

Differential Equations & *Mathematica*

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DE.01 Transition from Calculus to DiffEq: The Exponential Differential Equation Basics

B.1) The unforced exponential differential equation, the most important differential equation of all:

$$y'[t] + r y[t] = 0.$$

This is the boss diffeq.

$$y[t] = E^{-r t} \text{ starter}$$

solves the unforced exponential diffeq

$$y'[t] + r y[t] = 0 \text{ with } y[0] = \text{starter.}$$

□ **B.1.a)** $y[t] = a E^{-r t}$ solves $y'[t] + r y[t] = 0$

Given a number r , to solve the exponential differential equation

$$y'[t] + r y[t] = 0,$$

you want a function $y[t]$ with the property that

$$y'[t] + r y[t] = 0 \text{ for all } t\text{'s.}$$

How do you get it?

□ **Answer:**

You do educated guesswork.

The diffeq is

$$y'[t] + r y[t] = 0 \text{ for all } t\text{'s.}$$

You go with numbers K and z and put

$$\text{guessy}[t] = K E^{z t}$$

Because the diffeq is

$$y'[t] + r y[t] = 0,$$

you want to select z so that

$$\text{guessy}'[t] + r \text{guessy}[t] = 0.$$

This is the same as

$$K z E^{z t} + r K E^{z t} = 0.$$

This is the same as

$$z = -r$$

So

$$\text{guessy}[t] = K E^{-r t}.$$

Check it out:

```
Clear[guessy, K, t, r];
guessy[t_] = K E^{-r t}
E^{-r t} K
```

Check whether

$$\text{guessy}'[t] + r \text{guessy}[t] = 0:$$

```
guessy'[t] + r guessy[t]
0
```

Good.

This tells you and everyone else that

$$y[t] = a E^{-r t}$$

solves the exponential diffeq

$$y'[t] + r y[t] = 0.$$

You can get *Mathematica* to spit this out directly:

```
Clear[y, t, r]
DSolve[y'[t] + r y[t] == 0, y[t], t]
{{y[t] -> E^{-r t} C[1]}}
```

This means $C[1]$ can be any constant. You can reconcile the solution

$$y[t] = K E^{-r t}$$

above with the *Mathematica* solution

$$y[t] = C[1] E^{-r t}$$

by taking $K = C[1]$.

□ **B.1.b)**

Solve the diffeq

$$y'[t] + 0.6 y[t] = 0 \text{ with } y[0] = 12.8.$$

Give a nice plot of the solution.

□ **Answer:**

You already know

$$y[t] = K E^{-0.6 t}$$

solves

$$y'[t] + 0.6 y[t] = 0.$$

But when you look at $y[0]$, you get:

$$y[0] = K E^0 = K.$$

This tells you that to solve

$$y'[t] + 0.6 y[t] = 0 \text{ with } y[0] = 12.8,$$

all you have to do is to write down

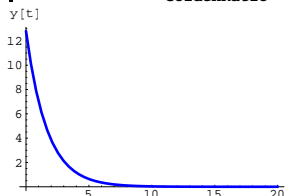
$$y[t] = 12.8 E^{-0.6 t}.$$

Check it out:

```
r = 0.6; starter = 12.8; Clear[y, t]
y[t_] = starter E^{-r t}
12.8 E^{-0.6 t}
```

See this solution:

```
endtime = 20;
Plot[y[t], {t, 0, endtime},
PlotStyle -> {{Blue, Thickness[0.01]}}, PlotRange -> All,
AspectRatio -> 1/GoldenRatio, AxesLabel -> {"t", "y[t]"}];
```



It falls off to 0 mighty fast.

That's what lots of folks like to call "exponential decay."

□ **B.1.c.i)**

Given the diffeq

$$y'[t] + r y[t] = 0 \text{ with } y[0] = 7.6,$$

how does the sign of r tell you whether the plot of the solution goes up or goes down to 0?

□ **Answer:**

If $r < 0$, solutions go up. If $r > 0$, solutions go down.

Reasons:

If $r < 0$, then you get

$$y'[0] + r y[0] = 0$$

or

$$y'[0] = -r y[0]$$

$$y'[0] = -7.6 r$$

Because $r < 0$ tells you that $y[t]$ goes up as t advances from 0.

Then when you look at

$$y'[t] = -r y[t],$$

you see that as $y[t]$, goes up so does $y'[t]$.

The upshot:

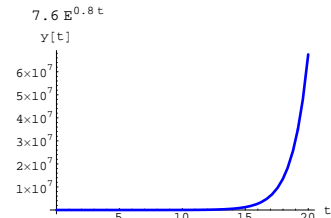
$y[t]$ grows at an increasingly rapid rate.

This increasingly rapid growth is called exponential growth.

See the solution of $y'[t] + r y[t] = 0$ with $y[0] = 7.6$

with $r = -0.8$:

```
starter = 7.6; r = -0.8; Clear[y, t]
y[t_] = starter E^-r t
endtime = 20;
Plot[y[t], {t, 0, endtime},
PlotStyle -> {{Blue, Thickness[0.01]}}, PlotRange -> All,
AspectRatio ->  $\frac{1}{\text{GoldenRatio}}$ , AxesLabel -> {"t", "y[t]"}];
```



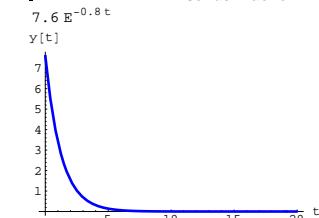
Look at that bastard grow. That's big time exponential growth.

On the other hand, if $r > 0$,

the solution is $y[t] = 7.6 E^{-t}$ which is guaranteed to decay to 0 pretty durn fast:

See the solution of $y'[t] + r y[t] = 0$ with $y[0] = 7.6$ and with $r = 0.8$:

```
starter = 7.6; r = 0.8; Clear[y, t]
y[t_] = starter E^-r t
endtime = 20;
Plot[y[t], {t, 0, endtime},
PlotStyle -> {{Blue, Thickness[0.01]}}, PlotRange -> All,
AspectRatio ->  $\frac{1}{\text{GoldenRatio}}$ , AxesLabel -> {"t", "y[t]"}];
```



Decaying to 0. Typical exponential decay.

B.2) $y[t] = E^{-rt}$ starter + $E^{-rt} \int_0^t E^{rt} f[t] dt$ solves the forced exponential diffeq

$y'[t] + r y[t] = f[t]$ with $y[0] = \text{starter}$.

□**B.2.a)**

Old hands at diffeq know that you can get a formula for the solution of the forced exponential diffeq

$$y'[t] + r y[t] = f[t], \text{ with } y[0] = \text{starter}$$

merely by calculating an integral.

The formula for the solution is

$$y[t] = E^{-rt} \text{starter} + E^{-rt} \int_0^t E^{rt} f[t] dt.$$

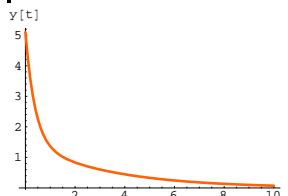
Here is the formula at work on

$$y'[t] + 2.6 y[t] = 3.4 E^{-0.3t}, \text{ with } y[0] = 5.1:$$

```
Clear[yformula, f, t]
f[t_] = 3.4 E^-0.3 t;
r = 2.6;
starter = 5.1;
yformula[t_] = E^-r t starter + E^-r t  $\int_0^t E^{rt} f[t] dt$ 
5.1 E^-2.6 t + E^-2.6 t (-1.47826 + 1.47826 E^{2.3 t})
```

See it:

```
endtime = 10;
Plot[yformula[t], {t, 0, endtime}, PlotStyle ->
{{CadmiumOrange, Thickness[0.01]}}, AxesLabel -> {"t", "y[t]"},
PlotRange -> All, AspectRatio ->  $\frac{1}{\text{GoldenRatio}}$ ];
```



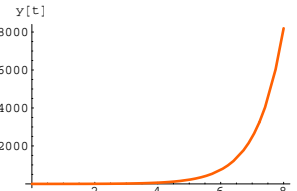
Here is the formula at work on

$$y'[t] - 1.2 y[t] = 0.8 t, \text{ with } y[0] = 0:$$

```
Clear[yformula, f, t]
f[t_] = 0.8 t;
r = -1.2; starter = 0;
yformula[t_] = E^-r t starter + E^-r t  $\int_0^t E^{rt} f[t] dt$ 
E^{1.2 t} (0.555556 - 0.666667 E^{-1.2 t} (0.833333 + 1. t))
```

See it:

```
endtime = 8;
Plot[yformula[t], {t, 0, endtime}, PlotStyle ->
{{CadmiumOrange, Thickness[0.01]}}, AxesLabel -> {"t", "y[t]"},
PlotRange -> All, AspectRatio ->  $\frac{1}{\text{GoldenRatio}}$ ];
```



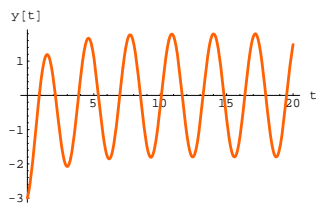
Here is the formula at work on

$$y'[t] + 0.5 y[t] = 3.7 \text{Sin}[2 t], \text{ with } y[0] = -3.0:$$

```
Clear[yformula, f, t]
f[t_] = 3.7 Sin[2 t]
r = 0.5;
starter = -3.0;
yformula[t_] = E^-r t starter + E^-r t  $\int_0^t E^{rt} f[t] dt$ 
3.7 Sin[2 t]
-3. E^-0.5 t + E^-0.5 t ((1.74118 + 0. I) -
(1.74118 + 0. I) E^{0.5 t} ((1. + 0. I) Cos[2 t] - (0.25 + 0. I) Sin[2 t]))
```

See it:

```
endtime = 20;
Plot[yformula[t], {t, 0, endtime}, PlotStyle ->
{{CadmiumOrange, Thickness[0.01]}}, AxesLabel -> {"t", "y[t]"},
PlotRange -> All, AspectRatio ->  $\frac{1}{\text{GoldenRatio}}$ ];
```



Explain why the formula

$y_{\text{formula}}[t] = E^{-rt} \text{starter} + E^{-rt} \int_0^t E^{rt} f[t] dt$
is guaranteed to work (if the integral can be done)

□Answer:

Start with

$$y'[t] + r y[t] = f[t], \text{ with } y[0] = \text{starter}.$$

Multiply both sides by E^{rt} to get:

$$E^{rt} y'[t] + E^{rt} r y[t] = E^{rt} f[t].$$

Lots of folks like to call E^{rt} by the name "integrating factor."

And you note (thanks to the product rule of derivatives) that the new equation is same as

$$D[E^{rt} y[t], t] = E^{rt} f[t].$$

Check:

```
Clear[x, t, f, y]
D[E^x t y[t]] == E^x t f[t]
E^x t r y[t] + E^x t y'[t] == E^x t f[t]
```

This tells you that the original diffeq

$$y'[t] + r y[t] = f[t]$$

is the same as

$$D[E^{rt} y[t], t] = E^{rt} f[t].$$

Next you integrate both sides of

$$D[E^{rt} y[t], t] = E^{rt} f[t]$$

from 0 to t to get

$$E^{rt} y[t] - E^0 y[0] = \int_0^t D(E^{rt} y[t], t) dt = \int_0^t E^{rt} f[t] dt.$$

This gives

$$E^{rt} y[t] - y[0] = \int_0^t E^{rt} f[t] dt.$$

So

$$y[t] - E^{-rt} y[0] = E^{-rt} \int_0^t E^{rt} f[t] dt$$

This gives the exact formula for y[t] in terms of f[t], r and y[0]:

$$y[t] = E^{-rt} y[0] + E^{-rt} \int_0^t E^{rt} f[t] dt.$$

□B.2) The downside

What's the downside of trying to integrate out?

□Answer:

If you're working by hand with pencil and paper, the integrals can drive you nuts.

