\ &= \int_a^b 2\,(f[t] - c\, g[t])\,(-g[t])\,dt\\ &= -2\int_a^b g[t]\,(f[t] - c\, g[t])\,dt\\ &= -2\int_a^b g[t]\,f[t]\,dt + 2\,c\int_a^b g[t]\,g[t]\,dt)\\ &= -2\,(f\bullet g - c\, g\bullet g)\end{aligned}$$

So:

$$\text{rmssquared}'[c] = 0 \text{ for } c = \frac{f\bullet g}{g\bullet g}.$$

The upshot:

$c = \frac{f\bullet g}{g\bullet g}$ minimizes the root-mean-square distance between f[t] and c g[t] on [a,b] .

In other words, the component of f[t] in the direction of g[t] is the multiple of g[t] that is closest to f[t].

B.2) Orthogonal sets of functions:

The Sine system on [0,π]

The Cosine system on [0,π]

The Sine -Cosine system on [-π,π]

The Sine -Cosine system on [0,2π]

The Legendre Polynomial system on [-1,1]

Function spaces spanned by orthogonal sets of functions

□B.2.a.i) Orthogonal families of functions

Folks like to say that a given set of non-zero functions

$\{s_1[t], s_2[t], s_3[t], \ldots, s_k[t], \ldots\}$

defined on an interval [a,b] are orthogonal (mutually perpendicular) on [a,b] if

$$s_p \bullet s_q = \int_a^b s_p[t]\, s_q[t]\, dt = 0 \quad \text{when } p \neq q.$$

and

$\int_a^b s_p[t]^2\, dt > 0$ for all the p's.
Give some examples of orthogonal sets all the pros know:

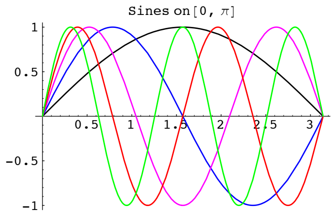
Answer:

Here you go:

- **Sine system on [0, π]:**

The set of functions
$\{s_1[t], s_2[t], s_3[t], \ldots, s_k[t], \ldots\}$
for
$s_k[t] = \mathrm{Sin}[k\, t]$
is an orthogonal set on the interval $a \le t \le b$ with $a = 0$ and $b = \pi$.
Here are plots of the first five.

```
Clear[s, Subscript, k, t];
s_k_[t_] := Sin[k t]
a = 0;
b = π;
Plot[{s_1[t], s_2[t], s_3[t], s_4[t], s_5[t]}, {t, a, b},
 PlotStyle -> {{Black}, {Blue}, {Magenta}, {Red}, {Green}},
 PlotLabel -> Sines on [a, b]];
```



To illustrate the fact that this set of functions
$\{s_1[t], s_2[t], s_3[t], \ldots, s_k[t], \ldots\}$
is an orthogonal set on the interval $a \le t \le b$ with $a = 0$ and $b = \pi$,
here is a calculation of
$s_p \bullet s_q = \int_a^b s_p[t]\, s_q[t]\, dt$
for random positive integers p and q with $p \neq q$

```
q = Random[Integer, {1, 6}];
p = Random[Integer, {q + 1, 3 q}];
∫_a^b s_q[t] s_p[t] dt
```

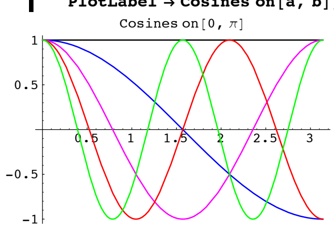
0

Rerun a couple of times.

- **Cosine system on [0, π]:**

The set of functions
$\{s_1[t], s_2[t], s_3[t], \ldots, s_k[t], \ldots\}$
for
$s_k[t] = \mathrm{Cos}[(k - 1)\, t]$
is an orthogonal set on the interval $a \le t \le b$ with $a = 0$ and $b = \pi$.
Here are plots of the first five.

```
Clear[Subscript, s, k, t];
s_k_[t_] := Cos[(k - 1) t]
a = 0;
b = π;
Plot[{s_1[t], s_2[t], s_3[t], s_4[t], s_5[t]}, {t, a, b},
  PlotStyle → {{Black}, {Blue}, {Magenta}, {Red}, {Green}},
  PlotLabel → Cosines on [a, b]];
```



To illustrate the fact that this set of functions
$\{s_1[t], s_2[t], s_3[t], \ldots, s_k[t], \ldots\}$
is an orthogonal set on the interval $a \le t \le b$ with $a = 0$ and $b = \pi$,
here is a calculation of
$s_p \bullet s_q = \int_a^b s_p[t]\, s_q[t]\, dt$
for random positive integers p and q with $p \neq q$

```
q = Random[Integer, {1, 6}];
p = Random[Integer, {q + 1, 3 q}];
∫_a^b s_q[t] s_p[t] dt
```

0

Rerun a couple of times.

- **Sine-Cosine system on [-π, π]:**

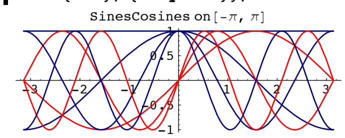
This set of functions
$\{s_1[t], s_2[t], s_3[t], ..\}$
$= \{1, \mathrm{Sin}[t], \mathrm{Cos}[t], \mathrm{Sin}[2\, t], \mathrm{Cos}[2\, t], \mathrm{Sin}[3\, t], \mathrm{Cos}[3\, t], \ldots\}$
is an orthogonal set on the interval $a \le t \le b$ with $a = -\pi$ and $b = \pi$.
Here are formulas for the first 13.

```
Clear[s, Subscript, k, t];
s_k_[t_] := If[EvenQ[k], Sin[k t / 2], Cos[1/2 (k - 1) t]];
Table[s_k[t], {k, 1, 13}]
```

{1, Sin[t], Cos[t], Sin[2 t], Cos[2 t], Sin[3 t], Cos[3 t], Sin[4 t], Cos[4 t], Sin[5 t], Cos[5 t], Sin[6 t], Cos[6 t]}

And plots of the first seven:

```
a = -π;
b = π;
Plot[{s_1[t], s_2[t], s_3[t], s_4[t], s_5[t], s_6[t], s_7[t]}, {t, a, b},
  PlotStyle → {{NavyBlue}, {Red}, {NavyBlue}, {Red}, {NavyBlue},
    {Red}, {NavyBlue}}, PlotLabel → SinesCosines on [a, b]];
```



To illustrate the fact that this set of functions
$\{s_1[t], s_2[t], s_3[t], \ldots, s_k[t], \ldots\}$
is an orthogonal set on the interval $a \le t \le b$ with $a = -\pi$ and $b = \pi$,
here is a calculation of
$s_p \bullet s_q = \int_a^b s_p[t]\, s_q[t]\, dt$
for random positive integers p and q with $p \neq q$

```
q = Random[Integer, {1, 6}];
p = Random[Integer, {q + 1, 3 q}];
∫_a^b s_q[t] s_p[t] dt
```

0

Rerun a couple of times.

- **Sine-Cosine system on [0, 2π]:**

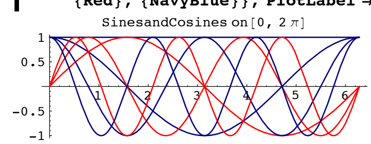
This set of functions
$\{s_1[t], s_2[t], s_3[t], ..\}$
$= \{1, \mathrm{Sin}[t], \mathrm{Cos}[t], \mathrm{Sin}[2\, t], \mathrm{Cos}[2\, t], \mathrm{Sin}[3\, t], \mathrm{Cos}[3\, t], \ldots\}$
is also orthogonal set on the interval $a \le t \le b$ with $a = 0$ and $b = 2\pi$.
Here are formulas for the first 15.

```
Clear[s, Subscript, k, t];
s_k_[t_] := If[EvenQ[k], Sin[k t / 2], Cos[1/2 (k - 1) t]];
Table[s_k[t], {k, 1, 15}]
```

{1, Sin[t], Cos[t], Sin[2 t], Cos[2 t], Sin[3 t], Cos[3 t], Sin[4 t], Cos[4 t], Sin[5 t], Cos[5 t], Sin[6 t], Cos[6 t], Sin[7 t], Cos[7 t]}

And plots of the first seven:

```
a = 0;
b = 2 π;
Plot[{s_1[t], s_2[t], s_3[t], s_4[t], s_5[t], s_6[t], s_7[t]}, {t, a, b},
  PlotStyle → {{NavyBlue}, {Red}, {NavyBlue}, {Red}, {NavyBlue},
    {Red}, {NavyBlue}}, PlotLabel → SinesandCosines on [a, b]];
```



To illustrate the fact that this set of functions
$\{s_1[t], s_2[t], s_3[t], \ldots, s_k[t], \ldots\}$
is an orthogonal set on the interval $a \le t \le b$ with $a = 0$ and $b = 2\pi$,
here is a calculation of
$s_p \bullet s_q = \int_a^b s_p[t]\, s_q[t]\, dt$
for random positive integers p and q with $p \neq q$

```
q = Random[Integer, {1, 6}];
p = Random[Integer, {q + 1, 3 q}];
Integrate[s_q[t] s_p[t], {t, a, b}]
```

0

Rerun a couple of times.

- **Legendre polynomial system on [-1,1]:**

The set of functions
$\{s_1[t], s_2[t], s_3[t], \ldots, s_k[t], \ldots\}$
for
$s_k[t] = \mathrm{LegendreP}[k - 1,\ t]$
is an orthogonal set on the interval $a \le t \le b$ with $a = -1$ and $b = 1$.
Here are the formulas for the first six:

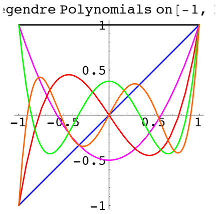
```
Clear[k];
ColumnForm[Table[LegendreP[k - 1, t], {k, 1, 6}]]
```

1
t
$-\frac{1}{2} + \frac{3 t^2}{2}$
$-\frac{3 t}{2} + \frac{5 t^3}{2}$
$\frac{3}{8} - \frac{15 t^2}{4} + \frac{35 t^4}{8}$
$\frac{15 t}{8} - \frac{35 t^3}{4} + \frac{63 t^5}{8}$

Here are plots of the first six:

```
Clear[s, Subscript, k, t];
s_k_[t_] := LegendreP[k - 1, t];
```

```
a = -1;
b = 1;
Plot[
  {s1[t], s2[t], s3[t], s4[t], s5[t], s6[t]}, {t, a, b}, PlotStyle →
   {{Black}, {Blue}, {Magenta}, {Red}, {Green}, {CadmiumOrange}},
  PlotLabel → Legendre Polynomials on[a, b]];
```



To illustrate the fact that this set of functions

$\{s_1[t], s_2[t], s_3[t], \ldots, s_k[t], \ldots\}$

is an orthogonal set on the interval $a \le t \le b$ with $a = -1$ and $b = 1$,
here is a calculation of

$$s_p \bullet s_q = \int_a^b s_p[t]\, s_q[t]\, dt$$

for random positive integers p and q with $p \neq q$

```
q = Random[Integer, {1, 6}];
p = Random[Integer, {q + 1, 3 q}];
Integrate[s_q[t] s_p[t], {t, a, b}]
```

0

Rerun a couple of times.

