Matrices, Geometry&*Mathematica*

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Producer: Bruce Carpenter

Publisher: Math Everywhere, Inc.

MGM.04 SVD Analysis of 2D Matrices
LITERACY

What you should be able to handle when you are away from the machine.

□L.1)

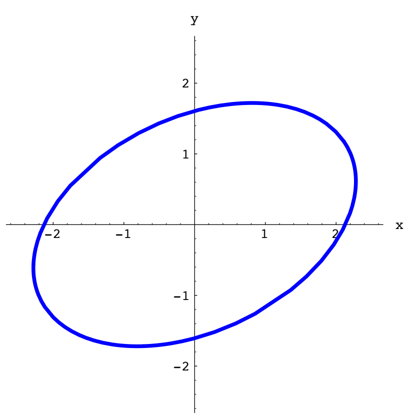
Explain this:
Given a 2D matrix A, if you can come up with an s (in radians) so that
A.{Cos[s], Sin[s]} is perpendicular to A.{Cos[s + $\frac{\pi}{2}$], Sin[s + $\frac{\pi}{2}$]}, then you can read off
1) SVD aligner for A
2) SVD stretchfactors for A
3) SVD hanger for A

□L.2)

Here's a random 2D matrix A:

$$\begin{pmatrix} 2.11002 & 0.86323 \\ 1.17559 & -1.25519 \end{pmatrix}$$

Here's the ellipse you get when you hit this matrix on the unit circle centered at {0,0}:



The SVD hanger matrix for A is:

$$\begin{pmatrix} -0.901522 & -0.432733 \\ -0.432733 & 0.901522 \end{pmatrix}$$

The SVD stretcher matrix for A is:

$$\begin{pmatrix} 2.42238 & 0 \\ 0 & 1.51227 \end{pmatrix}$$

The SVD aligner matrix for A is:

$$\begin{pmatrix} -0.995281 & -0.0970362 \\ 0.0970362 & -0.995281 \end{pmatrix}$$

Fill the blanks:
The length of the long axis of this ellipse is ....
The length of the short axis of this ellipse is....................
The perpendicular frame {perpframe[1],perpframe[2]} that defines the long and the short axes of this ellipse is
perpframe[1] = {.........................., ...}
perpframe[2] = {.........................., ...}
The area enclosed by this ellipse is (..............................) times (............................) times π.

□L.3)

Given a 2D matrix A how do you use the SVD stretch factors of A to determine whether A is invertible?

□L.4)

Given a 2D matrix A how do you use the SVD stretch factors of A to determine the rank of A?

□L.5)

Lots of folks like to say that a given 2D matrix A is:
→ invertible if Det[A] ≠ 0
→ not invertible if Det[A] = 0.
Explain why they are right.

□L.6)

Given a 2D matrix A you do an SVD analysis and get

A = hanger.$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$.aligner.

with a and b both positive.
This information tells you that given any 2D vector Y, there is exactly one 2D vector X with A.X = Y.
Agree.......................................
Disagree.......................................

□L.7)

Given a 2D matrix A you do an SVD analysis and get

A = hanger.$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$.aligner.

with a > 0.
This information tells you that the rank of A is ....

□L.8)

Given a 2D matrix A you do an SVD analysis and get

A = hanger.$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$.aligner.

with a >0.
This information tells you that if a 2D vector Y is on a certain line through {0,0}, then there is exactly one 2D vector X with A.X = Y.
Agree.......................................
Disagree.......................................

□L.9)

Given a 2D matrix A you do an SVD analysis and get

A = hanger.$\begin{pmatrix} 1.234 & 0 \\ 0 & 0.0 \end{pmatrix}$.aligner.

This information tells you that given a 2D vector Y, then either
A.X = Y has no solution X or A.X = Y has many solutions X.
.
Agree.......................................
Disagree.......................................