It can even drive mighty *Mathematica* nuts

Here is the formula at work on

$$y'[t] + 0.5 y[t] = 3.7 \text{Sin}[2t], \text{ with } y[0] = -3.0:$$

```
Clear[yformula, f, t]
f[t_] = 3.7 E^Sin[t]
r = 0.5;
starter = -3.0;
yformula[t_] = E^-x t starter + E^-x t \int_0^t E^x t f[t] dt
3.7 E^Sin[t]
-3. E^-0.5 t + 3.7 E^-0.5 t \int_0^t E^0.5 t + Sin[t] dt
```

Mathematica could not calculate $\int_0^t E^{0.5t + \text{Sin}[t]} dt$.

Neither can anything or anybody else.

If you can calculate the integral, the method is as easy as apple pie.

But if you can't do the integrals, the method is dead in the water.

Moral: Integrating out works best on contrived classroom problems.

B.3) Steady state for $y'[x] + r y[x] = f[t]$ with $r > 0$: Any solution eventually settles into the same steady state. If $r < 0$, then all bets are off

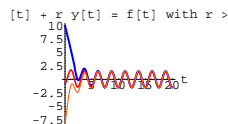
□B.3.a.i)

Here are three plots of solutions a random forced exponential diffeq

$$y'[t] + r y[t] = 4 \text{Sin}[2.5 t], \text{ (with } r > 0)$$

with random starting values on y[0]:

```
r = Random[Real, {0.5, 1.0}];
f[t_] = 4 Sin[2.5 t];
endtime = 20;
Clear[y, y1, y2, y3, t]
starter1 = Random[Real, {5, 10}];
starter2 = Random[Real, {-2, 2}]; starter3 = Random[Real, {-10, -5}];
y1[t_] = E^-x t starter1 + E^-x t \int_0^t E^x t f[t] dt;
y2[t_] = E^-x t starter2 + E^-x t \int_0^t E^x t f[t] dt;
y3[t_] = E^-x t starter3 + E^-x t \int_0^t E^x t f[t] dt;
plots1 = Plot[{y1[t], y2[t], y3[t]},
{t, 0, endtime}, PlotStyle -> {{Thickness[0.018], Blue},
{Thickness[0.014], Red}, {Thickness[0.01], CadmiumOrange}},
PlotRange -> All, AspectRatio -> Automatic, AxesLabel -> {"t", ""},
PlotLabel -> "y'[t] + r y[t] = f[t] with r > 0";
```



Rerun several times.

Describe what you see and explain why you see it.

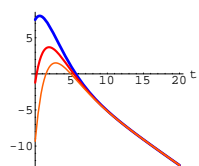
□Answer:

Take a look at another one - this time it will be

$$y'[t] + r y[t] = 6 E^{-0.4t} + p t$$

with random $r > 0$ random choice of p and random starting values on y[0]:

```
r = Random[Real, {0.5, 1.0}];
p = Random[Real, {-0.5, 0.5}]; f[t_] = 6 E^-0.4 t + p t;
endtime = 20;
Clear[y, y1, y2, y3, t]
y'[t] + r y[t] == f[t]
starter1 = Random[Real, {5, 10}];
starter2 = Random[Real, {-2, 2}]; starter3 = Random[Real, {-10, -5}];
y1[t_] = E^-x t starter1 + E^-x t \int_0^t E^x t f[t] dt;
y2[t_] = E^-x t starter2 + E^-x t \int_0^t E^x t f[t] dt;
y3[t_] = E^-x t starter3 + E^-x t \int_0^t E^x t f[t] dt;
plots2 = Plot[{y1[t], y2[t], y3[t]},
{t, 0, endtime}, PlotStyle -> {{Thickness[0.018], Blue},
{Thickness[0.014], Red}, {Thickness[0.01], CadmiumOrange}},
PlotRange -> All, AspectRatio -> Automatic, AxesLabel -> {"t", ""};
0.552598 y[t] + y'[t] == 6 E^-0.4 t - 0.384923 t
```



Rerun several times.

Every time the three solutions begin their trip in totally different ways, but when t is large, you need a scorecard to tell them apart.

In other words, they all settle into the same steady state behavior—regardless of the starting values on $y[0]$. The long term behavior of the solutions is totally insensitive to the starting value on $y[0]$.

To explain why this happens, look at the formula

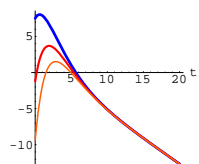
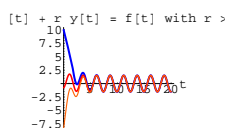
$$y[t] = E^{-rt} \text{ starter} + E^{-rt} \int_0^t E^{rt} f[t] dt$$

If $r > 0$, then as t gets large, you are guaranteed that $E^{-rt} \text{ starter} \rightarrow 0$.

The upshot: In the long run, $E^{-rt} \int_0^t E^{rt} f[t] dt$

That's what you're seeing here:

```
Show[plots1];
Show[plots2];
```



□B.3.a.ii)

Does this mean that when you have a forced exponential diffeq

$$y'[t] + r y[t] = f[t], \text{ with } r > 0$$

then the long term steady state behavior of solutions

→ depends only on the behavior of $f[t]$

and

→ does not depend not on the starting value for $y[0]$?

□Answer:

Yes!!!!

□B.3.a.ii)

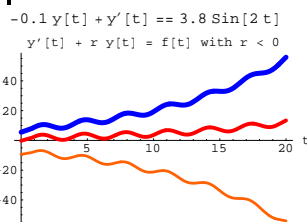
What happens when $r < 0$?

□Answer:

Try it and see:

```
r = -0.1; f[t_] = 3.8 Sin[2 t];
endtime = 20;
Clear[y, y1, y2, y3, t]
y'[t] + r y[t] == f[t]
starter1 = Random[Real, {5, 10}];
starter2 = Random[Real, {-2, 2}]; starter3 = Random[Real, {-10, -5}];
y1[t_] = E^{-rt} starter1 + E^{-rt} \int_0^t E^{rt} f[t] dt;
y2[t_] = E^{-rt} starter2 + E^{-rt} \int_0^t E^{rt} f[t] dt;
y3[t_] = E^{-rt} starter3 + E^{-rt} \int_0^t E^{rt} f[t] dt;
plots = Plot[{y1[t], y2[t], y3[t]},
```

```
{t, 0, endtime}, PlotStyle -> {{Thickness[0.018], Blue},
{Thickness[0.014], Red}, {Thickness[0.01], CadmiumOrange}},
PlotRange -> All, AspectRatio -> \frac{1}{GoldenRatio}, AxesLabel -> {"t", ""},
PlotLabel -> "y'[t] + r y[t] = f[t] with r < 0";
```



If $r < 0$, the solutions don't merge.

To see why, look at the formula

$$y[t] = E^{-rt} \text{ starter} + E^{-rt} \int_0^t E^{rt} f[t] dt.$$

When $r < 0$, the term $E^{-rt} \text{ starter}$ does not go away as t gets big.

In fact $E^{-rt} \text{ starter}$ gets bigger and bigger as t gets large.

That's why different starter values on $y[0]$ yield increasingly different plots as t gets large.

DE.01 Transition from Calculus to DiffEq: The Exponential Differential Equation

$y'[t] + r y[t] = f[t]$ Tutorials

T.1) The exponential diffeq and your personal finances

You deposit \$K in a savings account bearing interest rate 100 x percent per year compounded every instant. This means that if $\text{balance}[t]$ is the value in dollars of this untouched savings account at time t years after the deposit, then

$$\text{balance}[t] = K e^{xt}.$$

□T.1.a.i)

What differential equation does $\text{balance}[t]$ solve?

□Answer:

$$\text{balance}'[t] = K e^{xt} x$$

$$= \text{balance}[t] x$$

$$= x \text{ balance}[t];$$

so $\text{balance}[t]$ solves the unforced exponential differential equation

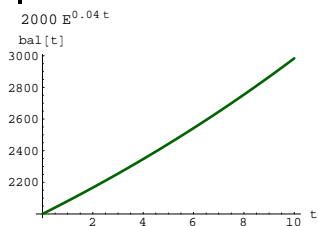
$$\text{balance}'[t] - x \text{ balance}[t] = 0.$$

with $\text{balance}[0] = K$.

See the balance grow over a ten year period in this sample case of an initial deposit of $K = \$2000$ with interest rate 100 x = 4%.

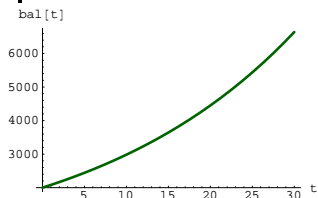
```
Clear[balance, t];
K = 2000;
x = 0.04;
balance[t_] = K E^{xt}
endtime = 10;
```

```
Plot[balance[t], {t, 0, endtime},
PlotStyle -> {{Thickness[0.01], DarkGreen}},
AxesLabel -> {"t", "bal[t]"}, AspectRatio ->  $\frac{1}{\text{GoldenRatio}}$ ];
```



See what happens over a 30 year period:

```
endtime = 30; Plot[balance[t],
{t, 0, endtime}, PlotStyle -> {{Thickness[0.01], DarkGreen}},
AxesLabel -> {"t", "bal[t]"}, AspectRatio ->  $\frac{1}{\text{GoldenRatio}}$ ];
```



□T.1.a.iii)

Plot balance[t] over the first 10 years for the case in which

$$K = \$5000,$$

$$d = \$500,$$

$$100\% = 5.3\% \text{ compounded every instant.}$$

□Answer:

It's a snap:

The diffeq is

$$\text{balance}'[t] - x \text{balance}[t] = d$$

with

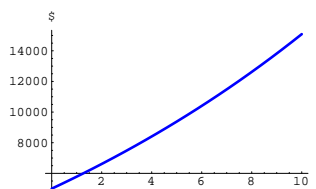
$$\text{balance}[0] = K.$$

Here's the formula for balance[t]:

```
Clear[balance, f, K, x, d, t]
endtime = 10;
K = 5000;
starter = K;
d = 500;
x = 0.053;
r = -x;
f[t_] = d;
balance[t_] = Expand[E-x t starter + E-x t ∫0t Ex t f[t] dt]
-9433.96 + 14434. E0.053 t
```

Here comes the plot:

```
endtime = 10;
plot = Plot[balance[t], {t, 0, endtime},
PlotStyle -> {{Blue, Thickness[0.01]}}, AspectRatio ->  $\frac{1}{\text{GoldenRatio}}$ ,
AxesLabel -> {"t", "$"}];
```



Better than \$14,000 in 10 years.

□T.1.b)

You and your significant other have just had a beautiful bouncing baby. You can already tell that the kid is as smart as all get-out, and you decide to set up a bank account geared toward producing big-time college money in 18 years.

You can come up with \$10,000 for the account now and you can lock in an interest rate of

$$100\% = 4.6\% \text{ compounded every instant.}$$

How do you set your yearly payment rate d (to result in \$ d after t years) to be sure of having \$120,000 dollars in your account by the end of 18 years?

□Answer:

Thanks to the magic of differential equations, coming up with the answer is a piece of cake.

The diffeq is:

$$\text{balance}'[t] + 0.046 \text{balance}[t] = d$$

with $\text{balance}[0] = 10000$.

Go with a cleared d to get a formula for balance[t]:

```
Clear[balance, f, K, x, d, t]
K = 10000; starter = K; x = 0.046; r = -x; f[t_] = d;
balance[t_] = Expand[E-x t starter + E-x t ∫0t Ex t f[t] dt]
```

□T.1.a.ii)

At the time of your original deposit of \$ K , you arrange with your bank to make a continuous deposit into your account at a rate of d dollars per year. This means that after t years,

$t d$ additional dollars

have been deposited. As the new money is deposited into the account, the new money begins to earn interest at the rate of $100x$ percent compounded every instant.

If $\text{balance}[t]$ is the amount in the account t years after the arrangement was set up, then what differential equation does $\text{balance}[t]$ satisfy and what is a formula for its solution?

□Answer:

The differential equation is

$$\text{balance}'[t] = x \text{balance}[t] + d$$

with

$$\text{balance}[0] = P.$$

The derivative of the sum is the sum of the derivatives; the $(r \text{balance}[t])$ term comes from interest and the d term comes from your continuous deposit.)