□B.2.a.ii) Using orthogonal families to make function spaces

How do you use orthogonal sets of functions to make function spaces?

□Answer:

Very easily.

You with a finite set of functions

$\{s_1[t], s_2[t], s_3[t], \ldots, s_k[t]\}$

orthogonal on the interval $a \le t \le b$ (so that $s_p \bullet s_q = \int_a^b s_p[t]\, s_q[t]\, dt = 0$ when $p \neq q$)

and then you make the function space S[a,b] spanned by

$\{s_1[t], s_2[t], s_3[t], \ldots, s_k[t]\}$.

This function space consists of all functions that are linear combinations of the spanning set

$\{s_1[t], s_2[t], s_3[t], \ldots, s_k[t]\}$.

In other words, S[a,b] consists of all functions f[t] of the form.

$$f[t] = \sum_{j=1}^{k} c_j\, s_j[t] \quad \text{with} \quad a \le t \le b.$$

The c_j's are numbers.

Here is a random member f[t] of the function space $S[0, \pi]$ spanned by

$\{s_1[t], s_2[t], s_3[t], s_4[t], s_5[t]\}$

for

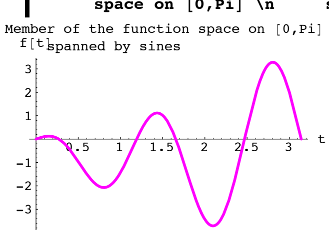
$s_k[t] = \mathrm{Sin}[k\, t]$

```
dim = 5;
Clear[f, s, g, k, t];
s_k_[t_] := Sin[k t];
a = 0;
b = π;
c_k_ := Random[Real, {-2, 2}]
f[t_] = Sum[c_k s_k[t], {k, 1, dim}]
```

-0.997406 Sin[t] - 0.174916 Sin[2 t] +
0.114702 Sin[3 t] - 1.67053 Sin[4 t] + 1.79011 Sin[5 t]

And its plot:

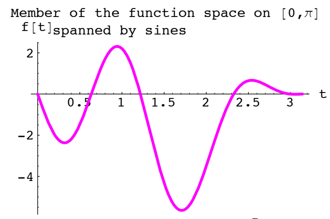
```
Plot[f[t], {t, a, b}, PlotStyle → {{Thickness[0.01], Magenta}},
  AxesLabel → {"t", "f[t]"}, AspectRatio → 1/GoldenRatio,
  PlotLabel → "Member of the function
    space on [0,Pi] \n      spanned by sines"];
```



See some more functions in this function space::

```
dim = 5;
Clear[f, s, g, k, t];
s_k_[t_] := Sin[k t];
a = 0;
b = π;
c_k_ := Random[Real, {-2, 2}];
f[t_] = Sum[c_k s_k[t], {k, 1, dim}]
Plot[f[t], {t, a, b}, PlotStyle → {{Thickness[0.01], Magenta}},
  AxesLabel → {"t", "f[t]"}, AspectRatio → 1/GoldenRatio,
  PlotLabel → "Member of the function
    space on [0,π] \n      spanned by sines"];
```

-1.90159 Sin[t] + 0.698589 Sin[2 t] +
1.46671 Sin[3 t] - 1.77681 Sin[4 t] - 1.68784 Sin[5 t]



Rerun a couple of times.

Here is a random member of the function space $S[0, \pi]$ spanned by

$\{s_1[t], s_2[t], s_3[t], s_4[t], s_5[t], s_6[t]\}$

for

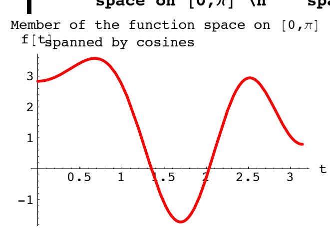
$s_k[t] = \mathrm{Cos}[(k - 1)\, t]$.

```
dim = 6;
Clear[f, s, g, k, t];
s_k_[t_] := Cos[(k - 1) t]
a = 0;
b = π;
c_k_ := Random[Real, {-2, 2}]
f[t_] = Sum[c_k s_k[t], {k, 1, dim}]
```

1.59484 + 0.846593 Cos[t] + 1.63982 Cos[2 t] -
0.290566 Cos[3 t] - 1.42125 Cos[4 t] + 0.465557 Cos[5 t]

And a plot:

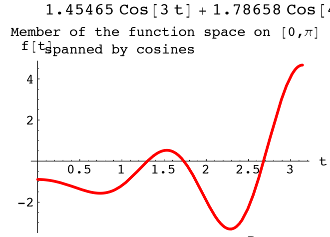
```
Plot[f[t], {t, a, b}, PlotStyle → {{Thickness[0.01], Red}},
  AxesLabel → {"t", "f[t]"}, AspectRatio → 1/GoldenRatio,
  PlotLabel → "Member of the function
    space on [0,π] \n      spanned by cosines"];
```



See some more functions in this function space::

```
dim = 6;
Clear[f, s, g, k, t];
s_k_[t_] := Cos[(k - 1) t]
a = 0;
b = π;
c_k_ := Random[Real, {-2, 2}]
f[t_] = Sum[c_k s_k[t], {k, 1, dim}]
Plot[f[t], {t, a, b}, PlotStyle → {{Thickness[0.01], Red}},
  AxesLabel → {"t", "f[t]"}, AspectRatio → 1/GoldenRatio,
  PlotLabel → "Member of the function
    space on [0,π] \n      spanned by cosines"];
```

-0.587073 - 0.812286 Cos[t] + 0.690917 Cos[2 t] -
1.45465 Cos[3 t] + 1.78658 Cos[4 t] - 0.516604 Cos[5 t]



Rerun a couple of times..

Here is a random member of the function space S[−1, 1] spanned by

$\{s_1[t], s_2[t], s_3[t], s_4[t]\}$

for

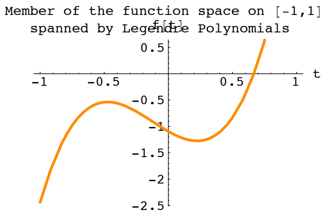
$s_k[t] = \text{LegendreP}[k-1, t]$.

```
dim = 4;
Clear[f, s, g, k, t];
s_k_[t_] := LegendreP[k - 1, t]
a = -1;
b = 1;
c_k_ := Random[Real, {-1, 2}]
f[t_] = Sum[c_k s_k[t], {k, 1, dim}]
```

$$-0.550849 + 1.23307\,t + 1.08799\left(-\frac{1}{2}+\frac{3t^2}{2}\right) + 1.74373\left(-\frac{3t}{2}+\frac{5t^3}{2}\right)$$

And its plot:

```
Plot[f[t], {t, a, b}, PlotStyle → {{Thickness[0.01], DarkOrange}},
  AxesLabel → {"t", "f[t]"}, AspectRatio → 1/GoldenRatio,
  PlotLabel → "Member of the function space on [
     -1,1] \n   spanned by Legendre Polynomials"];
```

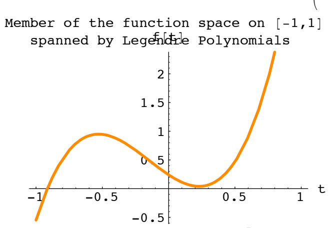


See more:

```
dim = 4;
Clear[f, s, g, k, t];
s_k_[t_] := LegendreP[k - 1, t]
a = -1;
b = 1;
c_k_ := Random[Real, {-1, 2}]
```

```
f[t_] = Sum[c_k s_k[t], {k, 1, dim}]
Plot[f[t], {t, a, b}, PlotStyle -> {{Thickness[0.01], DarkOrange}},
  AxesLabel -> {"t", "f[t]"}, AspectRatio -> 1 / GoldenRatio,
  PlotLabel -> "Member of the function space on [
     -1,1] \n   spanned by Legendre Polynomials"];
```

$$0.863125 + 0.985973\,t + 1.2454\left(-\frac{1}{2}+\frac{3t^2}{2}\right) + 1.66993\left(-\frac{3t}{2}+\frac{5t^3}{2}\right)$$



Rerun a couple of times..

□ B.2.a.iii) Reading the dimension of a function space

To make a function space S[a,b], you go with a finite set of functions

$\{s_1[t], s_2[t], s_3[t], \ldots, s_k[t]\}$

orthogonal on the interval $a \le t \le b$ and then you make the function space S[a,b] spanned by

$\{s_1[t], s_2[t], s_3[t], \ldots, s_k[t]\}$.

This function space consists of all functions f[t] of the form.

$$f[t] = \sum_{j=1}^{k} c_j\, s_j[t] \text{ with } a \le t \le b.$$

How do you read off the dimension of S[a,b]?

□ Answer:

The functions $f[t] = \sum_{j=1}^{k} c_j\, s_j[t]$ in S[a,b] allow freedom for k constants

$\{c_1, c_2, c_3, \ldots, c_k\}$.

So the dimension of S[a,b] is k.

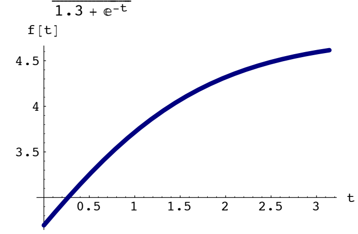
B.3) Fourier Approximations: Best root-mean-square approximation of a function by a member of a function space

□ B.3.a.i) Coming up with the member of a function space that is closest to a given function

Here's a function f[t] plotted on [0, π]:

```
Clear[f, t];
f[t_] = 6.2 / (1.3 + e^-t)
fplot =
  Plot[f[t], {t, 0, π}, PlotStyle → {{Thickness[0.015], NavyBlue}},
   AxesLabel → {"t", "f[t]"}];
```

$$\frac{6.2}{1.3 + e^{-t}}$$



The set of functions

$\{s_1[t], s_2[t], s_3[t], \ldots, s_k[t], \ldots\}$

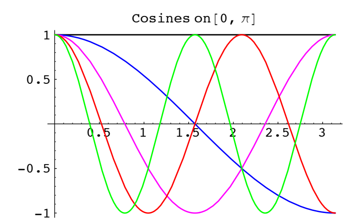
for

$s_k[t] = \text{Cos}[(k-1)\,t]$

is an orthogonal set on the interval $a \le t \le b$ with a = 0 and b = π.