□L.10)

Given a 2D matrix A you do an SVD analysis and get

A = hanger.$\begin{pmatrix} 3.0 & 0 \\ 0 & 2.0 \end{pmatrix}$.aligner.

You plot the SVD aligner frame and learn it is a right hand perpendicular frame.
You plot the SVD hanger frame and learn it is a right hand perpendicular frame.
Fill the blank:
Det[A] = ...

□L.11)

Given a 2D matrix A you do an SVD analysis and get

A = hanger.$\begin{pmatrix} 2.0 & 0 \\ 0 & 5.0 \end{pmatrix}$.aligner.

You plot the SVD aligner frame and learn it is a left hand perpendicular frame.
You plot the SVD hanger frame and learn it is a left hand perpendicular frame.
Fill the blank:
Det[A] = ...

□L.12)

You are given a 2D matrix A and learn that Det[A] < 0. This tells you that a hit with A incorporates a flip.
Agree....... Disagree........

□L.13)

You are given a 2D matrix A and learn that Det[A] > 0. This tells you that a hit with A incorporates no flip.
Agree....... Disagree........

□L.14)

Given a 2D matrix A you do an SVD analysis and get

$$A = \text{hanger}.\begin{pmatrix} 4.0 & 0 \\ 0 & 0.5 \end{pmatrix}.\text{aligner}.$$

You plot the SVD aligner frame and learn it is a right hand perpendicular frame.
You plot the SVD hanger frame and learn it is a left hand perpendicular frame.
Fill the blank:
Det[A] = ...

□L.15)

Given a 2D matrix A you do an SVD analysis and get

$$A = \text{hanger}.\begin{pmatrix} 3.0 & 0 \\ 0 & 2.0 \end{pmatrix}.\text{aligner}.$$

You plot the aligner frame and learn it is a left hand perpendicular frame.
You plot the hanger frame and learn it is a right hand perpendicular frame.
Fill the blank:
Det[A] = ...

□L.16)

Given a 2D matrix A you do an SVD analysis and get

$$A = \text{hanger}.\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.\text{ aligner}.$$

Using this, you can compute A^t via the formula:

a) $A^t = \text{aligner}.\begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}.\text{ hanger}$ b) $A^t = \text{hanger}^t.\begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}.\text{ aligner}^t$

c) $A^t = \text{aligner}^t.\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.\text{ hanger}^t$ d) $A^t = \text{aligner}^t.\begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}.\text{ hanger}^t$

My choice is ....

□L.17)

Given a 2D matrix A you do an SVD analysis and get

$$A = \text{hanger}.\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.\text{ aligner}.$$

The aligner frame for A serves as a hanger frame for A^t and the hanger frame for A serves as an aligner frame for A^t.
Agree........ Disagree.........

□L.18)

Given a 2D matrix A, explain why

$$\text{Det}[A^t] = \text{Det}[A].$$

□L.19)

Given an invertible 2D matrix A you do an SVD analysis and get

$$A = \text{hanger}.\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.\text{ aligner}.$$

Using this, you can compute A^{-1} via the formula:

a) $A^{-1} = \text{aligner}.\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{pmatrix}.\text{ hanger}$ b) $A^{-1} = \text{hanger}^t.\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{pmatrix}.\text{ aligner}^t$

c) $A^{-1} = \text{aligner}^t.\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{pmatrix}.\text{ hanger}^t$ d) $A^{-1} = \text{aligner}^t.\begin{pmatrix} \frac{1}{b} & 0 \\ 0 & \frac{1}{a} \end{pmatrix}.\text{ hanger}^t$

My choice is ....

□L.20)

Given an invertible 2D matrix A you do an SVD analysis and get

$$A = \text{hanger}.\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.\text{ aligner}.$$

The aligner frame for A serves as a hanger frame for A^{-1} and the hanger frame for A serves as an aligner frame for A^{-1}.
Agree......... Disagree..........