This diffeq is the forced exponential diffeq

$$\text{balance}'[t] - x \text{balance}[t] = d$$

with

$$\text{balance}[0] = K.$$

A formula for its solution is:

```
Clear[balance, f, K, x, d, t]
starter = K;
r = -x;
f[t_] = d;
balance[t_] = Expand[E-x t starter + E-x t ∫0t Ex t f[t] dt]
E-x t K -  $\frac{d}{x} + \frac{d E^{x t}}{x}$ 
```

$$-21.7391 d + 10000 e^{0.046 t} + 21.7391 d e^{0.046 t}$$

After 18 years, the balance is:

```
balance[18]
22887.4 + 28.016 d
```

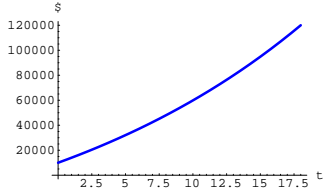
You want balance[18] = 120000:

```
Solve[balance[18] == 120000, d]
{{d -> 3466.33}}
```

You will need to put in about \$3500 per year into the college account.

See it grow:

```
d = 3467;
endtime = 18;
plot = Plot[balance[t], {t, 0, endtime},
PlotStyle -> {{Blue, Thickness[0.01]}}, AspectRatio -> 1/GoldenRatio,
AxesLabel -> {"t", "$"}];
```



There you go.

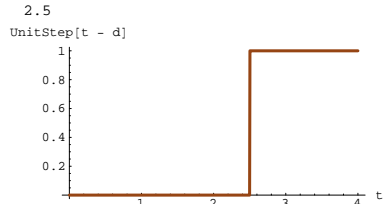
If you have a good job, this is a rather painless way of coming up with tuition, room and board.

T.2) The jump function UnitStep[t - d] and the impulse function DiracDelta[t - d]

□T.2.a.i)

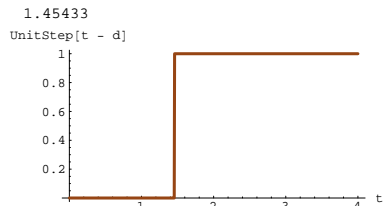
Here is a plot of the jump function UnitStep[t - d] for d = 2.5:

```
d = 2.5;
Clear[t];
Plot[UnitStep[t - d],
{t, 0, 4}, PlotStyle -> {{Thickness[0.01], MarsOrange}},
AspectRatio -> 1/2, AxesLabel -> {"t", "UnitStep[t - d]"},
Epilog -> {Red, Text["d", {d, -0.1}]}];
```



Here is a plot of the jump function UnitStep[t - d] for a random d:

```
d = Random[Real, {1, 3}]
Plot[UnitStep[t - d],
{t, 0, 4}, PlotStyle -> {{Thickness[0.01], MarsOrange}},
AspectRatio -> 1/2, AxesLabel -> {"t", "UnitStep[t - d]"},
Epilog -> {Red, Text["d", {d, -0.08}]}];
```



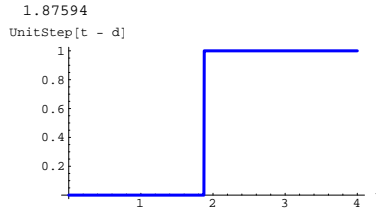
Rerun a couple of times.

Describe how UnitStep[t - d] plots out.

□Answer:

Take another look:

```
d = Random[Real, {1, 3}]
Plot[UnitStep[t - d], {t, 0, 4}, PlotStyle -> {{Thickness[0.01], Blue}},
AspectRatio -> 1/2, AxesLabel -> {"t", "UnitStep[t - d]"},
Epilog -> {Red, Text["d", {d, -0.08}]}];
```



UnitStep[t - d] plots out so that:

$$\text{UnitStep}[t - d] = 0 \text{ for } t < d$$

and

$$\text{UnitStep}[t - d] = 1 \text{ for } t > d.$$

At t = d,

UnitStep[t - d] jumps from 0 to 1.

□T.2.a.ii) The impulse DiracDelta[t - d] is the derivative of UnitStep[t - d]

Here is Mathematica's calculation of the derivative of UnitStep[t - d]:

```
Clear[f, t, d]
f[t_] = UnitStep[t - d];
f'[t]
DiracDelta[-d + t]
```

What does this tell you about DiracDelta[t - d] for t not equal to d?

□Answer:

Remember:

$$\text{UnitStep}[t - d] = 0 \text{ for } t < d$$

and

$$\text{UnitStep}[t - d] = 1 \text{ for } t > d.$$

So $\partial_t \text{UnitStep}[t - d] = 0$ for all t's not equal to d.

Consequently:

$$\text{DiracDelta}[t - d] = \partial_t \text{UnitStep}[t - d] = 0$$

for all t's not equal to d.

□T.2.b.i) Integrating DiracDelta[t - d]

Look at these experiments:

```
Clear[t, d]
Integrate[DiracDelta[t - d] dt, {t, -Infinity, Infinity}]
```

```
1
h = 2;
Integrate[DiracDelta[t - d] dt, {t, d - h, d + h}]
```

```
1
h = 1;
Integrate[DiracDelta[t - d] dt, {t, d - h, d + h}]
```

```
1
h = 0.1;
Integrate[DiracDelta[t - d] dt, {t, d - h, d + h}]
```

```
1
h = 0.00001;
Integrate[DiracDelta[t - d] dt, {t, d - h, d + h}]
```

```
1
h = 0.00001;
Integrate[DiracDelta[t - d] dt, {t, -Infinity, 0}]
```

```
0
h = 0.00001;
Integrate[DiracDelta[t - d] dt, {t, 0, Infinity}]
```

0
Interpret these experiments.

□ Answer:

Even though $\text{DiracDelta}[t - d] = 0$ for t not equal to d , when you
→ integrate $\text{DiracDelta}[t - d]$ over any interval containing d , you get 1.
→ integrate $\text{DiracDelta}[t - d]$ over any interval not containing d , you get 0.

Lots of folks like to say that

$$\text{DiracDelta}[d - d] = \text{DiracDelta}[0]$$

spikes at 0 to a certain order of infinity that makes all this work.

□ T.2.b.ii) Integrals involving $\text{DiracDelta}[t - d]$ are easy to do by hand

Look at these experiments:

```
Clear[t, f, d]
Integrate[f[t] DiracDelta[t - d] dt, {t, -∞, ∞}]
f[d]
h = 2;
Integrate[f[t] DiracDelta[t - d] dt, {t, d - h, d + h}]
f[d]
h = 1;
Integrate[f[t] DiracDelta[t - d] dt, {t, d - h, d + h}]
f[d]
h = 0.1;
Integrate[f[t] DiracDelta[t - d] dt, {t, d - h, d + h}]
f[d]
h = 0.00001;
Integrate[f[t] DiracDelta[t - d] dt, {t, d - h, d + h}]
f[d]
```

```
f[d]
h = 0.00001;
Integrate[f[t] DiracDelta[t - d] dt, {t, -∞, ∞}]
0
h = 0.00001;
Integrate[f[t] DiracDelta[t - d] dt, {t, 0, ∞}]
0
```

Interpret these experiments.

□ Answer:

Even though $\text{DiracDelta}[t - d] = 0$ for t not equal to d , when you
→ integrate $f[t] \text{DiracDelta}[t - d]$ over any interval containing d , you get $f[d]$.
→ integrate $f[t] \text{DiracDelta}[t - d]$ over any interval not containing d , you get 0.

It seems that $\text{DiracDelta}[d - d] = \text{DiracDelta}[0]$ spikes to a certain order of infinity that makes all this work.

□ T.2 .b.iii) $\text{DiracDelta}[t - d]$ isn't a normal function

Give a short history of $\text{DiracDelta}[t - d]$.

□ Answer:

Nobel prize winning physicist Paul Dirac developed what is now called the Dirac delta function a little more than fifty years ago. Scientists immediately put the Dirac delta function to use and got good results, but mathematicians objected because there was no firm mathematical foundation for the Dirac delta function.

Luckily for science, the French mathematician, Laurent Schwartz,

pulled off a 20th Century mathematical miracle by creating an extended theory of functions. Schwartz's theory includes all functions that are functions in the normal sense of the word "function" and Schwartz's theory includes additional "functions" which some advanced folks call by the name "distributions."

$\text{DiracDelta}[t - d]$ is in this class of additional "functions."

You are invited to use and enjoy this mathematical beauty with confidence and authority.

If you have the desire to understand Schwartz's theory in full, then you're a good candidate to go on in advanced math.

T.3) Impulse-forcing the exponential diffeq with a DiracDelta function:

The physical meaning of the impulse hit

Old hands at diffeq know that you can get a formula for the solution of the forced exponential diffeq

$$y'[t] + r y[t] = f[t], \text{ with } y[0] = \text{starter}.$$

The formula is:

$$y[t] = E^{-rt} \text{starter} + E^{-rt} \int_0^t E^{rt} f[t] dt.$$

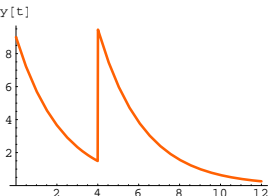
Here is the formula at work on

$$y'[t] + 0.45 y[t] = 8 \text{DiracDelta}[t - 4], \text{ with } y[0] = 9.0:$$

```
Clear[yformula, f, t]
f[t_] = 8 DiracDelta[t - 4]; r = 0.45; starter = 9.0;
yformula[t_] = E^-r t starter + E^-r t Integrate[E^r t f[t] dt, {t, 0, t}]
9. E^-0.45 t + 48.3972 E^-0.45 t UnitStep[-4 + t]
```

See it:

```
endtime = 12;
solutionplot = Plot[yformula[t], {t, 0, endtime}, PlotStyle -> {{CadmiumOrange, Thickness[0.01]}}, AxesLabel -> {"t", "y[t]"}, PlotRange -> All, AspectRatio -> 1/GoldenRatio];
```

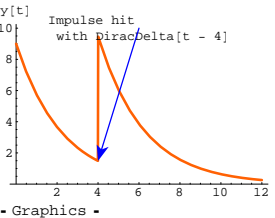


Describe what you see and explain why you see it.

□ Answer:

Look at this embellished picture

```
tail = {6, 10};
label = Graphics[Text["Impulse hit \n with DiracDelta[t - 4]", tail]];
pointer = Arrow[{4, yformula[3.9999]} - tail, Tail -> tail, HeadSize -> 1];
Show[solutionplot, label, pointer]
```



8 $\text{DiracDelta}[t - 4]$ fired at $t = 4$ and jumped $y[t]$ a whole 8 units up at $t = 4$:

```
yformula[4.00000001] - yformula[3.9999999]
8.
```

That's the effect of forcing the exponential diffeq with a DiracDelta function.

You can see why this happens by looking at

$$y_{\text{formula}}[t] = E^{-rt} \text{starter} + E^{-rt} \int_0^t E^{rt} 8 \text{DiracDelta}[t - 4] dt.$$

For $0 \leq t < 4$, $\text{DiracDelta}[t - 4] = 0$,

so if $t < 4$, then

$$\begin{aligned} y_{\text{formula}}[t] &= E^{-rt} \text{starter} + E^{-rt} \int_0^t E^{rt} 8 \text{DiracDelta}[t - 4] dt. \\ &= E^{-rt} \text{starter} + E^{-rt} \int_0^t E^{rt} 0 dt \\ &= E^{-rt} \text{starter} + 0 \\ &= E^{-rt} \text{starter} \end{aligned}$$

For $t > 4$, then $\text{DiracDelta}[t - 4]$ fires at $t = 4$ and

$$\begin{aligned} y_{\text{formula}}[t] &= E^{-rt} \text{starter} + E^{-rt} \int_0^t E^{rt} 8 \text{DiracDelta}[t - 4] dt. \\ &= E^{-rt} \text{starter} + E^{-rt} 8 E^{r4} \\ &= E^{-rt} \text{starter} + 8 E^{-r(t-4)} \end{aligned}$$

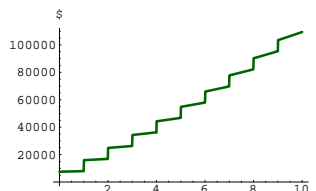
So as t advances through $t = 4$, $y_{\text{formula}}[t]$ jumps from

$$E^{-r4} \text{starter}$$

to

$$E^{-r4} \text{starter} + 8 E^{-r(4-4)} = E^{-r4} \text{starter} + 8.$$

This is a jump up of length 8.