Here are plots of the first five.

```
Clear[s, Subscript, k, t];
s_k_[t_] := Cos[(k - 1) t]
a = 0;
b = π;
Plot[{s_1[t], s_2[t], s_3[t], s_4[t], s_5[t]}, {t, a, b},
  PlotStyle → {{Black}, {Blue}, {Magenta}, {Red}, {Green}},
  PlotLabel → Cosines on[a, b]];
```



Go with the function space S[a,b] spanned by:

```
khigh = 3;
Table[ s_k[t], {k, 1, khigh}]
```

{1, Cos[t], Cos[2 t]}

Come up with the function Sclosest[t] in S[0,Pi] so that the root-mean-square distance between f[t] and Sclosest[t] is as small as possible

□ Answer:

It's not bad at all.

For each k, you take the component of f[t] in the direction of $s_k[t]$:

$$\frac{f \bullet s_k}{s_k \bullet s_k}\, s_k[t]$$

and then you add them up to get

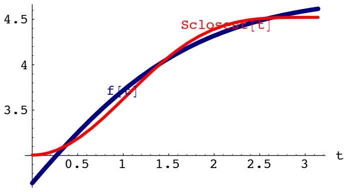
$$\text{Sclosest}[t] = \sum_{k=1}^{khigh} \frac{f \bullet s_k}{s_k \bullet s_k}\, s_k[t]:$$

```
Clear[fouriercoeff, Sclosest, k];
fouriercoeff[k_] :=
  fouriercoeff[k] = NIntegrate[f[t] s_k[t], {t, a, b}] / NIntegrate[s_k[t] s_k[t], {t, a, b}];
Sclosest[t_] = Sum[fouriercoeff[k] s_k[t], {k, 1, khigh}]
```

3.95273 − 0.758521 Cos[t] − 0.189524 Cos[2 t]

See the quality of the approximation of f[t] by Sclosest[t]:

```
fitplot = Plot[{f[t], Sclosest[t]},
   {t, a, b}, PlotStyle -> {{Thickness[0.015], NavyBlue},
    {Thickness[0.01], Red}}, AxesLabel -> {"t", ""},
   Epilog -> {{NavyBlue, Text["f[t]", {a + 1, f[a + 1]}]},
    {Red, Text["Sclosest[t]", {b - 1, Sclosest[b - 1]}]}}];
```



Not bad.

The root-mean-square distance between f[t] and Sclosest[t] is:

```
Sqrt[NIntegrate[(f[t] - Sclosest[t])^2, {t, a, b}]]
```

0.146193

□B.3.a.ii) Increasing the quality of the approximation

Stay with the same set up as above.
What can you do to increase the quality of the approximation of f[t] by a combination of Cosines?

□Answer:

Do the same thing but use more cosines, by raising khigh from 3 (as it was above) to 5 (or any other integer bigger than 3).

The function in the function space S[a,b] spanned by

```
khigh = 8;
Table[ s_k[t], {k, 1, khigh}]
```

{1, Cos[t], Cos[2 t], Cos[3 t], Cos[4 t], Cos[5 t], Cos[6 t], Cos[7 t]}

that is closest to f[t] is

$$\text{Sclosest}[t] = \sum_{k=1}^{\text{khigh}} \frac{f \bullet s_k}{s_k \bullet s_k}\, s_k[t]$$

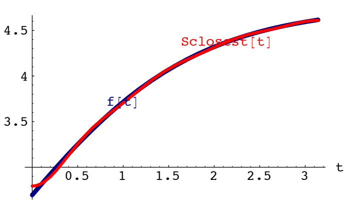
See the formula:

```
Clear[fouriercoeff, Sclosest, k];
fouriercoeff[k_] :=
  fouriercoeff[k] = NIntegrate[f[t] s_k[t], {t, a, b}] / NIntegrate[s_k[t] s_k[t], {t, a, b}];
Sclosest[t_] = Sum[fouriercoeff[k] s_k[t], {k, 1, khigh}]
```

3.95273 - 0.758521 Cos[t] - 0.189524 Cos[2 t] -
0.097787 Cos[3 t] - 0.0425985 Cos[4 t] -
0.0341212 Cos[5 t] - 0.0184477 Cos[6 t] - 0.0172782 Cos[7 t]

See the quality of the approximation of f[t] by Sclosest[t]:

```
fitplot = Plot[{f[t], Sclosest[t]},
   {t, a, b}, PlotStyle -> {{Thickness[0.015], NavyBlue},
     {Thickness[0.01], Red}}, AxesLabel -> {"t", ""},
   Epilog -> {{NavyBlue, Text["f[t]", {a + 1, f[a + 1]}]},
     {Red, Text["Sclosest[t]", {b - 1, Sclosest[b - 1]}]}}];
```



Sharing lots of ink.

That's very good.

You can do even better if you use more cosines by raising khigh:

The function in the function space S[a,b] spanned by

```
khigh = 15;
Table[ s_k[t], {k, 1, khigh}]
```

{1, Cos[t], Cos[2 t], Cos[3 t], Cos[4 t],
Cos[5 t], Cos[6 t], Cos[7 t], Cos[8 t], Cos[9 t],
Cos[10 t], Cos[11 t], Cos[12 t], Cos[13 t], Cos[14 t]}

that is closest to f[t] is

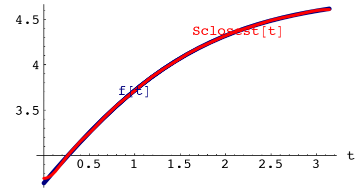
$$\text{Sclosest}[t] = \sum_{k=1}^{\text{khigh}} \frac{f \bullet s_k}{s_k \bullet s_k}\, s_k[t]:$$

```
Clear[fouriercoeff, Sclosest, k];
fouriercoeff[k_] :=
  fouriercoeff[k] = NIntegrate[f[t] s_k[t], {t, a, b}] / NIntegrate[s_k[t] s_k[t], {t, a, b}];
Sclosest[t_] = Sum[fouriercoeff[k] s_k[t], {k, 1, khigh}]
```

3.95273 - 0.758521 Cos[t] - 0.189524 Cos[2 t] -
0.097787 Cos[3 t] - 0.0425985 Cos[4 t] - 0.0341212 Cos[5 t] -
0.0184477 Cos[6 t] - 0.0172782 Cos[7 t] - 0.010289 Cos[8 t] -
0.0104222 Cos[9 t] - 0.0065597 Cos[10 t] - 0.00696701 Cos[11 t] -
0.00454596 Cos[12 t] - 0.00498423 Cos[13 t] - 0.00333577 Cos[14 t]

See the quality of the approximation of f[t] by Sclosest[t]:

```
fitplot = Plot[{f[t], Sclosest[t]},
   {t, a, b}, PlotStyle -> {{Thickness[0.015], NavyBlue},
     {Thickness[0.01], Red}}, AxesLabel -> {"t", ""},
   Epilog -> {{NavyBlue, Text["f[t]", {a + 1, f[a + 1]}]},
     {Red, Text["Sclosest[t]", {b - 1, Sclosest[b - 1]}]}}];
```



Very,very good.

Cool fit.

Sharing ink almost all the way.

The root-mean-square distance between f[t] and Sclosest[t] is:

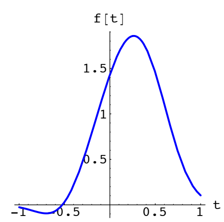
```
Sqrt[NIntegrate[(f[t] - Sclosest[t])^2, {t, a, b}]]
```

0.00998227

□B.3.a.iii) Legendre

Here is a function f[t] plotted on [-1,1]:

```
Clear[f, t];
f[t_] = 3.29 (Sin[2 t] + Cos[t]) / (1.3 + E^(3.8 t^2));
fplot = Plot[f[t], {t, -1, 1},
   PlotStyle -> {{Thickness[0.01], Blue}}, AxesLabel -> {"t", "f[t]"}];
```



The set of functions

$\{s_1[t], s_2[t], s_3[t], \ldots, s_k[t], \ldots\}$

for

$s_k[t] = \text{LegendreP}[k-1, t]$

is an orthogonal set on the interval $a \le t \le b$ with $a = -1$ and $b = 1$.
Here are the formulas for the first six:

```
Clear[k];
ColumnForm[Table[LegendreP[k - 1, t], {k, 1, 6}]]
```

1

t

$-\frac{1}{2} + \frac{3 t^2}{2}$

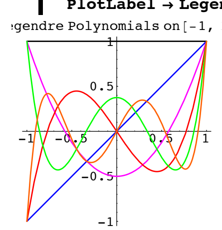
$-\frac{3 t}{2} + \frac{5 t^3}{2}$

$\frac{3}{8} - \frac{15 t^2}{4} + \frac{35 t^4}{8}$

$\frac{15 t}{8} - \frac{35 t^3}{4} + \frac{63 t^5}{8}$

Here are plots of the first six:

```
Clear[s, g, k, t]; s_k_[t_] := LegendreP[k - 1, t]; a = -1; b = 1;
Plot[{s_1[t], s_2[t], s_3[t], s_4[t], s_5[t], s_6[t]}, {t, a, b}, PlotStyle ->
   {{Black}, {Blue}, {Magenta}, {Red}, {Green}, {CadmiumOrange}},
  PlotLabel -> Legendre Polynomials on[a, b]];
```



Go with the function space S[a,b] spanned by:

```
khigh = 9;
Table[ s_k[t], {k, 1, khigh}]
```

$$\left\{1,\ t,\ -\frac{1}{2}+\frac{3t^2}{2},\ -\frac{3t}{2}+\frac{5t^3}{2},\ \frac{3}{8}-\frac{15t^2}{4}+\frac{35t^4}{8},\ \frac{15t}{8}-\frac{35t^3}{4}+\frac{63t^5}{8},\ -\frac{5}{16}+\frac{105t^2}{16}-\frac{315t^4}{16}+\frac{231t^6}{16},\ -\frac{35t}{16}+\frac{315t^3}{16}-\frac{693t^5}{16}+\frac{429t^7}{16},\ \frac{35}{128}-\frac{315t^2}{32}+\frac{3465t^4}{64}-\frac{3003t^6}{32}+\frac{6435t^8}{128}\right\}$$

Come up with the function Sclosest[t] in S[a,b] so that the root-mean-square distance between f[t] and Sclosest[t] is as small as possible

□ Answer:

You do it the same way you did it with Cosines in part i).