□L.21)

Given a 2D invertible matrix A, explain why

$$\text{Det}[A^{-1}] = \frac{1}{\text{Det}[A]}.$$

□L.22)

You are given a 2D matrix A and after you do your SVD analysis of it, you learn that both xstretch and ystretch are positive.
You make the call:
Can there be an {x,y} with
{x,y} ≠ {0,0}
and with
A.{x,y} = {0,0}?

□L.23)

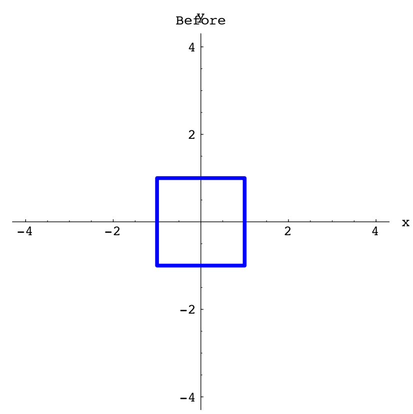
You are given a 2D matrix A and after you do your SVD analysis of it, you learn that both xstretch is positive and ystretch = 0.
You make the call:
Can there be an {x,y} with
{x,y} ≠ {0,0}
and with
A.{x,y} = {0,0}?

□L.24)

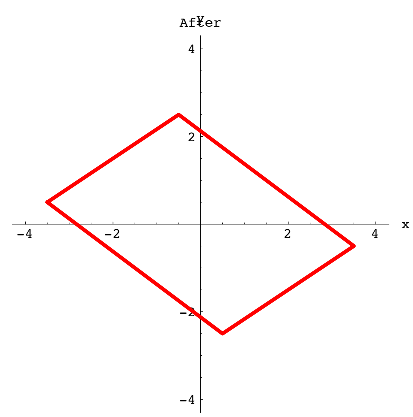
Here's a 2D matrix A:

$$\begin{pmatrix} 1.5 & 2. \\ 1. & -1.5 \end{pmatrix}$$

Here is the square with corners at {-1,-1}, {1,-1},{1,1}, and {-1,1}:



Here's what you get when you hit this square with A:



The determinant of A is:

-4.25

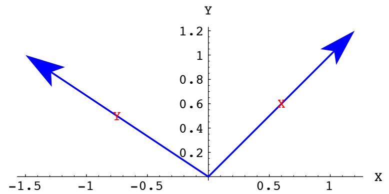
The area enclosed by the parallelogram is......................

□L.25)

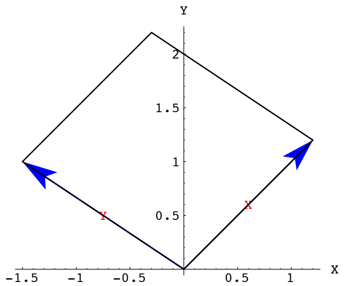
Here are two vectors X and Y and their plot:

{1.2, 1.2}

{-1.5, 1.}



Here is the parallelogram these two vectors define:



Tell how you would use X and Y to set up a matrix A so that |Det[A]| calculates the area of this parallelogram.

$A = \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}$

□L.26)

Here's a 2D matrix A:

$\begin{pmatrix} 0.7 & 1.8 \\ -0.8 & 0 \end{pmatrix}$

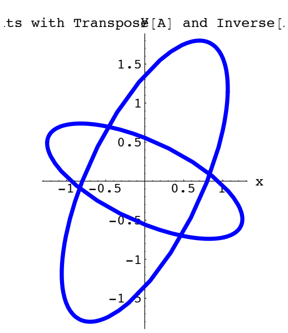
Here are A^t, the transpose of A, and A^{-1}, the inverse of A:

$\begin{pmatrix} 0.7 & -0.8 \\ 1.8 & 0 \end{pmatrix}$

$\begin{pmatrix} 0 & -1.25 \\ 0.555556 & 0.486111 \end{pmatrix}$

They don't look much alike.