Better than \$100,000 in 10 years.

DE.01 Transition from Calculus to DiffEq: The Exponential Differential Equation

$y'[t] + r y[t] = f[t]$
Give It a Try!

G.1) The exponential diffeq

□G.1.a)

The differential equation is

$$y'[t] + 0.2 y[t] = 0 \text{ with } y[0] = 15.3.$$

Your job is to come up with the formula for $y[t]$ and then to plot it for $0 \leq t \leq 10$.

□G.1.b)

The differential equation is

$$y'[t] + 0.2 y[t] = \text{Sin}[t] \text{ with } y[0] = 15.3.$$

Your job is to come up with the formula for $y[t]$ and then to plot it for $0 \leq t \leq 10$.

T.4) The DiracDelta function in your personal finances

□T.4.a)

You deposit \$7500 into an account paying 5.6 % interest over 10 years. You plan to make an annual additional deposit of \$8000 at the end of each of the 10 years. The same interest rate of 5.6 % is guaranteed over the 10 year period.

Use the DiracDelta function to help you to plot the projected status of your account:

□Answer:

The diffeq is

$$\text{balance}'[t] - 0.056 \text{balance}[t] = \sum_{k=1}^{10} d \text{DiracDelta}[t - k]$$

with

$$\text{balance}[0] = K.$$

Here's the formula for $\text{balance}[t]$:

```
Clear[balance, f, K, x, d, t]
endtime = 10; K = 7500; starter = K; d = 8000; r = -0.056;
f[t_] = Sum[d DiracDelta[t - k], {k, 1, 10}];
balance[t_] = Expand[E^-x t starter + E^-x t Integrate[E^x t f[t] dt, {t, 0, t}]]
7500 E^0.056 t + 4569.67 E^0.056 t UnitStep[-10 + t] +
4832.88 E^0.056 t UnitStep[-9 + t] + 5111.24 E^0.056 t UnitStep[-8 + t] +
5405.63 E^0.056 t UnitStep[-7 + t] + 5716.98 E^0.056 t UnitStep[-6 + t] +
6046.27 E^0.056 t UnitStep[-5 + t] + 6394.52 E^0.056 t UnitStep[-4 + t] +
6762.83 E^0.056 t UnitStep[-3 + t] + 7152.35 E^0.056 t UnitStep[-2 + t] +
7564.31 E^0.056 t UnitStep[-1 + t]
```

Here comes the plot:

```
endtime = 10;
plot = Plot[balance[t],
{t, 0, endtime}, PlotStyle -> {{DarkGreen, Thickness[0.01]}},
AspectRatio -> 1/GoldenRatio, AxesLabel -> {"t", "$"}];
```

□G.1.c)

The differential equation is

$$y'[t] = -0.2 y[t] + \text{Sin}[t] \text{ with } y[0] = 15.3.$$

Your job is to come up with the formula for $y[t]$ and then to plot it for $0 \leq t \leq 10$.

□G.1.d)

The differential equation is

$$y'[t] - 0.5 y[t] = 0 \text{ with } y[0] = 0.3.$$

Your job is to write down the formula for $y[t]$ and then to plot it for $0 \leq t \leq 6$.

□G.1.e)

The differential equation is

$$y'[t] = 0.5 y[t] \text{ with } y[0] = 0.3.$$

Your job is to write down the formula for $y[t]$ and then to plot it for $0 \leq t \leq 6$.

□G.1.f)

The differential equation is

$$y'[t] + 0.3 y[t] = 6.5 E^{-0.3t} \text{ with } y[0] = 15.3.$$

Your job is to write down the formula for $y[t]$ and then to plot it for $0 \leq t \leq 10$.

□G.1.g)

The differential equation is

$$y'[t] - 0.3 y[t] = 6 \text{DiracDelta}[t - 2] \text{ with } y[0] = 15.3.$$

Your job is to write down the formula for $y[t]$ and then to plot it for $0 \leq t \leq 6$.

G.2) Steady state for $y'[t] + r y[t] = f[t]$

□G.2.a.i)

Here are three plots of solutions a random forced damped exponential diffeq

$$y'[t] + r y[t] = 5 E^{-(t-3)^2} \quad (\text{with } r > 0)$$

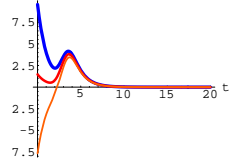
with random starting values on $y[0]$:

```
r = Random[Real, {0.5, 1.0}];
f[t_] = 5 E^{-(t-3)^2};
endtime = 20;
Clear[y, y1, y2, y3, t]
starter1 = Random[Real, {5, 10}];
starter2 = Random[Real, {-2, 2}]; starter3 = Random[Real, {-10, -5}];

y1[t_] = E^{-r t} starter1 + E^{-r t} \int_0^t E^{r t} f[t] dt;
y2[t_] = E^{-r t} starter2 + E^{-r t} \int_0^t E^{r t} f[t] dt;
y3[t_] = E^{-r t} starter3 + E^{-r t} \int_0^t E^{r t} f[t] dt;

setup = Plot[{y1[t], y2[t], y3[t]},
  {t, 0, endtime}, PlotStyle -> {{Thickness[0.018], Blue},
  {Thickness[0.014], Red}, {Thickness[0.01], CadmiumOrange}},
  PlotRange -> All, AspectRatio -> Automatic, AxesLabel -> {"t", ""},
  PlotLabel -> "y'[t] + r y[t] = f[t] with r > 0";
```

'[t] + r y[t] = f[t] with r >



Rerun several times.

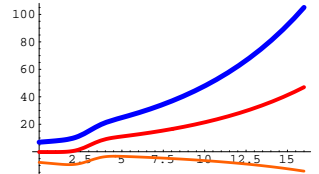
Describe what you see and explain why you see it.

```
r = Random[Real, {-0.3, -0.1}];
f[t_] = 5 E^{-(t-3)^2};
endtime = 16;
Clear[y, y1, y2, y3, t]
starter1 = Random[Real, {5, 10}];
starter2 = Random[Real, {-2, 2}];
starter3 = Random[Real, {-10, -5}];

y1[t_] = E^{-r t} starter1 + E^{-r t} \int_0^t E^{r t} f[t] dt;
y2[t_] = E^{-r t} starter2 + E^{-r t} \int_0^t E^{r t} f[t] dt;
y3[t_] = E^{-r t} starter3 + E^{-r t} \int_0^t E^{r t} f[t] dt;

setup = Plot[{y1[t], y2[t], y3[t]}, {t, 0, endtime},
  PlotStyle -> {{Thickness[0.018], Blue},
  {Thickness[0.014], Red}, {Thickness[0.01], CadmiumOrange}},
  PlotRange -> All, AspectRatio -> \frac{1}{GoldenRatio}, AxesLabel -> {"t", ""},
  PlotLabel -> "y'[t] + r y[t] = f[t] with r < 0";
```

y'[t] + r y[t] = f[t] with r < 0



Rerun several times.

Describe what you see and explain why you see it.

□G.2.a.iv)

Here are three plots of solutions a random forced exponential diffeq

$$y'[t] + r y[t] = 5 E^{-(t-3)^2} \quad (\text{with } r = 0)$$

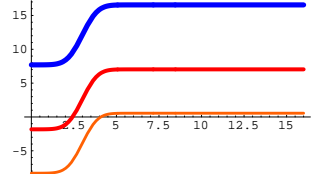
with random starting values on $y[0]$:

```
r = 0;
f[t_] = 5 E^{-(t-3)^2};
endtime = 16;
Clear[y, y1, y2, y3, t]
starter1 = Random[Real, {5, 10}];
starter2 = Random[Real, {-2, 2}];
```

```
starter3 = Random[Real, {-10, -5}];
y1[t_] = E^{-r t} starter1 + E^{-r t} \int_0^t E^{r t} f[t] dt;
y2[t_] = E^{-r t} starter2 + E^{-r t} \int_0^t E^{r t} f[t] dt;
y3[t_] = E^{-r t} starter3 + E^{-r t} \int_0^t E^{r t} f[t] dt;

setup = Plot[{y1[t], y2[t], y3[t]},
  {t, 0, endtime}, PlotStyle -> {{Thickness[0.018], Blue},
  {Thickness[0.014], Red}, {Thickness[0.01], CadmiumOrange}},
  PlotRange -> All, AspectRatio -> \frac{1}{GoldenRatio}, AxesLabel -> {"t", ""},
  PlotLabel -> "y'[t] + r y[t] = f[t] with r = 0";
```

y'[t] + r y[t] = f[t] with r = 0



Rerun several times.

Describe what you see and explain why you see it.

Click on the right for a tip

The solution formulas are

$$\begin{aligned} & E^{-r t} \text{starter1} + E^{-r t} \int_0^t E^{r t} f[t] dt, \\ & E^{-r t} \text{starter2} + E^{-r t} \int_0^t E^{r t} f[t] dt \text{ and} \\ & E^{-r t} \text{starter3} + E^{-r t} \int_0^t E^{r t} f[t] dt. \end{aligned}$$

In this case $r = 0$, so the solution formulas are

$$\begin{aligned} & \text{starter1} + \int_0^t f[t] dt, \\ & \text{starter2} + \int_0^t f[t] dt \text{ and} \\ & \text{starter3} + \int_0^t f[t] dt. \end{aligned}$$

□G.2.a.ii)

Here are three plots of solutions a random forced damped exponential diffeq

$$y'[t] + r y[t] = 4 \text{DiracDelta}[t - 3] + \text{Sin}[\pi t] \quad (\text{with } r > 0)$$

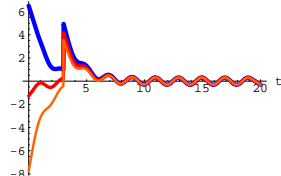
with random starting values on $y[0]$:

```
r = Random[Real, {0.3, 0.8}];
f[t_] = 4 DiracDelta[t - 3] + Sin[\pi t];
endtime = 20;
Clear[y, y1, y2, y3, t]
starter1 = Random[Real, {5, 10}];
starter2 = Random[Real, {-2, 2}];
starter3 = Random[Real, {-10, -5}];

y1[t_] = E^{-r t} starter1 + E^{-r t} \int_0^t E^{r t} f[t] dt;
y2[t_] = E^{-r t} starter2 + E^{-r t} \int_0^t E^{r t} f[t] dt;
y3[t_] = E^{-r t} starter3 + E^{-r t} \int_0^t E^{r t} f[t] dt;

setup = Plot[{y1[t], y2[t], y3[t]},
  {t, 0, endtime}, PlotStyle -> {{Thickness[0.018], Blue},
  {Thickness[0.014], Red}, {Thickness[0.01], CadmiumOrange}},
  PlotRange -> All, AspectRatio -> Automatic, AxesLabel -> {"t", ""},
  PlotLabel -> "y'[t] + r y[t] = f[t] with r > 0";
```

y'[t] + r y[t] = f[t] with r > 0



Rerun several times.

Describe what you see and explain why you see it.