For each k you take the component of f[t] in the direction of $s_k[t]$:

$$\frac{f \bullet s_k}{s_k \bullet s_k}\, s_k[t]$$

and then you add them up to get

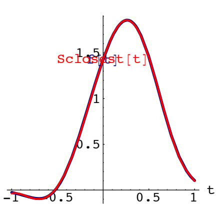
$$\text{Sclosest}[t] = \sum_{k=1}^{\text{khigh}} \frac{f \bullet s_k}{s_k \bullet s_k}\, s_k[t]:$$

```
Clear[fouriercoeff, Sclosest, k];
fouriercoeff[k_] :=
  fouriercoeff[k] = NIntegrate[f[t] s_k[t], {t, a, b}] / NIntegrate[s_k[t] s_k[t], {t, a, b}];
Sclosest[t_] = Sum[fouriercoeff[k] s_k[t], {k, 1, khigh}]
```

$$0.74298 + 0.602838\,t - 1.05741\left(-\frac{1}{2}+\frac{3t^2}{2}\right) - 0.88445\left(-\frac{3t}{2}+\frac{5t^3}{2}\right) + 0.40944\left(\frac{3}{8}-\frac{15t^2}{4}+\frac{35t^4}{8}\right) + 0.430588\left(\frac{15t}{8}-\frac{35t^3}{4}+\frac{63t^5}{8}\right) - 0.0423636\left(-\frac{5}{16}+\frac{105t^2}{16}-\frac{315t^4}{16}+\frac{231t^6}{16}\right) - 0.0804116\left(-\frac{35t}{16}+\frac{315t^3}{16}-\frac{693t^5}{16}+\frac{429t^7}{16}\right) - 0.0219155\left(\frac{35}{128}-\frac{315t^2}{32}+\frac{3465t^4}{64}-\frac{3003t^6}{32}+\frac{6435t^8}{128}\right)$$

See the quality of the approximation of f[t] by Sclosest[t]:

```
fitplot = Plot[{f[t], Sclosest[t]},
   {t, a, b}, PlotStyle -> {{Thickness[0.015], NavyBlue},
     {Thickness[0.01], Red}}, AxesLabel -> {"t", ""},
   Epilog -> {{NavyBlue, Text["f[t]", {a + 1, f[a + 1]}]},
     {Red, Text["Sclosest[t]", {b - 1, Sclosest[b - 1]}]}}];
```



Lookin' good and feeling good.

The root-mean-square distance between f[t] and closest[t] is:

```
Sqrt[NIntegrate[(f[t] - Sclosest[t])^2, {t, a, b}]]
```

0.0051227

□ B.3.a.iv) Other orthogonal families

Does the same technique work for all function spaces S[a,b] spanned by an orthogonal set on [a,b]?

□ Answer:

Sure does.

□ B.3.a.v) Why it works

Why does this technique work for all function spaces S[a,b] spanned by an orthogonal set on[a,b]?

□ Answer:

□ Perpendicularity explanation

Put $\text{Sclosest}[t] = \sum_{k=1}^{n} \frac{f \bullet s_k}{s_k \bullet s_k}\, s_k[t]$

and take p with $1 \le p \le n$ and look at

$$(f - \text{Sclosest}) \bullet s_p = f \bullet s_p - s \bullet s_p$$

$$= f \bullet s_p - \sum_{k=1}^{n} \frac{f \bullet s_k}{s_k \bullet s_k}\, s_k \bullet s_p$$

$$= f \bullet s_p - \frac{f \bullet s_p}{s_p \bullet s_p}\, s_p \bullet s_p \qquad (\text{because } s_k \bullet s_p = 0 \text{ for } k \neq p)$$

$$= f \bullet s_p - f \bullet s_p = 0.$$

This little manipulation reveals that

(f[t] - Sclosest[[t])

is perpendicular to each of the functions

$\{s_1[t], s_2[t], s_3[t], \ldots, s_n[t]\}$

and so (f[t] - Sclosest[t]) is perpendicular to every function

$$g[t] = \sum_{k=1}^{n} c_k\, s_k[t]$$

in the function space S spanned by $\{s_1[t], s_2[t], s_3[t], \ldots, s_n[t]\}$.

And because perpendicular distance is the shortest distance,

$$\text{Sclosest}[t] = \sum_{k=1}^{n} \frac{f \bullet s_k}{s_k \bullet s_k}\, s_k[t]$$

is the function in S closest to f[t].

□ Calculus explanation

Agree that

$$g[t] = \sum_{k=1}^{n} c_k\, s_k[t]$$

is the generic member of the function space S spanned by $\{s_1[t], s_2[t], s_3[t], \ldots, s_n[t]\}$.

Look at the square of the root-mean-square distance from f to g:

$$\| f - g \|^2 = (f - g) \bullet (f - g) = f \bullet f - 2 f \bullet g + g \bullet g)$$

This is the same as

$$\| f - g \|^2 = f \bullet f - 2 f \bullet \sum_{k=1}^{n} c_k\, s_k + \sum_{k=1}^{n} c_k\, s_k \bullet \sum_{k=1}^{n} c_k\, s_k\ .$$

This is the same as

$$\| f - g \|^2 = f \bullet f - 2 f \bullet \sum_{k=1}^{n} c_k\, s_k + \sum_{k=1}^{n} c_k\, s_k \bullet \sum_{k=1}^{n} c_p\, s_p\ .$$

This is the same as

$$\| f - g \|^2 = f \bullet f - 2 \sum_{k=1}^{n} c_k\, f \bullet s_k + \sum_{k=1}^{n} c_k^{\,2}\, s_k \bullet s_k \qquad \text{because } s_k \bullet s_p = 0 \text{ for } k \neq p.$$

Now remember that $g[t] = \sum_{k=1}^{n} c_k\, s_k[t]$.

To see what c_k's make $\| f - g \|^2$ as small as possible,

differentiate $\| f - g \|^2$ with respect to one of the c_k's - say c_p- with $1 \le p \le n$ and set the result equal to 0.

This gives

$$\partial_{c_p} \| f - g \|^2 = 0 - 2 f \bullet s_p + 2 c_p\, s_p \bullet s_p = 0.$$

Clean this up to get

$$c_p = \frac{f \bullet s_p}{s_p \bullet s_p}$$

This tells you that the minimizing c_k 's are given by

$$c_k = \frac{f \bullet s_k}{s_k \bullet s_k}\ .$$

And this says that in no uncertain terms that the function g[t] in the function space S spanned by $\{s_1[t], s_2[t], s_3[t], \ldots, s_n[t]\}$ closest to f[t] is

$$\text{Sclosest}[t] = \sum_{k=1}^{n} \frac{f \bullet s_k}{s_k \bullet s_k}\, s_k[t]$$

B.4) Complete orthogonal sets of functions

□ B.4.a.i) Complete orthogonal families

What do folks mean when they say that a given orthogonal family is complete?

□ Answer:

Folks say that a given orthogonal family

$\{s_1[t], s_2[t], s_3[t], \ldots, s_k[t], \ldots\}$

on the interval $a \le t \le b$ is complete if you are guaranteed that you can get great root-mean-square approximations on [a,b] of any function f[t] on [a,b] with $\int_a^b f[t]^2\, dt <$ Infinity via the approximations

$$\sum_{k=1}^{\text{khigh}} \frac{f \bullet s_k}{s_k \bullet s_k}\, s_k[t]$$

for large khigh.

This is the same as saying that the root mean-square distance on [a,b] between

$\sum_{k=1}^{\text{khigh}} \frac{f \bullet s_k}{s_k \bullet s_k}\, s_k[t]$ and f[t] goes to 0 as khigh gets large.

In other words.

$$\sqrt{\int_a^b \left(\sum_{k=1}^{khigh} \frac{f \bullet s_k\, s_k[t]}{s_k \bullet s_k} - f[t]\right)^2 dt} \to 0 \text{ as khigh gets large.}$$

In this case, folks like to say that the Fourier series

$$\sum_{k=1}^{\infty} \frac{f \bullet s_k}{s_k \bullet s_k}\, s_k[t]$$

converges to f[t] in the sense of root-mean-square distance on the interval [a,b].

□B.4.a.ii) Examples of complete orthogonal families

Give some examples of orthogonal systems that are known to be complete.

□Answer:

Some examples:

The Sine system on [0,π]

The Cosine system on [0,π]

The Sine -Cosine system on [-π,π]

The Sine -Cosine system on [0,2π]

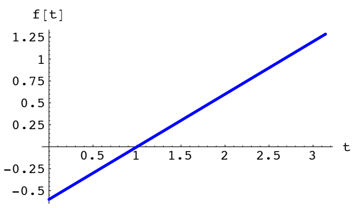
The Legendre Polynomial system on [-1,1]

Proofs of the completeness of these systems are given in graduate level math courses.

□B.4.a.iii) Sometimes the approximation breaks down in isolated spots

Here is a function f[t] plotted on [0, π]:

```
a = 0;
b = π;
Clear[f, t];
f[t_] = 0.6 (t - 1);
fplot = Plot[f[t], {t, a, b}, PlotStyle -> {{Thickness[0.01], Blue}},
  AxesLabel -> {"t", "f[t]"}];
```



The set of functions

$\{s_1[t], s_2[t], s_3[t], \ldots, s_k[t], \ldots\}$

for

$s_k[t] = \text{Sin}[k\, t]$

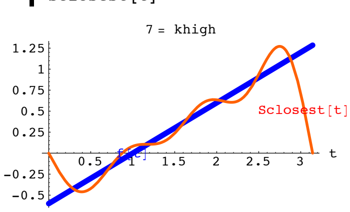
is a complete orthogonal set on the interval $a \le t \le b$ with a = 0 and b = π.