But see what they do when you hit both on the unit circle:



Describe what you see and try to explain why you see it.

Some questions to ponder:

Both ellipses seem to be hanging on the same perpendicular frame. What perpendicular frame is it?

Why does the long axis of each ellipse line up with the short axis of the other?

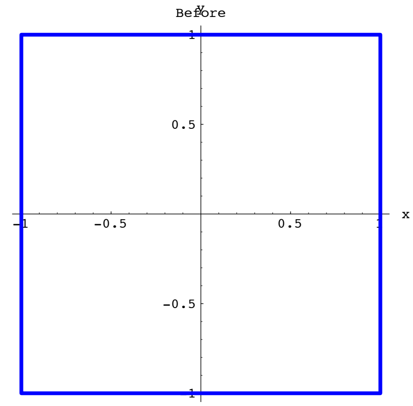
When you multiply the area enclosed by one of the ellipses times the area enclosed by other ellipse, what do you get?

□L.27)

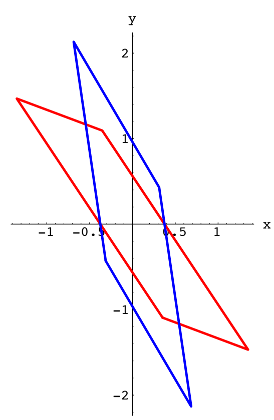
Here's a random matrix A:

$\begin{pmatrix} 0.498942 & -0.851264 \\ -0.187056 & 1.28118 \end{pmatrix}$

Here is the square with corners at {-1,-1}, {1,-1},{1,1}, and {-1,1}:



Here are the parallelograms you get what you get when you hit this square with this matrix and its transpose:



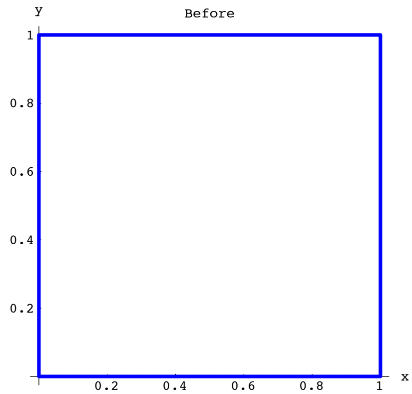
Express the measurement of area enclosed by one of these parallelograms in terms of the measurement of the area enclosed by the other.

□L.28)

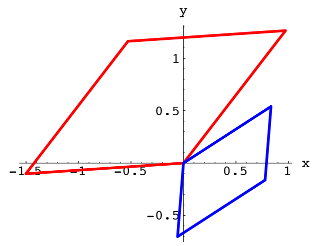
Here's a random matrix A:

$\begin{pmatrix} 0.972616 & -1.50148 \\ 1.26461 & -0.101578 \end{pmatrix}$

Here is the square with corners at {0,0},{1,0},{1,1}, and{0,1}:



Here are the parallelograms you get what you get when you hit this square with this matrix and its inverse:



Express the measurement of area enclosed by one of these parallelograms in terms of the measurement of the area enclosed by the other.

Hand Calculation Literacy

LHC.1) Determinants

Here's a random 2D matrix:

$$\begin{pmatrix} 2 & -1 \\ -2 & -3 \end{pmatrix}$$

Give a hand calculation of the determinant of A.

LHC.2) Solving 2D systems by hand via the Cramer's rule formula

Here is a 2D linear system:

```
x[1] - 3 x[2] == 1
-1. x[1] - 8 x[2] == -1
```

Use the hand-style Cramer's rule to come up with the solution {x[1], x[2]}.

LHC.3) Inverting a 2D matrices by hand via the Cramer's rule formula

Here is a 2D matrix:

$$\begin{pmatrix} 1 & 2 \\ -0.5 & -8 \end{pmatrix}$$

Use hand-style Cramer's rule to come up with the inverse of A.