□G.2.a.iii)

Here are three plots of solutions a random forced exponential diffeq

$$y'[t] + r y[t] = 5 E^{-(t-3)^2} \quad (\text{with } r < 0)$$

with random starting values on $y[0]$:

G.3) Visualizing the effect of the forcing functions on solutions of $y'[t] + r y[t] = f[t]$ with $r > 0$

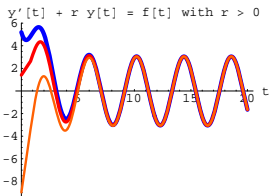
□G.3.a.i)

Here are three plots of solutions of a random forced exponential diffeq $y'[t] + r y[t] = 5 \sin[1.5 t]$ (with $r > 0$) with random starting values on $y[0]$:

```
r = Random[Real, {0.5, 1.0}];
f[t_] = 5 Sin[1.5 t];
endtime = 20;
Clear[y, y1, y2, y3, t]
starter1 = Random[Real, {5, 10}];
starter2 = Random[Real, {-2, 2}];
starter3 = Random[Real, {-10, -5}];

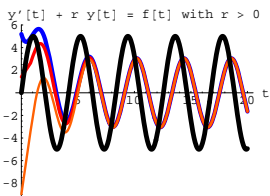
y1[t_] = E^-r t starter1 + E^-r t ∫₀ᵗ E^r t f[t] dt;
y2[t_] = E^-r t starter2 + E^-r t ∫₀ᵗ E^r t f[t] dt;
y3[t_] = E^-r t starter3 + E^-r t ∫₀ᵗ E^r t f[t] dt;

setup = Plot[{y1[t], y2[t], y3[t]},
  {t, 0, endtime}, PlotStyle -> {{Thickness[0.018], Blue},
  {Thickness[0.014], Red}, {Thickness[0.01], CadmiumOrange}},
  PlotRange -> All, AspectRatio -> Automatic, AxesLabel -> {"t", ""},
  PlotLabel -> "y'[t] + r y[t] = f[t] with r > 0";
```



Throw in this plot of the forcing function $f[t]$:

```
fplot = Plot[f[t], {t, 0, endtime}, PlotRange -> All,
  PlotStyle -> {{Thickness[0.02], Black}}, DisplayFunction -> Identity];
Show[setup, fplot, DisplayFunction -> $DisplayFunction];
```



Using only your eyes, explain in general terms how the solution plots are related to the plot of the forcing function $f[t]$.

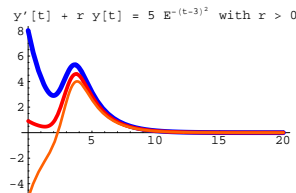
□G.3.a.ii)

Here are three plots of solutions a random forced exponential diffeq $y'[t] + r y[t] = 5 E^{-(t-3)^2}$ (with $r > 0$) with random starting values on $y[0]$:

```
r = Random[Real, {0.5, 1.0}];
f[t_] = 5 E^-(t-3)^2;
endtime = 20;
Clear[y, y1, y2, y3, t]
starter1 = Random[Real, {5, 10}];
starter2 = Random[Real, {-2, 2}];
starter3 = Random[Real, {-10, -5}];

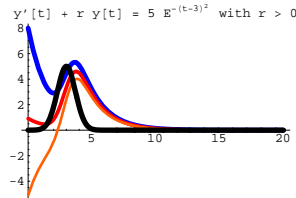
y1[t_] = E^-r t starter1 + E^-r t ∫₀ᵗ E^r t f[t] dt;
y2[t_] = E^-r t starter2 + E^-r t ∫₀ᵗ E^r t f[t] dt;
y3[t_] = E^-r t starter3 + E^-r t ∫₀ᵗ E^r t f[t] dt;

setup = Plot[{y1[t], y2[t], y3[t]},
  {t, 0, endtime}, PlotStyle -> {{Thickness[0.018], Blue},
  {Thickness[0.014], Red}, {Thickness[0.01], CadmiumOrange}},
  PlotRange -> All, AspectRatio -> Automatic, AxesLabel -> {"t", ""},
  PlotLabel -> "y'[t] + r y[t] = 5 E^-(t-3)^2 with r > 0";
```



Throw in this plot of the forcing function $f[t]$:

```
fplot = Plot[f[t], {t, 0, endtime}, PlotRange -> All,
  PlotStyle -> {{Thickness[0.02], Black}}, DisplayFunction -> Identity];
Show[setup, fplot, DisplayFunction -> $DisplayFunction];
```



Using only your eyes and no calculation, explain in general terms how the solution plots are related to the plot of the forcing function

□G.3.a.iii)

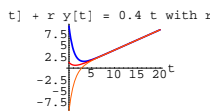
Here are three plots of solutions a random forced exponential diffeq $y'[t] + r y[t] = 0.4 t$ (with $r > 0$) with random starting values on $y[0]$:

```
r = Random[Real, {0.5, 1.0}];
f[t_] = 0.4 t;
endtime = 20;
Clear[y, y1, y2, y3, t]
starter1 = Random[Real, {5, 10}];
starter2 = Random[Real, {-2, 2}];
starter3 = Random[Real, {-10, -5}];

y1[t_] = E^-r t starter1 + E^-r t ∫₀ᵗ E^r t f[t] dt;
y2[t_] = E^-r t starter2 + E^-r t ∫₀ᵗ E^r t f[t] dt;
```

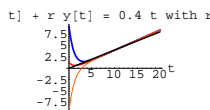
```
y3[t_] = E^-r t starter3 + E^-r t ∫₀ᵗ E^r t f[t] dt;

setup = Plot[{y1[t], y2[t], y3[t]},
  {t, 0, endtime}, PlotStyle -> {{Thickness[0.018], Blue},
  {Thickness[0.014], Red}, {Thickness[0.01], CadmiumOrange}},
  PlotRange -> All, AspectRatio -> Automatic, AxesLabel -> {"t", ""},
  PlotLabel -> "y'[t] + r y[t] = 0.4 t with r > 0";
```



Throw in this plot of the forcing function $f[t]$:

```
fplot = Plot[f[t], {t, 0, endtime}, PlotRange -> All,
  PlotStyle -> {{Thickness[0.015], Black}}, DisplayFunction -> Identity];
Show[setup, fplot, DisplayFunction -> $DisplayFunction];
- Graphics -
```



Using only your eyes and no calculation, explain in general terms how the solution plots are related to the plot of the forcing function $f[t]$.

□G.3.a.iv)

Here are three plots of solutions a random forced exponential diffeq $y'[t] + r y[t] = -5 \text{DiracDelta}[t - 3] + 6 \text{DiracDelta}[t - 6]$ (with $r > 0$) with random starting values on $y[0]$:

```
r = Random[Real, {0.2, 0.5}];
f[t_] = -5 DiracDelta[t - 3] + 6 DiracDelta[t - 6];
endtime = 20; Clear[y, y1, y2, y3, t]
```

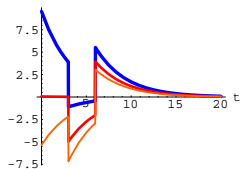
```

starter1 = Random[Real, {5, 10}];
starter2 = Random[Real, {-2, 2}];
starter3 = Random[Real, {-10, -5}];

y1[t_] = E^-r t starter1 + E^-r t ∫₀ᵗ E^r t f[t] dt;
y2[t_] = E^-r t starter2 + E^-r t ∫₀ᵗ E^r t f[t] dt;
y3[t_] = E^-r t starter3 + E^-r t ∫₀ᵗ E^r t f[t] dt;

setup = Plot[{y1[t], y2[t], y3[t]},
  {t, 0, endtime}, PlotStyle -> {{Thickness[0.018], Blue},
  {Thickness[0.014], Red}, {Thickness[0.01], CadmiumOrange}},
  PlotRange -> All, AspectRatio -> Automatic, AxesLabel -> {"t", ""}];

```



Using only your eyes and no calculation, explain in general terms how the solution plots are related to the action of the forcing function

$$f[t] = -5 \text{DiracDelta}[t - 3] + 6 \text{DiracDelta}[t - 6].$$

How big are the jumps at $t = 3$ and $t = 6$?

□G.3.a.v)

Here are three plots of solutions the forced exponential diffeq

$$y'[t] + 0.5 y[t] = 3$$

with random starting values on $y[0]$:

```

r = 0.5;
f[t_] = 3;
endtime = 20;
Clear[y, y1, y2, y3, t]
starter1 = Random[Real, {5, 10}];
starter2 = Random[Real, {-2, 2}]; starter3 = Random[Real, {-10, -5}];

y1[t_] = E^-r t starter1 + E^-r t ∫₀ᵗ E^r t f[t] dt;

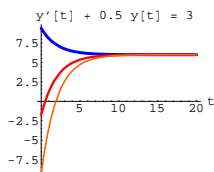
```

```

y2[t_] = E^-r t starter2 + E^-r t ∫₀ᵗ E^r t f[t] dt;
y3[t_] = E^-r t starter3 + E^-r t ∫₀ᵗ E^r t f[t] dt;

setup = Plot[{y1[t], y2[t], y3[t]},
  {t, 0, endtime}, PlotStyle -> {{Thickness[0.018], Blue},
  {Thickness[0.014], Red}, {Thickness[0.01], CadmiumOrange}},
  PlotRange -> All, AspectRatio -> Automatic, AxesLabel -> {"t", ""},
  PlotLabel -> "y'[t] + 0.5 y[t] = 3 ";

```

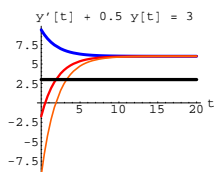


Throw in this plot of the forcing function $f[t]$:

```

fplot = Plot[f[t], {t, 0, endtime}, PlotRange -> All,
  PlotStyle -> {{Thickness[0.02], Black}}, DisplayFunction -> Identity];
Show[setup, fplot, DisplayFunction -> $DisplayFunction];

```



The forcing function runs at 3, the solutions gravitate to 6.

Is this just an accident?

If not then why not?

[Click on the right for a tip](#)

The diffeq is

$$y'[t] + 0.5 y[t] = 3.$$

Is $y[t] = 6$ a solution?

Because $r > 0$, all solutions eventually merge.

G.4) Sometimes the starter value on $y[0]$ makes little difference; sometimes it does

□G.4.a.i)

Here are three plots of solutions of the forced exponential diffeq

$$y'[t] + 1.85 y[t] = 0.7 t^2$$

with starter values on $y[0]$ equal to $-6, 0$, and $+7$:

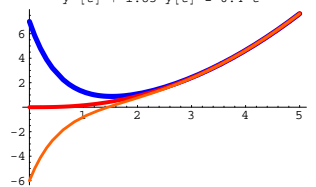
```

r = 1.85;
Clear[f, t]
f[t_] = 0.7 t^2;
endtime = 5;
Clear[y, y1, y2, y3, t]
starter1 = 7;
starter2 = 0;
starter3 = -6;

y1[t_] = E^-r t starter1 + E^-r t ∫₀ᵗ E^r t f[t] dt;
y2[t_] = E^-r t starter2 + E^-r t ∫₀ᵗ E^r t f[t] dt;
y3[t_] = E^-r t starter3 + E^-r t ∫₀ᵗ E^r t f[t] dt;

solutionplots = Plot[{y1[t], y2[t], y3[t]},
  {t, 0, endtime}, PlotStyle -> {{Thickness[0.018], Blue},
  {Thickness[0.014], Red}, {Thickness[0.01], CadmiumOrange}},
  PlotRange -> All, AspectRatio -> 1/GoldenRatio, AxesLabel -> {"t", ""},
  PlotLabel -> "y'[t] + 1.85 y[t] = 0.4 t^2";

```



The formulas for the plotted curves are:

[Expand\[y1\[t\]\]](#)

$$0.221112 + 6.77889 E^{-1.85 t} - 0.409058 t + 0.378378 t^2$$

[Expand\[y2\[t\]\]](#)

$$0.221112 - 0.221112 E^{-1.85 t} - 0.409058 t + 0.378378 t^2$$

[Expand\[y3\[t\]\]](#)

$$0.221112 - 6.22111 E^{-1.85 t} - 0.409058 t + 0.378378 t^2$$

All these solution eventually merge onto a certain parabola.

Give the formula for this parabola and add its plot to the solution plot.