Look at these attempts to approximate f[t] in terms of these sines:

```
khigh = 7;
Clear[s, k, t];
s_k_[t_] := Sin[k t];

Clear[fouriercoeff, Sclosest, k];
fouriercoeff[k_] := NIntegrate[f[t] s_k[t], {t, a, b}] / NIntegrate[s_k[t] s_k[t], {t, a, b}];
Sclosest[t_] = Sum[fouriercoeff[k] s_k[t], {k, 1, khigh}];
fitplot = Plot[{f[t], Sclosest[t]},
  {t, a, b}, PlotStyle -> {{Thickness[0.02], Blue},
    {Thickness[0.01], CadmiumOrange}}, AxesLabel -> {"t", ""},
  Epilog -> {{Blue, Text["f[t]", {a + 1, f[a + 1]}]},
    {Red, Text["Sclosest[t]", {b - 0.1, Sclosest[b - 0.1]}]}},
  PlotRange -> All, PlotLabel -> khigh "= khigh"];

Sclosest[t]
```



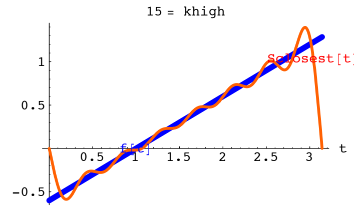
0.436056 Sin[t] - 0.6 Sin[2 t] + 0.145352 Sin[3 t] - 0.3 Sin[4 t] +
0.0872113 Sin[5 t] - 0.2 Sin[6 t] + 0.0622938 Sin[7 t]

Use more sines:

```
khigh = 15;

Sclosest[t_] = Sum[fouriercoeff[k] s_k[t], {k, 1, khigh}];

fitplot = Plot[{f[t], Sclosest[t]},
  {t, a, b}, PlotStyle -> {{Thickness[0.02], Blue},
    {Thickness[0.01], CadmiumOrange}}, AxesLabel -> {"t", ""},
  Epilog -> {{Blue, Text["f[t]", {a + 1, f[a + 1]}]},
    {Red, Text["Sclosest[t]", {b - 0.1, Sclosest[b - 0.1]}]}},
  PlotRange -> All, PlotLabel -> khigh "= khigh"];
Sclosest[
 t]
```



0.436056 Sin[t] - 0.6 Sin[2 t] + 0.145352 Sin[3 t] - 0.3 Sin[4 t] +
0.0872113 Sin[5 t] - 0.2 Sin[6 t] + 0.0622938 Sin[7 t] - 0.15 Sin[8 t] +
0.0484507 Sin[9 t] - 0.12 Sin[10 t] + 0.0396415 Sin[11 t] - 0.1 Sin[12 t] +
0.0335428 Sin[13 t] - 0.0857143 Sin[14 t] + 0.0290704 Sin[15 t]

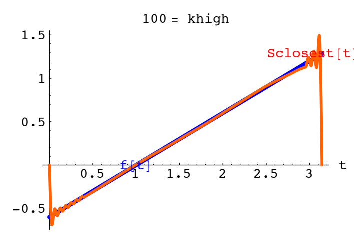
Use a helluva lot of sines:

```
khigh = 100;

Sclosest[t_] = Sum[fouriercoeff[k] s_k[t], {k, 1, khigh}];

fitplot = Plot[{f[t], Sclosest[t]},
  {t, a, b}, PlotStyle -> {{Thickness[0.015], Blue},
    {Thickness[0.01], CadmiumOrange}}, AxesLabel -> {"t", ""},
  Epilog -> {{Blue, Text["f[t]", {a + 1, f[a + 1]}]},
    {Red, Text["Sclosest[t]", {b - 0.1, Sclosest[b - 0.1]}]}},
  PlotRange -> All, PlotLabel -> khigh "= khigh"];

Sclosest[t]
```



0.436056 Sin[t] - 0.6 Sin[2 t] + 0.145352 Sin[3 t] - 0.3 Sin[4 t] +
0.0872113 Sin[5 t] - 0.2 Sin[6 t] + 0.0622938 Sin[7 t] -
0.15 Sin[8 t] + 0.0484507 Sin[9 t] - 0.12 Sin[10 t] +
0.0396415 Sin[11 t] - 0.1 Sin[12 t] + 0.0335428 Sin[13 t] -
0.0857143 Sin[14 t] + 0.0290704 Sin[15 t] - 0.075 Sin[16 t] +
0.0256504 Sin[17 t] - 0.0666667 Sin[18 t] + 0.0229503 Sin[19 t] -
0.06 Sin[20 t] + 0.0207646 Sin[21 t] - 0.0545455 Sin[22 t] +
0.018959 Sin[23 t] - 0.05 Sin[24 t] + 0.0174423 Sin[25 t] -
0.0461538 Sin[26 t] + 0.0161502 Sin[27 t] - 0.0428571 Sin[28 t] +
0.0150364 Sin[29 t] - 0.04 Sin[30 t] + 0.0140663 Sin[31 t] -
0.0375 Sin[32 t] + 0.0132138 Sin[33 t] - 0.0352941 Sin[34 t] +
0.0124588 Sin[35 t] - 0.0333333 Sin[36 t] + 0.0117853 Sin[37 t] -
0.0315789 Sin[38 t] + 0.0111809 Sin[39 t] - 0.03 Sin[40 t] +
0.0106355 Sin[41 t] - 0.0285714 Sin[42 t] + 0.0101408 Sin[43 t] -
0.0272727 Sin[44 t] + 0.00969014 Sin[45 t] - 0.026087 Sin[46 t] +
0.00927779 Sin[47 t] - 0.025 Sin[48 t] + 0.00889911 Sin[49 t] -
0.024 Sin[50 t] + 0.00855012 Sin[51 t] - 0.0230769 Sin[52 t] +
0.00822748 Sin[53 t] - 0.0222222 Sin[54 t] + 0.0079283 Sin[55 t] -
0.0214286 Sin[56 t] + 0.00765011 Sin[57 t] - 0.0206897 Sin[58 t] +
0.00739078 Sin[59 t] - 0.02 Sin[60 t] + 0.00714846 Sin[61 t] -
0.0193548 Sin[62 t] + 0.00692153 Sin[63 t] - 0.01875 Sin[64 t] +
0.00670856 Sin[65 t] - 0.0181818 Sin[66 t] + 0.0065083 Sin[67 t] -
0.0176471 Sin[68 t] + 0.00631966 Sin[69 t] - 0.0171429 Sin[70 t] +
0.00614164 Sin[71 t] - 0.0166667 Sin[72 t] + 0.00597337 Sin[73 t] -
0.0162162 Sin[74 t] + 0.00581408 Sin[75 t] - 0.0157895 Sin[76 t] +
0.00566307 Sin[77 t] - 0.0153846 Sin[78 t] + 0.0055197 Sin[79 t] -
0.015 Sin[80 t] + 0.00538341 Sin[81 t] - 0.0146341 Sin[82 t] +
0.00525369 Sin[83 t] - 0.0142857 Sin[84 t] + 0.00513007 Sin[85 t] -
0.0139535 Sin[86 t] + 0.00501214 Sin[87 t] - 0.0136364 Sin[88 t] +
0.00489951 Sin[89 t] - 0.0133333 Sin[90 t] + 0.00479183 Sin[91 t] -
0.0130435 Sin[92 t] + 0.00468878 Sin[93 t] - 0.012766 Sin[94 t] +
0.00459007 Sin[95 t] - 0.0125 Sin[96 t] + 0.00449543 Sin[97 t] -
0.0122449 Sin[98 t] + 0.00440461 Sin[99 t] - 0.012 Sin[100 t]

The approximation is great except at the endpoints.
Does this fly in the face of the fact that

{Sin[t], Sin[2 t], Sin[3 t], Sin[4 t], .. .}

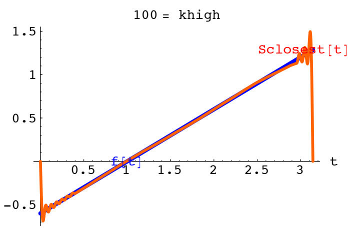
is a complete orthogonal family on $[0, \pi]$?

□ Answer:

Not at all.

Look again:

```
Show[fitplot];
```



The root-mean-square distance between f[t] and Sclosest[t] is very small:

```
√NIntegrate[(f[t] - Sclosest[t])^2, {t, a, b}]
```

0.112653

The point here is that the root mean square distance between f[t] and Sclosest[t]

$$\sqrt{\int_a^b (f[t] - \text{Sclosest}[t])^2 \, dt}$$

can be very small without having the plots of f[t] and Sclosest[t] sharing ink the whole way across the interval.

But when the root mean square distance between f[t] and Sclosest[t]

$$\sqrt{\int_a^b (f[t] - \text{Sclosest}[t])^2 \, dt}$$

is very small ,

the plots of f[t] and Sclosest[t] do have to share lots of ink except at some at some isolated parts of [a,b].

B.5) Linear independence and linear dependence for sets of functions

Orthogonal families are guaranteed to be linearly independent.

The determinant test for linear independence or linear dependence

□ B.5.a.i) Definitions of linear independence and linear dependence for sets of functions

What do folks mean when they say that a given set of functions is linearly independent?
What do the same folks mean when they say that a given set of functions is linearly dependent?

□ Answer:

Here's the inside scoop:

▪ A set of functions { $f_1[t], f_2[t], f_3[t], \ldots, f_n[t]$ } is linearly independent on an interval [a,b] if none of the functions in this set are in the function space spanned by the others.

▪ A set of functions { $f_1[t], f_2[t], f_3[t], \ldots, f_n[t]$ } is linearly dependent on an interval [a,b] at least one of functions in this set are in the function space spanned by the others.

□ B.5.a.ii) Orthogonal families are linearly independent

A given set of non-zero functions

$\{s_1[t], s_2[t], s_3[t], \ldots, s_k[t]\}$

defined on an interval [a,b] are orthogonal (mutually perpendicular) on [a,b]
if

$$s_p \bullet s_q = \int_a^b s_p[t]\, s_q[t]\, dt = 0 \quad \text{when } p \neq q.$$

and

$$\int_a^b s_p[t]^2 \, dt > 0 \text{ for all the p's.}$$

Explain this:

If $\{s_1[t], s_2[t], s_3[t], \ldots, s_k[t]\}$ are orthogonal on [a,b], then $\{s_1[t], s_2[t], s_3[t], \ldots, s_k[t]\}$ is linearly independent on [a,b].

□ Answer:

This will be an explanation by contradiction.

Assume $\{s_1[t], s_2[t], s_3[t], \ldots, s_k[t]\}$ are orthogonal on [a,b] but not linearly independent on [a,b].

This tells you that one of the functions is in the function space spanned by the others,
Relabel the set so that this one function is $s_1[t]$. And because $s_1[t]$. is in the function space spanned by the others, you know that

$$s_1[t]. = \sum_{j=2}^{k} c_j\, s_j[t]. \text{ for certain numbers } \{c_2, c_3, \ldots, c_k\}.$$

Multiply through by $s_1[t]$. to get

$$s_1[t]^2. = \sum_{j=2}^{k} c_j\, s_1[t].\, s_j[t].$$

Integrate from a to b to get:

$$\int_a^b s_1[t]^2 \, dt = \sum_{j=2}^{k} c_j \int_a^b s_1[t]\, s_j[t]\, dt$$

This gives

$$\int_a^b s_1[t]^2 \, dt = \sum_{j=2}^{k} c_j\, 0 = 0.$$

because $s_p \bullet s_q = \int_a^b s_p[t]\, s_q[t]\, dt = 0$ when $p \neq q$.

This leads to the conclusion that $\int_a^b s_1[t]^2 \, dt = 0$, which is a contradiction because $\int_a^b s_p[t]^2 \, dt > 0$ for all the p's.