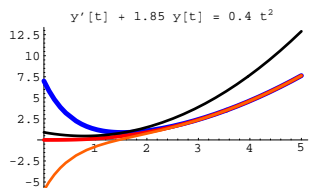
[Click on the right for a tip.](#)

To plot your parabola, modify this code:

```

Clear[parabola, t]
parabola[t_] = 0.9 - 1.1 t + 0.7 t^2;
parabolaplot = Plot[parabola[t], {t, 0, endtime},
  PlotStyle -> {{Black, Thickness[0.01]}, DisplayFunction -> Identity];
Show[solutionplots, parabolaplot,
  DisplayFunction -> $DisplayFunction];

```



The plotted parabola formula is not correct. Your job is to enter the correct formula.

□G.4.a.ii)

Keep everything the same as in part i) above. The solution calculate above are all solutions of:

$$y'[t] + r y[t] = f[t]$$

$$1.85 y[t] + y'[t] = 0.7 t^2$$

The solution formulas are:

[Expand\[y1\[t\]\]](#)

$$0.221112 + 6.77889 E^{-1.85 t} - 0.409058 t + 0.378378 t^2$$

[Expand\[y2\[t\]\]](#)

$$0.221112 - 0.221112 E^{-1.85 t} - 0.409058 t + 0.378378 t^2$$

| Expand[y3[t]]

$$0.221112 - 6.22111 E^{-1.85 t} - 0.409058 t + 0.378378 t^2$$

Where does the value of $r = 1.85$ show up vividly in these formulas?

How does the fact that $r > 0$ explain why the plots try to merge?

□G.4.b.i)

Here are three plots of solutions of the forced exponential diffeq

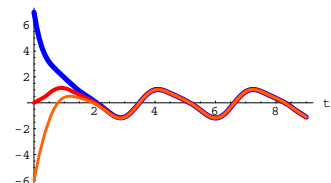
$$y'[t] + 2.13 y[t] = 3 \sin[2 t] + \sin[4 t]$$

with starter values on $y[0]$ equal to $-6, 0,$ and $+7$:

```
r = 2.13; Clear[f, t]
f[t_] = 3 Sin[2 t] + Sin[4 t];
endtime = 9;
Clear[y, y1, y2, y3, t];
starter1 = 7;
starter2 = 0;
starter3 = -6;

y1[t_] = E^{-r t} starter1 + E^{-r t} \int_0^t E^{r t} f[t] dt;
y2[t_] = E^{-r t} starter2 + E^{-r t} \int_0^t E^{r t} f[t] dt;
y3[t_] = E^{-r t} starter3 + E^{-r t} \int_0^t E^{r t} f[t] dt;

solutionplots = Plot[{y1[t], y2[t], y3[t]},
  {t, 0, endtime}, PlotStyle -> {{Thickness[0.018], Blue},
  {Thickness[0.014], Red}, {Thickness[0.01], CadmiumOrange}},
  PlotRange -> All, AspectRatio -> \frac{1}{GoldenRatio},
  AxesLabel -> {"t", ""}];
```



The formulas for the plotted curves are:

| Chop[Expand[y1[t]]]

$$7.8976 E^{-2.13 t} - 0.702831 \cos[2 t] - 0.194771 \cos[4 t] + 0.748515 \sin[2 t] + 0.103716 \sin[4 t]$$

| Chop[Expand[y2[t]]]

$$0.897603 E^{-2.13 t} - 0.702831 \cos[2 t] - 0.194771 \cos[4 t] + 0.748515 \sin[2 t] + 0.103716 \sin[4 t]$$

| Chop[Expand[y3[t]]]

$$-5.1024 E^{-2.13 t} - 0.702831 \cos[2 t] - 0.194771 \cos[4 t] + 0.748515 \sin[2 t] + 0.103716 \sin[4 t]$$

All these solutions eventually merge onto a certain curve.

Give the formula for this curve and add its plot to the solution plot.

□G.4.b.ii)

Keep everything the same as in part i) above.

The solutions calculated above are all solutions of:

$$y'[t] + r y[t] == f[t]$$

$$2.13 y[t] + y'[t] == 3 \sin[2 t] + \sin[4 t]$$

The solution formulas are:

| Chop[Expand[y1[t]]]

$$7.8976 E^{-2.13 t} - 0.702831 \cos[2 t] - 0.194771 \cos[4 t] + 0.748515 \sin[2 t] + 0.103716 \sin[4 t]$$

| Chop[Expand[y2[t]]]

$$0.897603 E^{-2.13 t} - 0.702831 \cos[2 t] - 0.194771 \cos[4 t] + 0.748515 \sin[2 t] + 0.103716 \sin[4 t]$$

| Chop[Expand[y3[t]]]

$$-5.1024 E^{-2.13 t} - 0.702831 \cos[2 t] - 0.194771 \cos[4 t] + 0.748515 \sin[2 t] + 0.103716 \sin[4 t]$$

Where does the value of $r = 2.13$ show up vividly in these formulas?

How does the fact that $r > 0$ explain why the plots try to merge?

□G.4.c.i)

Here are three plots of solutions of the forced exponential diffeq

$$y'[t] - 0.6 y[t] = 0.2 (t - t^2)$$

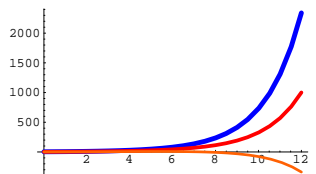
with starter values on $y[0]$ equal to 1, 2 and 3:

```
r = -0.6; Clear[f, t]
f[t_] = 0.2 (t - t^2);
endtime = 12;
Clear[y, y1, y2, y3, t];
starter1 = 3;
```

```
starter2 = 2;
starter3 = 1;

y1[t_] = E^{-r t} starter1 + E^{-r t} \int_0^t E^{r t} f[t] dt;
y2[t_] = E^{-r t} starter2 + E^{-r t} \int_0^t E^{r t} f[t] dt;
y3[t_] = E^{-r t} starter3 + E^{-r t} \int_0^t E^{r t} f[t] dt;

solutionplots = Plot[{y1[t], y2[t], y3[t]},
  {t, 0, endtime}, PlotStyle -> {{Thickness[0.018], Blue},
  {Thickness[0.014], Red}, {Thickness[0.01], CadmiumOrange}},
  PlotRange -> All, AspectRatio -> \frac{1}{GoldenRatio}, AxesLabel -> {"t", ""}];
```



This time the solution plots pull apart instead of merging.

The solution formulas are:

| Chop[Expand[y1[t]]]

$$1.2963 + 1.7037 E^{0.6 t} + 0.777778 t + 0.333333 t^2$$

| Chop[Expand[y2[t]]]

$$1.2963 + 0.703704 E^{0.6 t} + 0.777778 t + 0.333333 t^2$$

| Chop[Expand[y3[t]]]

$$1.2963 - 0.296296 E^{0.6 t} + 0.777778 t + 0.333333 t^2$$

Where does the value of $r = -0.6$ show up vividly in these formulas?

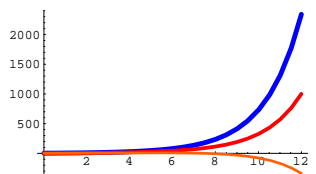
How does the fact that $r < 0$ explain why the solution plots go wildly separate ways as t gets large?

Remember that exponential growth eventually dominates power growth.

□G.4.c.ii)

Look at the three solution plots from part i) immediately above:

| Show[solutionplots];



Notice that some solutions eventually go up and others eventually go down.

The formula for the solution of

$$y'[t] - 0.6 y[t] = 0.2 (t - t^2) \quad \text{with } y[0] = \text{starter} \text{ is:}$$

Clear[starter, y]

$$y[t_] = \text{Expand}[E^{-r t} \text{starter} + E^{-r t} \int_0^t E^{r t} f[t] dt]$$

$$1.2963 - 1.2963 E^{0.6 t} + E^{0.6 t} \text{starter} + 0.777778 t + 0.333333 t^2$$

Remembering that exponential growth eventually dominates power growth, estimate a number d so that:

→ If $\text{starter} > d$, then the corresponding solution eventually goes up.

→ If $\text{starter} < d$, then the corresponding solution eventually goes down.

Show off your answer with a couple of nice plots.

G.5) Using the forced exponential diffeq to manage your money

G.5.a) Your red BMW convertible

You graduate and get one of those high-paying yuppie jobs everyone hopes to get. To announce your success, you decide to buy that new red BMW 325i convertible you've always had your heart set on. You go to the bank to talk about financing a \$25,000 loan for the BMW. Here are the details:

→ The interest on the unpaid part of the loan is to be compounded every instant.
 → The payoff is to be continuous in the sense that a number d (the yearly rate) is to be set and then after t (whole or fractional) years, you are to pay $\$t d$. Thus after the first day you will have paid $\$ \frac{d}{365}$, after the first month of the year you will have paid $\$ \frac{d}{12}$, and after two years, you will have paid $\$2 d$. The bank can transfer money from your account to their coffers every time they run their computers.

You can have the following options:

- 2-year loan at 4.9 % per year interest compounded every instant,
- 3-year loan at 6.9 % per year interest compounded every instant,
- 5-year loan at 10.9 % per year interest compounded every instant.

How much money $\$d$ will you have to pay each year under each of the options given?

What is your total payout in each case?

Click on the right for a tip.

For the 2-year loan at 4.9 % per year interest compounded every instant, the diffeq is

$$P'[t] = 0.049 P[t] - d \text{ with } P[0] = 25000$$

This is the same as

$$P'[t] - 0.049 P[t] = d \text{ with } P[0] = 25000.$$

Solve this with d cleared.

You want to set d so that $P[2] = 0$.

G.5.b) Planning ahead

You deposit \$10000 into an account paying 4.8 % interest over 12 years. You plan to make an annual additional deposit of $\$d$ at the end of every year. The same interest rate of 4.8 % is guaranteed. You want to set d so that the account balance is \$150000 in 12 years.

You decide to use the DiracDelta function; so you copy, paste and edit this from one of the Tutorials:

The diffeq is

$$\text{balance}'[t] - 0.048 \text{balance}[t] = \sum_{k=1}^{12} d \text{DiracDelta}[t - k]$$

with

$$\text{balance}[0] = 10000.$$

The formula for $\text{balance}[t]$ is:

```
Clear[balance, f, K, x, d, t]
endtime = 12;
starter = 10000;
r = -0.048;
```

$$f[t_] = \sum_{k=1}^{12} d \text{DiracDelta}[t - k];$$

$$\text{balance}[t_] = \text{Expand}[E^{-x t} \text{starter} + E^{-x t} \int_0^t E^{x t} f[t] dt]$$

$$10000 E^{0.048 t} + 0.562142 d E^{0.048 t} \text{UnitStep}[-12 + t] + 0.589783 d E^{0.048 t} \text{UnitStep}[-11 + t] + 0.618783 d E^{0.048 t} \text{UnitStep}[-10 + t] + 0.649209 d E^{0.048 t} \text{UnitStep}[-9 + t] + 0.681131 d E^{0.048 t} \text{UnitStep}[-8 + t] + 0.714623 d E^{0.048 t} \text{UnitStep}[-7 + t] + 0.749762 d E^{0.048 t} \text{UnitStep}[-6 + t] + 0.786628 d E^{0.048 t} \text{UnitStep}[-5 + t] + 0.825307 d E^{0.048 t} \text{UnitStep}[-4 + t] + 0.865888 d E^{0.048 t} \text{UnitStep}[-3 + t] + 0.908464 d E^{0.048 t} \text{UnitStep}[-2 + t] + 0.953134 d E^{0.048 t} \text{UnitStep}[-1 + t]$$

You look at:

```
balance[12]
17789.1 + 15.8409 d
```

Now you take over and set d so that $\text{balance}[12] = 150000$.

G.5.c.i) You inherit \$50,000

Your Aunt Matilda passes away and leaves you with the tidy sum of \$50,000. You find a bank that will pay you 5.15 % per year compounded every instant. You want to enjoy some of this money, but you don't want to blow it away too fast.

You decide to deposit the \$50,000 but to pull out d dollars per year so that after t years, you will have blown $d t$ dollars off.

If $P[t]$ is the amount in the account, then why does $P[t]$ solve the differential equation

$$P'[t] - 0.0515 P[t] = -d?$$

G.5.c.ii)

Continue with the set-up in part i) above. After some thought, you decide:

→ d should be as big as possible.