□ B.5.b.i) The determinant test for linear independence or dependence

What is the determinant test for linear independence and linear dependence?
And how do you use it?

□ Answer:

▪Saying a set of three functions { $f_1[t], f_2[t], f_3[t]$ } is linearly **dependent** on an interval [a,b] is the same as saying

$$\text{Det}\left[\begin{pmatrix} f_1[t_1] & f_2[t_1] & f_3[t_1] \\ f_1[t_2] & f_2[t_2] & f_3[t_2] \\ f_1[t_3] & f_2[t_3] & f_3[t_3] \end{pmatrix}\right] = 0 \text{ no matter what } t_1, t_2 \text{ and } t_3 \text{ in [a,b] are.}$$

▪Saying a set of three functions { $f_1[t], f_2[t], f_3[t]$ } is linearly **independent** on an interval [a,b] is the same as saying

$$\text{Det}\left[\begin{pmatrix} f_1[t_1] & f_2[t_1] & f_3[t_1] \\ f_1[t_2] & f_2[t_2] & f_3[t_2] \\ f_1[t_3] & f_2[t_3] & f_3[t_3] \end{pmatrix}\right] \neq 0 \text{ for at least one choice of } t_1, t_2 \text{ and } t_3 \text{ in [a,b].}$$

Here's how you use it on the sample case

{ $f_1[t], f_2[t], f_3[t]$ } = { e^t, e^{-t}, Sinh[t] } on [a,b] = [-2,2]

Make the test matrix:

```
Clear[s, f, Subscript, k, t];
{f₁[t_], f₂[t_], f₃[t_]} = {eᵗ, e⁻ᵗ, Sinh[t]};
testmatrix = Table[fⱼ[tᵢ], {i, 1, 3}, {j, 1, 3}];
MatrixForm[testmatrix]
```

$$\begin{pmatrix} e^{t_1} & e^{-t_1} & \text{Sinh}[t_1] \\ e^{t_2} & e^{-t_2} & \text{Sinh}[t_2] \\ e^{t_3} & e^{-t_3} & \text{Sinh}[t_3] \end{pmatrix}$$

Evaluate the determinant of the test matrix at some random selections of t_1, t_2 and t_3 in [-2,2]:

```
{a, b} = {-2, 2};
Det[testmatrix] /. {t₁ -> Random[Real, {a, b}],
  t₂ -> Random[Real, {a, b}], t₃ -> Random[Real, {a, b}]}
```

0

Rerun many times.

Zero every time.

All indications are that { $f_1[t], f_2[t], f_3[t]$ } is a linearly dependent set.

Give it the acid test by calculating Det[testmatrix] and letting *Mathematica* try to simplify:

```
Simplify[Det[testmatrix]]
```

0

Zero.

The call:

{ $f_1[t], f_2[t], f_3[t]$ } = { e^t, e^{-t}, Sinh[t] }: is linearly **dependent** on any interval [a,b].

This isn't terribly surprising because $\text{Sinh}[t] = \frac{1}{2} e^t - \frac{1}{2} e^{-t}$.

Here's how you use it on the sample case

{ $f_1[t], f_2[t], f_3[t]$ } = { e^t, t, e^{-t} } and [a,b] = [-1,1]

Make the test matrix:

```
Clear[s, f, Subscript, k, t];
{f₁[t_], f₂[t_], f₃[t_]} = {eᵗ, t², e⁻ᵗ};
testmatrix = Table[fⱼ[tᵢ], {i, 1, 3}, {j, 1, 3}];
MatrixForm[testmatrix]
```

$$\begin{pmatrix} e^{t_1} & t_1^2 & e^{-t_1} \\ e^{t_2} & t_2^2 & e^{-t_2} \\ e^{t_3} & t_3^2 & e^{-t_3} \end{pmatrix}$$

Evaluate the determinant of the test matrix at some random selections of t_1, t_2 and t_3 in [-1,1]:

```
{a, b} = {-1, 1};
Det[testmatrix] /. {t1 -> Random[Real, {a, b}],
  t2 -> Random[Real, {a, b}], t3 -> Random[Real, {a, b}]}
```

0.330211

Rerun a couple of times.

Not zero.

The call:

{ $f_1[t]$, $f_2[t]$, $f_3[t]$} = {e^t, t^2, e^{-t}}: is linearly **independent** on any interval [a,b] that contains [-1,1].

B.5.b.ii) The determinant test works for families of two or more functions

Does the determinant test for linear dependence or independence work for sets of functions with any number of members?

Answer:

You betcha!

- Here's how you use it on the sample case

{ $f_1[t]$, $f_2[t]$, $f_3[t]$, $f_4[t]$} = {Sin[t], t, t^3, t^5} on [a,b] = [0, 2 π]:

Make the test matrix:

```
Clear[s, f, Subscript, k, t];
{f1[t_], f2[t_], f3[t_], f4[t_]} = {Sin[t], t, t^3, t^5};
testmatrix = Table[fj[ti], {i, 1, 4}, {j, 1, 4}];
MatrixForm[testmatrix]
```

$$\begin{pmatrix} \text{Sin}[t_1] & t_1 & t_1^3 & t_1^5 \\ \text{Sin}[t_2] & t_2 & t_2^3 & t_2^5 \\ \text{Sin}[t_3] & t_3 & t_3^3 & t_3^5 \\ \text{Sin}[t_4] & t_4 & t_4^3 & t_4^5 \end{pmatrix}$$

Evaluate the determinant of the test matrix at some random selections of t_1, t_2, t_3 and t_4 in [0, 2 π]:

```
{a, b} = {0, 2 π};
Det[testmatrix] /.
 {t1 -> Random[Real, {a, b}], t2 -> Random[Real, {a, b}],
  t3 -> Random[Real, {a, b}], t4 -> Random[Real, {a, b}]}
```

-54111.6

Rerun a couple of times.

Not zero.

The call:

{$f_1[t]$, $f_2[t]$, $f_3[t]$, $f_4[t]$} = {Sin[t], t, t^3, t^5}: is linearly **independent** on any interval [a,b] containing [0, 2 π].

.

- Here's how you use it on the sample case

{ $f_1[t]$, $f_2[t]$, $f_3[t]$, $f_4[t]$, $f_5[t]$, $f_6[t]$} = {1, Cos[t], Cos[2 t], Cos[3 t], Cos[4 t], $\text{Cos}[t]^4$}

on [a,b] = [0, 2 π]:

Make the test matrix:

```
Clear[s, f, Subscript, k, t];
{f1[t_], f2[t_], f3[t_], f4[t_], f5[t_], f6[t_]} =
  {1, Cos[t], Cos[2 t], Cos[3 t], Cos[4 t], Cos[t]^4};
testmatrix = Table[fj[ti], {i, 1, 6}, {j, 1, 6}];
MatrixForm[testmatrix]
```

$$\begin{pmatrix} 1 & \text{Cos}[t_1] & \text{Cos}[2 t_1] & \text{Cos}[3 t_1] & \text{Cos}[4 t_1] & \text{Cos}[t_1]^4 \\ 1 & \text{Cos}[t_2] & \text{Cos}[2 t_2] & \text{Cos}[3 t_2] & \text{Cos}[4 t_2] & \text{Cos}[t_2]^4 \\ 1 & \text{Cos}[t_3] & \text{Cos}[2 t_3] & \text{Cos}[3 t_3] & \text{Cos}[4 t_3] & \text{Cos}[t_3]^4 \\ 1 & \text{Cos}[t_4] & \text{Cos}[2 t_4] & \text{Cos}[3 t_4] & \text{Cos}[4 t_4] & \text{Cos}[t_4]^4 \\ 1 & \text{Cos}[t_5] & \text{Cos}[2 t_5] & \text{Cos}[3 t_5] & \text{Cos}[4 t_5] & \text{Cos}[t_5]^4 \\ 1 & \text{Cos}[t_6] & \text{Cos}[2 t_6] & \text{Cos}[3 t_6] & \text{Cos}[4 t_6] & \text{Cos}[t_6]^4 \end{pmatrix}$$

Evaluate the determinant of the test matrix at some random selections of t_1, t_2, t_3, t_4, t_5 and t_6 in [0, 2 π]:

```
{a, b} = {0, 2 π};
Det[testmatrix] /.
 {t1 -> Random[Real, {a, b}], t2 -> Random[Real, {a, b}],
  t3 -> Random[Real, {a, b}], t4 -> Random[Real, {a, b}],
  t5 -> Random[Real, {a, b}], t6 -> Random[Real, {a, b}]}
```

0

Rerun a couple of times.

Zero every time.

All indications are that { $f_1[t]$, $f_2[t]$, $f_3[t]$, $f_4[t]$, $f_5[t]$, $f_6[t]$} is a linearly dependent set.

Give it the acid test by calculating Det[testmatrix] and letting *Mathematica* try to simplify:

```
Simplify[Det[testmatrix]]
```

0

Zero.

The call:

{ $f_1[t]$, $f_2[t]$, $f_3[t]$, $f_4[t]$, $f_5[t]$, $f_6[t]$} = {1, Cos[t], Cos[2 t], Cos[3 t], Cos[4 t], $\text{Cos}[t]^4$}

is linearly **dependent** on any interval [a,b].

B.5.b.iii) Finding the explicit dependence

Thanks go to Professor Bruce Reznick of University of Illinois at Urbana-Champaign for some helpful comments

Go with

{$f_1[t]$, $f_2[t]$, $f_3[t]$, $f_4[t]$, $f_5[t]$, $f_6[t]$} = {Sin[t], Sin[2 t], Sin[3 t], Sin[4 t], Sin[5 t], $\text{Sin}[t]^5$}

and make the test matrix:

```
Clear[s, f, Subscript, k, t];
{f1[t_], f2[t_], f3[t_], f4[t_], f5[t_], f6[t_]} =
  {Sin[t], Sin[2 t], Sin[3 t], Sin[4 t], Sin[5 t], Sin[t]^5};
testmatrix = Table[fj[ti], {i, 1, 6}, {j, 1, 6}];
MatrixForm[testmatrix]
```

$$\begin{pmatrix} \text{Sin}[t_1] & \text{Sin}[2 t_1] & \text{Sin}[3 t_1] & \text{Sin}[4 t_1] & \text{Sin}[5 t_1] & \text{Sin}[t_1]^5 \\ \text{Sin}[t_2] & \text{Sin}[2 t_2] & \text{Sin}[3 t_2] & \text{Sin}[4 t_2] & \text{Sin}[5 t_2] & \text{Sin}[t_2]^5 \\ \text{Sin}[t_3] & \text{Sin}[2 t_3] & \text{Sin}[3 t_3] & \text{Sin}[4 t_3] & \text{Sin}[5 t_3] & \text{Sin}[t_3]^5 \\ \text{Sin}[t_4] & \text{Sin}[2 t_4] & \text{Sin}[3 t_4] & \text{Sin}[4 t_4] & \text{Sin}[5 t_4] & \text{Sin}[t_4]^5 \\ \text{Sin}[t_5] & \text{Sin}[2 t_5] & \text{Sin}[3 t_5] & \text{Sin}[4 t_5] & \text{Sin}[5 t_5] & \text{Sin}[t_5]^5 \\ \text{Sin}[t_6] & \text{Sin}[2 t_6] & \text{Sin}[3 t_6] & \text{Sin}[4 t_6] & \text{Sin}[5 t_6] & \text{Sin}[t_6]^5 \end{pmatrix}$$

The determinant of the test matrix is

```
Simplify[Det[testmatrix]]
```

0

Zero.

This tells you that

{$f_1[t]$, $f_2[t]$, $f_3[t]$, $f_4[t]$, $f_5[t]$, $f_6[t]$} = {Sin[t], Sin[2 t], Sin[3 t], Sin[4 t], Sin[5 t], $\text{Sin}[t]^5$}

is linearly dependent on any interval [a,b].

How do you determine the functions in {$f_1[t]$, $f_2[t]$, $f_3[t]$, $f_4[t]$, $f_5[t]$, $f_6[t]$} that are in the function spaces determined by the others?

Answer:

Fairly easily.

At this stage you know that the determinant of the test matrix is 0 for any and all choices of t_1, t_2, t_3, t_4, t_5 and t_6.

When you replace t_1, t_2, t_3, t_4, t_5 with random numbers and replace t_6 with t, you are guaranteed the resulting determinant is 0 for all t.

```
{a, b} = {0, 2 π};
Clear[t]
result = (Det[testmatrix] /. {t1 -> Random[Real, {a, b}],
     t2 -> Random[Real, {a, b}], t3 -> Random[Real, {a, b}],
     t4 -> Random[Real, {a, b}], t5 -> Random[Real, {a, b}], t6 -> t})
```

-0.878122 Sin[t] + 1.405 $\text{Sin}[t]^5$ + 0.439061 Sin[3 t] - 0.0878122 Sin[5 t]

Knowing that this is **0** for all t, you can see at a glance that

- $\text{Sin}[t]^5$ is in the function space determined by the others.
- Sin[t] is in the function space determined by the others.
- Sin[3 t] is in the function space determined by the others.
- Sin[5 t] is in the function space determined by the others.

If this makes no sense to you, click on the right.

To see why $\text{Sin}[t]^5$ is in the function space determined by the others, take the coefficient of $\text{Sin}[t]^5$:

```
Coefficient[result, Sin[t]^5]
```

1.405

Divide everything by this coefficient:

```
Expand[1/Coefficient[result, Sin[t]^5] result]
```

-0.625 Sin[t] + 1. $\text{Sin}[t]^5$ + 0.3125 Sin[3 t] - 0.0625 Sin[5 t]

Knowing that this is 0 for all t, you see that

$$\text{Sin}[t]^5 = 0.625\ \text{Sin}[t] - 0.3125\ \text{Sin}[3 t] + 0.0625\ \text{Sin}[5 t]$$

for all t., confirming that $\text{Sin}[t]^5$ is in the function space spanned by the others.

B.5.b.iv) Why the determinant test works

Thanks go to Professor Joseph Rosenblatt of University of Illinois at Urbana-Champaign for some helpful suggestions.

Two functions:

Here's why it works for two functions{f[t],g[t]} on an interval [a,b]:

If $\text{Det}[\begin{pmatrix} f[x] & g[x] \\ f[y] & g[y] \end{pmatrix}] \neq 0$ for some x and y in [a,b], then $\begin{pmatrix} f[x] & g[x] \\ f[y] & g[y] \end{pmatrix}$ is of full rank; so neither function is a multiple of the other. This is enough to proclaim that {f[t],g[t]} is linearly independent on any interval including [a,b].

On the other hand, if $\text{Det}[\begin{pmatrix} f[x] & g[x] \\ f[y] & g[y] \end{pmatrix}] = 0$ for some x and y in [a,b], then you can solve:

```
Clear[f, g, x, y];
Solve[Det[( f[x]  g[x]
            f[y]  g[y] )] == 0, f[y]]
```

$\{\{f[y] \to \frac{f[x]\, g[y]}{g[x]}\}\}$

The upshot :
If there is an x with $g[x] \neq 0$, plug in this x to see

$f[y] = \frac{f[x]}{g[x]} g[y]$

as a definite multiple ($\frac{f[x]}{g[x]}$) of g[y] for all y's in [a,b]. This tells you that the set consisting of these two functions is a linearly independent set.
This tells you that for all y's in [a,b] with the result that {f[t],g[t]} is linearly dependent.

On the other hand, if g[x] = 0 for all x's in [a,b], then {f[t],g[t]} is automatically linearly dependent..

Three functions:

Here's why it works for three functions {f[t],g[t],h[t]} on an interval [a,b]:

If $\text{Det}[\begin{pmatrix} f[x] & g[x] & h[x] \\ f[y] & g[y] & h[y] \\ f[z] & g[z] & h[z] \end{pmatrix}] \neq 0$ for some x, y and z in [a,b], then $\begin{pmatrix} f[x] & g[x] & h[x] \\ f[y] & g[y] & h[y] \\ f[z] & g[z] & h[z] \end{pmatrix}$ is of full rank; so neither This rules out the possibilities that one of the functions is in the function space spanned by the others. This signals that {f[t],g[t],h[t]} is linearly independent on [a.b] (and on any interval containing [a,b]).

On the other hand, if $\text{Det}[\begin{pmatrix} f[x] & g[x] & h[x] \\ f[y] & g[y] & h[y] \\ f[z] & g[z] & h[z] \end{pmatrix}] = 0$ for all x, y and z in [a,b], then you can solve:

```
Clear[f, g, h, x, y, z]
Simplify[Solve[Det[( f[x]  g[x]  h[x]
                     f[y]  g[y]  h[y]
                     f[z]  g[z]  h[z] )] == 0, f[z]]]
```

$\{\{f[z] \to \frac{f[y]\, g[z]\, h[x] - f[x]\, g[z]\, h[y] - f[y]\, g[x]\, h[z] + f[x]\, g[y]\, h[z]}{g[y]\, h[x] - g[x]\, h[y]}\}\}$

This says

$$f[z] = \frac{f[y]\; g[z]\; h[x] - f[x]\; g[z]\; h[y] - f[y]\; g[x]\; h[z] + f[x]\; g[y]\; h[z]}{g[y]\, h[x] - g[x]\, h[y]}$$

for all x,y and z in [a,b].
This is the same as

$$f[z] = \left(\frac{f[y]\;\; h[x] - f[x]\;\; h[y]}{g[y]\, h[x] - g[x]\, h[y]}\right) g[z] + \left(\frac{-f[y]\; g[x] + f[x]\; g[y]}{g[y]\, h[x] - g[x]\, h[y]}\right) h[z].$$

for all x,y and z in [a,b].
The upshot :
If there are x and y with

$g[y]\, h[x] - g[x]\, h[y] \neq 0,$

you can plug in this choice of x and y to write f[z] s an explicit linear combination of g[z] and h[z] with the result that f[t] is in the function space spanned by g[t] and h[t]. This signals that {f[t],g[t],h[t]} is linearly dependent on [a,b]

On the other hand if

$$g[y]\, h[x] - g[x]\, h[y] = \text{Det}[\begin{pmatrix} g[y] & h[y] \\ g[x] & h[x] \end{pmatrix}] = 0$$

for all y and z in [a,b] then the two function case tells you {g[t],h[t]} is linearly dependent on [a,b] and so {f[t],g[t],h[t]} is linearly dependent as well.

More than three functions

It is possible to extend the explanation given above to more than 3 functions via Mathematical Induction.
Math mavens should think about this.

B.6) The Gram-Schmidt Process: Just the ticket for dealing with function spaces spanned by non-orthogonal sets of functions

□B.6.a.i) Function spaces spanned ny non-orthogonal families

Given a set of functions

$\{f_1[t], f_2[t], f_3[t], \ldots f_k[t]\}$

on an interval [a, b], you make the function space S[a, b] spanned by

$\{f_1[t], f_2[t], f_3[t], \ldots f_k[t]\}.$

This function space S[a, b] consists of all functions s[t] of the form.

$$s[t] = \sum_{j=1}^{k} c_j\, f_j[t] \text{ with } a \le t \le b.$$

One good example is the function space S[−1, 1] of all fourth degree polynomials on [−1, 1].
You get this by going with

$\{f_1[t], f_2[t], f_3[t], f_4[t], f_5[t]\} = \{1, t, t^2, t^3, t^4\}.$

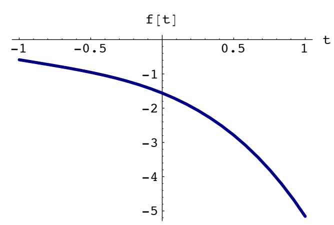
See a random member of this function space :

```
Clear[f, Subscript, k, t];
f1[t_] = 1;
f2[t_] = t;
f3[t_] = t^2;
f4[t_] = t^3;
f5[t_] = t^4;
a = -1;
b = 1;
ck_ := Random[Real, {-2, 2}]
s[t_] = Sum[ck fk[t], {k, 1, 5}]
```

$-1.55943 - 1.68131\, t - 1.22393\, t^2 - 0.604767\, t^3 - 0.0898116\, t^4$

And its plot:

```
Plot[s[t], {t, a, b}, PlotStyle -> {{Thickness[0.01], NavyBlue}},
  AxesLabel -> {"t", "f[t]"}, AspectRatio -> 1 / GoldenRatio];
```



Now look at this:

```
Integrate[f2[t] f4[t], {t, a, b}]
```

$\frac{2}{5}$

Interpret the result.