→ But the balance in the account should never dip below \$30,000 for 10 years.

Estimate your d .

Click on the right for a tip.

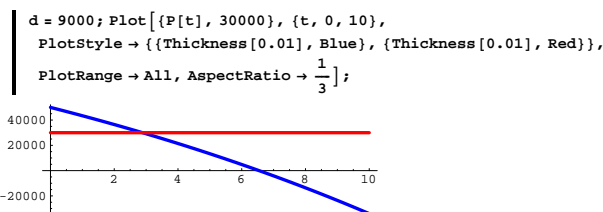
Tip:

You might want to look at:

```
Clear[P, f, K, x, d, t]
starter = 50000;
x = 0.0515;
r = -x;
f[t_] = -d;

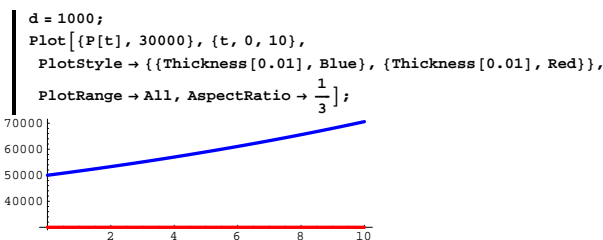
P[t_] = Expand[E^{-x t} starter + E^{-x t} \int_0^t E^{x t} f[t] dt]
19.4175 d + 50000 E^{0.0515 t} - 19.4175 d E^{0.0515 t}
```

For $d = 9000$, your balance looks this way:



$d = 9000$ is way too much.

Try $d = 1000$:



You have room to raise d .

The d you are after lies somewhere between 1000 and 9000. Estimate your d by continued trial and error.

G.6) Exponential models

Many of the problems here were influenced by problems in Ethan Bolker's book Using Algebra, Little, Brown, 1983.

G.6.a.i) Inflation

When the government announces an inflation rate of x % per year, this means that if the inflation rate never changes, then the price of an item that costs \$100 today is given by is the solution $P[t]$ of

$$P'[t] = x P[t], \text{ with } P[0] = 100$$

Interpretation:

The rate at which the price goes up is directly proportional to the

current
price.

Take the current year for $t = 0$ and assume that the Federal Reserve Board holds inflation at 3.5 % per year for the next 50 years. Give a plot of the projected price over the next 50 years of an item that costs \$100 today.

□G.6.a.ii)

Take the current year for $t = 0$ and assume that the Federal Reserve Board holds inflation at 5.5 % per year for the next 50 years. Give a plot of the projected price over the next 50 years of an item that costs \$100 today.

□G.6.a.iii)

Take the current year for $t = 0$ and assume that the Federal Reserve Board holds inflation at 7.5 % per year for the next 50 years. Give a plot of the projected price over the next 50 years of an item that costs \$100 today.

□G.6.a.iv)

Take the current year for $t = 0$ and assume that the Federal Reserve Board holds inflation at 9.5 % per year for the next 50 years. Give a plot of the projected price over the next 50 years of an item that costs \$100 today.

□G.6.a.v)

Discuss how you expect the Federal Reserve Board's policies on inflation to influence your future. Are a couple of extra points on the inflation rate anything to worry about?

What is it about the function

$$f[t] = E^{xt} \text{ (for } x > 0 \text{)}$$

that makes long-term high inflation rates really scary?

□G.6.b)

When a condenser discharges electricity, the instantaneous rate of change of the voltage is proportional to the voltage in the condenser. Suppose you have a discharging condenser and the instantaneous rate of change of the voltage is -0.01 of the voltage (in volts per second). How many seconds does it take for the voltage to decrease by 90 %?

□Tip:

As appealing as the answer of 90 seconds may be, it is not correct.

G.7) Examining the formula E^{-rt} starter + $E^{-rt} \int_0^t E^{rt} f[t] dt$ and helping out a clueless EE student

When you are given

$$y'[t] + r y[t] = f[t] \text{ with } y[0] = \text{starter},$$

you copy paste and edit

$$y[t] = E^{-rt} \text{ starter} + E^{-rt} \int_0^t E^{rt} f[t] dt.$$

Enter it:

```

Clear[y, t, starter, r];
y[t_] = E^-r t starter + E^-r t Integrate[E^r t f[t], t]
E^-r t (1000/r - 1000 E^r t/r) + E^-r t starter

```

Check whether $y'[t] + r y[t] = f[t]$:

```

y'[t] + r y[t]
-1000 - E^-r t (1000/r - 1000 E^r t/r) r - E^-r t r starter +
r (E^-r t (1000/r - 1000 E^r t/r) + E^-r t starter)

```

Multiply it out:

```

Expand[y'[t] + r y[t]]
-1000

```

Yes.

$$y[t] = E^{-rt} \text{ starter} + E^{-rt} \int_0^t E^{rt} f[t] dt$$

solves $y'[t] + r y[t] = f[t]$.

$$\begin{aligned} \text{And } y[0] &= E^{-r \cdot 0} \text{ starter} + E^{-0t} \int_0^0 E^{rt} f[t] dt \\ &= \text{starter} + \int_0^0 E^{rt} f[t] dt \\ &= \text{starter} + 0 = \text{starter} \end{aligned}$$

So this $y[t]$ also satisfies $y[0] = \text{starter}$:

Now go back to

$$y[t] = E^{-rt} \text{ starter} + E^{-rt} \int_0^t E^{rt} f[t] dt$$

Put:

$$\text{yunforced}[t] = E^{-rt} \text{ starter}$$

and

$$\text{yzeroinput}[t] = E^{-rt} \int_0^t E^{rt} f[t] dt$$

so that

$$y[t] = \text{yunforced}[t] + \text{yzeroinput}[t].$$

Explain these facts:

□G.7.a.i)

$\text{yunforced}[t]$ solves $y'[t] + r y[t] = 0$, with $y[0] = \text{starter}$.

□G.7.a.ii)

$\text{yzeroinput}[t]$ solves $y'[t] + r y[t] = f[t]$, with $y[0] = 0$.

□G.7.a.iii)

→ If $r > 0$, then $\text{yunforced}[t] \rightarrow 0$ as t gets large.

□G.7.a.iv)

→ If $r > 0$, then, as t gets large, the plot of

$$y[t] = \text{yunforced}[t] + \text{yzeroinput}[t]$$

merges with the plot of $\text{yzeroinput}[t]$.

□G.7.b.i) Helping out a clueless EE student

You don't have to understand the electrical jargon to be able to do this problem.

You are fortunate enough to have your own computer set-up right in your dorm room. One night, Brian, the EE student who lives across

the hall comes in and says, "Part of my EE 207 Circuits homework for tomorrow is to analyze some circuits and I don't have a clue about where to start."

At first, you think, "I'm scared because I'm a life science major and I don't know diddly squat about electrical circuits." But you don't let on. Instead, you say, "Let me see that EE 207 textbook." It turns out to be thicker and heavier than you had expected. In the text book you see that in a certain driven circuit, the voltage $v[t]$ satisfies

$$c v'[t] + \frac{v[t]}{r} = f[t], \text{ with } v[0] \text{ given.}$$

where $f[t]$ is called the driving or forcing function.

You see also some words about capacitors, switches and nodal equations which you don't understand.

But you barrel ahead, converting this diffeq to

$$v'[t] + \frac{v[t]}{rc} = \frac{f[t]}{c} \text{ with } v[0] = \text{starter}$$

and you type:

```

Clear[v, r, c, f, starter];
v[t_] = E^-t/c starter + E^-t/c Integrate[E^t/c f[t], t]
E^-t/c starter + E^-t/c Integrate[E^t/c f[t], t]

```

Why did you do this?

□G.7.b.ii)

In the book, you see that

→ $E^{-t/rc}$ starter is called the natural (unforced) response of the driven circuit

and

→ $E^{-(t/rc)} \int_0^t E^{t/(rc)} \frac{f[t]}{c} dt$ is called the forced response of the driven circuit

Brian asks you what this means.

What do you say?

□G.7.b.iii)

Next Brian gets out his homework assignment and asks you to solve

$$c v'[t] + \frac{v[t]}{r} = f[t] \text{ with}$$

with $c = 0.5$, $r = 0.2$, $f[t] = 1.2$ and $v[0] = 0$.

Do it for him.

And give him a nice plot.

□G.7.b.iv)

Next Brian gets out his homework assignment again and asks you to solve

$$c v'[t] + \frac{v[t]}{r} = f[t] \text{ with}$$

with $c = 0.5$, $r = 0.2$, $f[t] = 1.2 \cos[t]$ and $v[0] = 2$.

Do it for him.

And give him a nice plot.

□G.7.b.v)

And then Brian gets out his homework assignment again and asks you to solve

$$c v'[t] + \frac{v[t]}{r} = f[t] \text{ with}$$

with $c = 0.5$, $r = 0.2$, $f[t] = 3 \text{ DiracDelta}[t - 2]$ and $v[0] = 1$.

Do it for him.

Explain why it is guaranteed that $v[t] \rightarrow 0$ as t gets large.

□G.8.a.ii)

If $y_{\text{zeroinput}}[t]$ solves

$$y'[t] + 7y[t] = f[t], \text{ with } y[0] = 0,$$

and $y_{\text{unforced}}[t]$ solves

$$y'[t] + 7y[t] = 0, \text{ with } y[0] = 12,$$

then what differential equation does

$$y[t] = y_{\text{zeroinput}}[t] + y_{\text{unforced}}[t]$$

solve?

Make sure you give the starting data.

□G.8.a.iii)

If $y_1[t]$ solves

$$y'[t] + 7y[t] = \sin[t], \text{ with } y[0] = 0,$$

and $y_2[t]$ solves

$$y'[t] + 7y[t] = -\cos[t], \text{ with } y[0] = 12,$$

then what differential equation does

$$y_3[t] = y_1[t] + y_2[t]$$

solve?

Make sure you give the starting data.

□G.8.a.iv)

That dimwit Calculus Cal told Careful Carrie that if $y_1[t]$ solves

$$y'[t] + 8y[t] = \text{DiracDelta}[t - 1], \text{ with } y[0] = 3,$$

and if $y_2[t]$ solves

$$y'[t] + 8y[t] = \text{DiracDelta}[t - 1], \text{ with } y[0] = 7,$$

then

$$y_1[t] + y_2[t]$$

solves

$$y'[t] + 8y[t] = \text{DiracDelta}[t - 1], \text{ with } y[0] = 10.$$

What did Carrie say to Cal?

G.8) Add'em up: The superposition principle

The superposition principle says:

If $y_1[t]$ solves

$$y'[t] + r y[t] = f[t] \text{ with } y[0] = p$$

and if $y_2[t]$ solves

$$y'[t] + r y[t] = g[t] \text{ with } y[0] = q$$

then

$$y_3[t] = y_1[t] + y_2[t]$$

solves

$$y'[t] + r y[t] = f[t] + g[t] \text{ with } y[0] = p + q$$

Click on the right for an explanation

$$y_3'[t] + r y_3[t]$$

$$= (y_1'[t] + y_2'[t]) + r(y_1[t] + y_2[t])$$

$$= (y_1'[t] + r y_1[t]) + (y_2'[t] + r y_2[t]) = f[t] + g[t]$$

and

$$y_3[0] = y_1[0] + y_2[0] = p + q.$$

None of these problems require formulas or other extensive calculation.