□**Answer:**

Look again:

```
Integrate[f2[t] f4[t], {t, a, b}]
```

$\frac{2}{5}$

This tells you that

$$f_2 \bullet f_4 = \int_a^b f_2[t]\, f_4[t]\, dt \neq 0.$$

The interpretation : The given family

$\{f_1[t], f_2[t], f_3[t], f_4[t], f_5[t]\} = \{1, t, t^2, t^3, t^4\}$

is **not** orthogonal on [a, b] = [−1, 1]

□B.6.a.ii) Using Gram-Schmidt to get an orthogonal spanning set

Come up with an orthogonal family

$\{s_1[t], s_2[t], s_3[t], s_4[t], s_5[t]\}$ on [a,b]

so that function space spanned by

$\{s_1[t], s_2[t], s_3[t], s_4[t], s_5[t]\}$

is the same as the function space spanned by

$\{f_1[t], f_2[t], f_3[t], f_4[t], f_5[t]\} = \{1, t, t^2, t^3, t^4\}$

□**Answer:**

Apply something called the Gram − Schmidt process to $\{f_1[t], f_2[t], f_3[t], f_4[t], f_5[t]\}$:

```
orthospanners = GramSchmidt[
  {f1[t], f2[t], f3[t], f4[t], f5[t]}, InnerProduct → (∫_a^b #1 #2 ⅆt &)]
```

$$\left\{\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\,t, \frac{3}{2}\sqrt{\frac{5}{2}}\left(-\frac{1}{3}+t^2\right), \frac{5}{2}\sqrt{\frac{7}{2}}\left(-\frac{3t}{5}+t^3\right), \frac{105\left(-\frac{1}{5}+t^4-\frac{6}{7}\left(-\frac{1}{3}+t^2\right)\right)}{8\sqrt{2}}\right\}$$

Put:

```
sj_[t_] := orthospanners[[j]];
ColumnForm[Table[sj[t], {j, 1, Length[orthospanners]}]]
```

$$\frac{1}{\sqrt{2}}$$

$$\sqrt{\frac{3}{2}}\,t$$

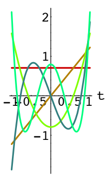
$$\frac{3}{2}\sqrt{\frac{5}{2}}\left(-\frac{1}{3}+t^2\right)$$

$$\frac{5}{2}\sqrt{\frac{7}{2}}\left(-\frac{3t}{5}+t^3\right)$$

$$\frac{105\left(-\frac{1}{5}+t^4-\frac{6}{7}\left(-\frac{1}{3}+t^2\right)\right)}{8\sqrt{2}}$$

See this orthogonal family:

```
Plot[{s1[t], s2[t], s3[t], s4[t], s5[t]}, {t, a, b},
  PlotStyle → {{Thickness[0.02], RGBColor[0.8, 0, 0]},
    {Thickness[0.02], RGBColor[0.7, 0.5, 0]},
    {Thickness[0.02], RGBColor[0.5, 1, 0]},
    {Thickness[0.02], RGBColor[0.2, 0.5, 0.5]},
    {Thickness[0.02], RGBColor[0, 1, 0.5]}}, AxesLabel → {"t", ""}];
```



The function space spanned by

$\{s_1[t], s_2[t], s_3[t], s_4[t], s_5[t]\}$

is guaranteed to be the same as the function space spanned by

$\{f_1[t], f_2[t], f_3[t], f_4[t], f_5[t]\} = \{1, t, t^2, t^3, t^4\}$

If you want to see how the Gram-Schmidt process works step-by-step, click and the right.

Gram-Schmidt Step 1:

Put $g_1[t] = f_1[t]$.

:

```
Clear[g, s, t];
g1[t_] = f1[t]
```

1

Gram-Schmidt Step 2:

Put

$g_2[t] = f_2[t] -$ (component of $f_2[t]$ in the direction of $g_1[t]$).

```
g2[t_] = f2[t] - (∫_a^b f2[t] g1[t] ⅆt) / (∫_a^b g1[t] g1[t] ⅆt) g1[t]
```

t

Gram-Schmidt Step 3:

Put

$g_3[t] = f_3[t] -$ component of $f_3[t]$ in the direction of $g_1[t]$ – .

– component of $f_3[t]$ in the direction of $g_2[t]$

```
g3[t_] = f3[t] - (∫_a^b f3[t] g1[t] ⅆt) / (∫_a^b g1[t] g1[t] ⅆt) g1[t] - (∫_a^b f3[t] g2[t] ⅆt) / (∫_a^b g2[t] g2[t] ⅆt) g2[t]
```

$$-\frac{1}{3}+t^2$$

Gram-Schmidt Step 4:

Put

$$g_4[t] = f_4[t] - \sum_{j=1}^{3} \text{component of } f_4[t] \text{ in the direction of } g_j[t].$$

```
g4[t_] = f4[t] - Σ_{j=1}^{3} ((∫_a^b f4[t] gj[t] ⅆt) / (∫_a^b gj[t] gj[t] ⅆt) gj[t])
```

$$-\frac{3t}{5}+t^3$$

Gram-Schmidt Step 5:

Put

$$g_5[t] = f_5[t] - \sum_{j=1}^{4} \text{component of } f_5[t] \text{ in the direction of } g_j[t].$$

```
g5[t_] = f5[t] - Σ_{j=1}^{4} ((∫_a^b f5[t] gj[t] ⅆt) / (∫_a^b gj[t] gj[t] ⅆt) gj[t])
```

$$-\frac{1}{5}+t^4-\frac{6}{7}\left(-\frac{1}{3}+t^2\right)$$

Now you stop because you have used the whole family

$\{f_1[t], f_2[t], f_3[t], f_4[t], f_5[t]\} = \{1, t, t^2, t^3, t^4\}$.

By its construction:

→ The new family $\{g_1[t], g_2[t], g_3[t], g_4[t], g_5[t]\}$ determines the same function space S[a,b] as the original family $\{f_1[t], f_2[t], f_3[t], f_4[t], f_5[t]\}$.

→ The new family $\{g_1[t], g_2[t], g_3[t], g_4[t], g_5[t]\}$ is orthogonal on [a,b].

Spot check:

```
q = Random[Integer, {1, 3}];
p = Random[Integer, {q + 1, 5}];
Integrate[gq[t] gp[t], {t, a, b}]
```

0

Now finish it off by setting

$$s_j[t] = \frac{g_j[t]}{\sqrt{g_j \bullet g_j}}:$$

```
sj_[t_] := gj[t] / Sqrt[∫_a^b gj[t] gj[t] ⅆt]
```

```
Table[sj[t], {j, 1, 5}]
```

$$\left\{\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\,t, \frac{3}{2}\sqrt{\frac{5}{2}}\left(-\frac{1}{3}+t^2\right), \frac{5}{2}\sqrt{\frac{7}{2}}\left(-\frac{3t}{5}+t^3\right), \frac{105\left(-\frac{1}{5}+t^4-\frac{6}{7}\left(-\frac{1}{3}+t^2\right)\right)}{8\sqrt{2}}\right\}$$

Compare:

```
GramSchmidt[{f1[t], f2[t], f3[t], f4[t], f5[t]}, InnerProduct ->
  (Integrate[#1 #2, {t, a, b}] &)]
```

$$\left\{\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\,t, \frac{3}{2}\sqrt{\frac{5}{2}}\left(-\frac{1}{3}+t^2\right), \frac{5}{2}\sqrt{\frac{7}{2}}\left(-\frac{3t}{5}+t^3\right), \frac{105\left(-\frac{1}{5}+t^4-\frac{6}{7}\left(-\frac{1}{3}+t^2\right)\right)}{8\sqrt{2}}\right\}$$

That's it.

B.6.a.iii) Using Gram-Schmidt to come up with root-mean-square approximations

Come up with the best root-mean-square approximation on [-1 ,1] of $f[t] = e^t$ by a fourth degree polynomial.

Answer:

This is the same as going with the function space S[-1,1] spanned by

$\{f_1[t], f_2[t], f_3[t], f_4[t], f_5[t]\} = \{1, t, t^2, t^3, t^4\}$

and asking for the function

$$\text{Sclosest}[t] = \sum_{k=1}^{5} c_k f_k[t]$$

that is closest to

$f[t] = e^t$

with respect to root-mean-square distance on [-1,1].

Thanks to the work in the last part, you know that the family

$\{s_1[t], s_2[t], s_3[t], s_4[t], s_5[t]\}$

resulting from running Gram-Schmidt on

$\{f_1[t], f_2[t], f_3[t], f_4[t], f_5[t]\}$

is orthogonal on [-1,1] and spans the same function space as the original family

$\{f_1[t], f_2[t], f_3[t], f_4[t], f_5[t]\} = \{1, t, t^2, t^3, t^4\}$.

```
khigh = 5;
Clear[s, f, Subscript, k, t];
f_k_[t_] = t^(k-1);
a = -1;
b = 1;
orthospanners = GramSchmidt[{f_1[t], f_2[t], f_3[t], f_4[t], f_5[t]},
    InnerProduct -> (∫_a^b #1 #2 ⅆt &)];
s_j_[t_] := orthospanners[[j]];
Table[s_j[t], {j, 1, Length[orthospanners]}]
```

$$\left\{\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\, t, \frac{3}{2}\sqrt{\frac{5}{2}}\left(-\frac{1}{3}+t^2\right), \frac{5}{2}\sqrt{\frac{7}{2}}\left(-\frac{3t}{5}+t^3\right), \frac{105\left(-\frac{1}{5}+t^4-\frac{6}{7}\left(-\frac{1}{3}+t^2\right)\right)}{8\sqrt{2}}\right\}$$

So the question reduces to finding the function of the form and asking for the function

$$\text{Sclosest}[t] = \sum_{k=1}^{5} c_k\, s_k[t]$$

that is closest to

$$f[t] = e^t$$

with respect to root-mean-square distance on [-1,1].

And this is easy.

The function you are after is:

```
khigh = 5;
Clear[f, t];
f[t_] = ⅇ^t;
Clear[fouriercoeff, Sclosest];
fouriercoeff[k_] :=
  fouriercoeff[k] = NIntegrate[f[t] s_k[t], {t, a, b}] / NIntegrate[s_k[t] s_k[t], {t, a, b}];
Sclosest[t_] = Sum[fouriercoeff[k] s_k[t], {k, 1, khigh}]
```

$$1.1752 + 1.10364\, t + 0.536722\left(-\frac{1}{3}+t^2\right) + 0.176139\left(-\frac{3t}{5}+t^3\right) + 0.0435974\left(-\frac{1}{5}+t^4-\frac{6}{7}\left(-\frac{1}{3}+t^2\right)\right)$$

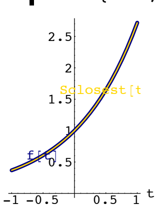
Multiply it out:

```
Expand[Sclosest[t]]
```

$$1.00003 + 0.997955\, t + 0.499352\, t^2 + 0.176139\, t^3 + 0.0435974\, t^4$$

See the quality of the approximation:

```
fitplot = Plot[{f[t], Sclosest[t]},
   {t, a, b}, PlotStyle -> {{Thickness[0.03], NavyBlue},
     {Thickness[0.01], Gold}}, AxesLabel -> {"t", ""},
   Epilog -> {{NavyBlue, Text["f[t]", {a + 0.5, f[a + 0.5]}]},
     {Gold, Text["Sclosest[t]", {b - 0.5, Sclosest[b - 0.5]}]}}];
```



Copacetic.