□G.8.a.i) Add them up

Use the superposition principle to set a number p that if $y_1[t]$ solves

$$y'[t] + 8y[t] = f[t], \text{ with } y[0] = 1,$$

and $y_2[t]$ solves

$$y'[t] + 8y[t] = 0, \text{ with } y[0] = p,$$

then $y[t] = y_1[t] + y_2[t]$

solves

$$y'[t] + 8y[t] = f[t], \text{ with } y[0] = 3.$$

□G.8.a.v)

Given:

→ $y_1[t]$ solves

$$y'[t] + 9.7y[t] = E^{-0.4t} \cos[t], \text{ with } y[0] = 0,$$

and

→ $y_2[t]$ solves

$$y'[t] + 9.7y[t] = 0, \text{ with } y[0] = 1.$$

What numbers p and q do you pick to make

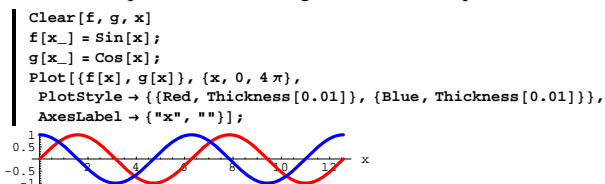
$$y[t] = p y_1[t] + q y_2[t]$$

solve

$$y'[t] + 9.7y[t] = 3 E^{-0.4t} \cos[t], \text{ with } y[0] = 3?$$

G.9) Formulas and diffeqs**□G.9.a.i)**

Here are a couple of functions together with their plots.



The red one is the sine function, and the blue one is the cosine function. Would you say these functions are the same?

There are values of x where the function values are the same. Does that mean that the functions are the same?

□G.9.a.ii)

Given two functions $f[t]$ and $g[t]$:
What's the difference between these two statements?

Statement 1):

$$f[t] = g[t].$$

Statement 2):

There are values of t for which $f[t] = g[t]$

What could you do to check the first statement?
What would you do about the second statement?

□G.9.a.iii)

What do you think it means to say, "Two given functions are the same"?

What do you think it means to say, "Two given functions are reasonable approximations of each other?"

□G.9.b.i)

Here's the exponential diffeq:

Lots of folks say that this is the
BOSS differential equation ... Numero Uno ... The heart of it all.

```
Clear[y, r, t, diffeq];
diffeq = y'[t] - r y[t] == 0
-r y[t] + y'[t] == 0
```

Here's a formula for a function $y[t]$:

```
y[t_] = Sin[r t]
Sin[r t]
```

For this $y[t]$, the right hand side of the differential equation

$$y'[t] = r y[t]$$

is:

```
right = r y[t]
r Sin[r t]
```

The left hand side is:

```
left = y'[t]
```

```
r Cos[r t]
```

You make the call.

Is

$$y[t] = \text{Sin}[r t]$$

a solution of the differential equation?

□G.9.b.ii)

Here's the exponential diffeq:

```
Clear[y, r, t, diffeq];
diffeq = y'[t] == r y[t]
y'[t] == r y[t]
```

Here's a formula for new function $y[t]$:

```
y[t_] = Exp[-r t]
Exp[-r t]
```

For this $y[t]$, the right hand side of the differential equation

$$y'[t] = r y[t]$$

is:

```
right = r y[t]
Exp[-r t] r
```

The left hand side is:

```
left = y'[t]
-Exp[-r t] r
```

You make the call.

Is

$$y[t] = \text{Exp}[-r t]$$

a solution of the differential equation?

□G.9.b.iii)

Here's the exponential diffeq:

```
Clear[y, r, t, k, diffeq];
diffeq = y'[t] == r y[t]
y'[t] == r y[t]
```

Here's a formula for new function $y[t]$:

```
y[t_] = k Exp[r t]
Exp[r t] k
```

For this $y[t]$, the right hand side of the differential equation

$$y'[t] = r y[t]$$

is:

```
right = r y[t]
Exp[r t] k r
```

The left hand side is:

```
left = y'[t]
Exp[r t] k r
```

You make the call.

Is

$$y[t] = k \text{Exp}[r t]$$

a solution of the differential equation?

□G.9.c)

Here's a string of differential equations and a string of functions to go with them. In each case, decide whether or not the given function is a solution of the given differential equation.

□G.9.c.i)

```
Clear[f, x, diffeq]
diffeq = f'[x] == x + 2 Sin[x]
f'[x] == x + 2 Sin[x]
f[x_] = x^2/2 - Cos[x]
x^2/2 - Cos[x]
f'[x]
x + Sin[x]
```

□G.9.c.ii)

```
Clear[f, x, diffeq]
diffeq = f'[x] == x + Sin[x]
f'[x] == x + Sin[x]
f[x_] = x^2/2 - Cos[x]
x^2/2 - Cos[x]
f'[x]
```

```
x + Sin[x]
```

□G.9.c.iii)

```
Clear[f, x, diffeq]
diffeq = f'[x] == x
f'[x] == x
f[x_] = x^2/2 + 10
10 + x^2/2
f'[x]
x
```

□G.9.c.iv)

```
Clear[f, x, diffeq]
diffeq = f'[x] == x f[x]
f'[x] == x f[x]
f[x_] = pi Exp[x^2/2]
Exp[x^2/2] pi
f'[x]
Exp[x^2/2] pi x
```

□G.9.c.v)

```
Clear[x, t, diffeq]
diffeq = x''[t] + 9 x[t] == 0
9 x[t] + x''[t] == 0
x[t_] = Sin[3 t]
Sin[3 t]
x''[t]
-9 Sin[3 t]
```

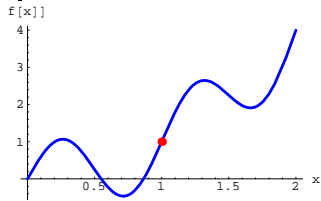
G.10 Ups and downs for functions

DiffEq's always have something to do with derivatives. This problem is a chance for you to refresh yourself on the true meaning of derivatives.

□G.10.a.i)

Here's the plot of a function $f[x]$, and a point on the plot at $\{1, f[1]\}$.


```
Clear[f, x]
f[x_] = Sin[2 π x] + x2;
Plot[f[x], {x, 0, 2}, AspectRatio →  $\frac{1}{\text{GoldenRatio}}$ ,
PlotStyle → {{Thickness[0.01], Blue}}, AxesLabel → {"x", "f[x]"},
Epilog → {PointSize[0.03], Red, Point[{1, f[1]}}];
```



Just by looking at the plot:

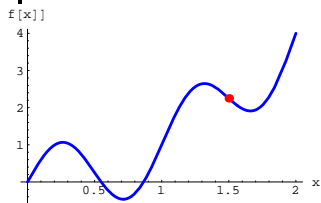
Do you say that the curve is going up or going down at the point {1, f[1]}?

Do you say $f'[1]$ is positive or negative at the point $x = 1$?

□G.10.a.ii)

Here's the plot of the same function shown with a new point at {1.5, f[1.5]}.

```
Plot[f[x], {x, 0, 2}, AspectRatio →  $\frac{1}{\text{GoldenRatio}}$ ,
PlotStyle → {{Thickness[0.01], Blue}}, AxesLabel → {"x", "f[x]"},
Epilog → {PointSize[0.03], Red, Point[{1.5, f[1.5]}}];
```



Just by looking at the plot:

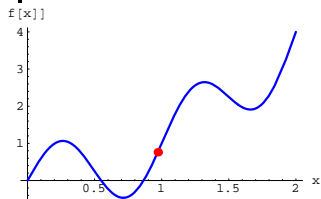
Do you say that the curve is going up or going down at the point {1.5, f[1.5]}?

Do you say that $f'[1.5]$ is positive or negative?

□G.10.a.iii)

Here's the plot of the same function f[x], and a point on the graph at a randomly chosen point {u, f[u]}.

```
u = Random[Real, {0, 2}];
Plot[f[x], {x, 0, 2}, AspectRatio →  $\frac{1}{\text{GoldenRatio}}$ ,
PlotStyle → {{Thickness[0.008], Blue}}, AxesLabel → {"x", "f[x]"},
Epilog → {PointSize[0.03], Red, Point[{u, f[u]}}];
```



Just by looking at the plot:

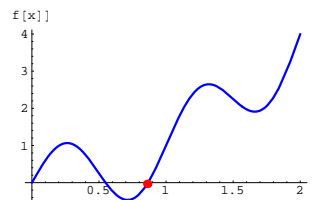
Do you say that the curve is going up or going down at the plotted point {u, f[u]}?

Do you say that $f'[u]$ is positive, negative or zero?

□G.10.a.iv)

Here's the plot of the same function f[x], and a point on the graph at a new randomly chosen point {u, f[u]}.

```
u = Random[Real, {0, 2}];
Plot[f[x], {x, 0, 2}, AspectRatio →  $\frac{1}{\text{GoldenRatio}}$ ,
PlotStyle → {{Thickness[0.008], Blue}}, AxesLabel → {"x", "f[x]"},
Epilog → {PointSize[0.03], Red, Point[{u, f[u]}}];
```



Just by looking at the plot:

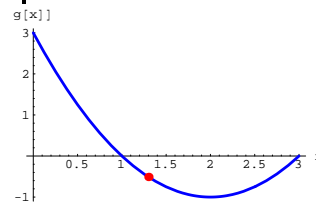
Do you say that the curve is going up or going down at the plotted point {u, f[u]}?

Do you say that $f'[u]$ is positive, negative or zero?

□G.10.b.i)

Here's the plot of a new function g[x] together with the point at {1.3, g[1.3]}.

```
Clear[g, x]
g[x_] = (x - 1) (x - 3);
Plot[g[x], {x, 0, 3},
PlotStyle → {{Thickness[0.01], Blue}}, AspectRatio →  $\frac{1}{\text{GoldenRatio}}$ ,
PlotRange → All, AxesLabel → {"x", "g[x]"},
Epilog → {PointSize[0.03], Red, Point[{1.3, g[1.3]}}];
```



Given that f[x] is another function with

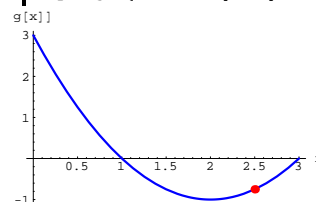
$$f'[x] = g[x],$$

do you say that the plot of f[x] is going up or going down at the point {1.3, f[1.3]}?

□G.10.b.ii)

Here's the plot of the same function g[x] together with a point at {2.5, g[2.5]}:

```
Plot[g[x], {x, 0, 3},
PlotStyle → {{Thickness[0.008], Blue}}, AspectRatio →  $\frac{1}{\text{GoldenRatio}}$ ,
PlotRange → All, AxesLabel → {"x", "g[x]"},
Epilog → {PointSize[0.03], Red, Point[{2.5, g[2.5]}}];
```



Given that f[x] is another function with

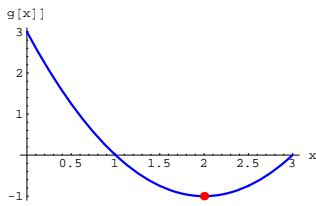
$$f'[x] = g[x],$$

do you say that the plot of f[x] is going up or going down at the point {2.5, f[2.5]}?

□G.10.b.iii)

Here's the plot of the same function g[x] together with the point at {2.0, g[2.0]}:

```
Plot[g[x], {x, 0, 3},
PlotStyle → {{Thickness[0.008], Blue}}, AspectRatio →  $\frac{1}{\text{GoldenRatio}}$ ,
PlotRange → All, AxesLabel → {"x", "g[x]"},
Epilog → {PointSize[0.03], Red, Point[{2.0, g[2.0]}}];
```



Given that $f[x]$ is another function with
 $f'[x] = g[x]$,
 do you say that the plot of $f[x]$ is going up or going down at the point
 $\{2.0, f[2.0]\}$?

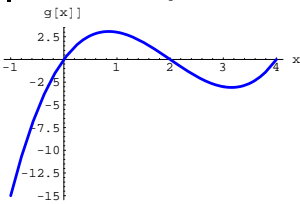
□G.10.c)

Make

$$g[x] = x(x - 2)(x - 4)$$

and plot:

```
Clear[g, x]
g[x_] = x (x - 2) (x - 4);
Plot[g[x], {x, -1, 4}, PlotStyle -> {{Thickness[0.01], Blue}},
  AspectRatio ->  $\frac{1}{\text{GoldenRatio}}$ , AxesLabel -> {"x", "g[x]"},
  PlotRange -> All];
```



List the intervals inside

$$-1 \leq x \leq 4$$

on which $g[x] > 0$.

List the intervals inside

$$-1 \leq x \leq 4$$

on which $g[x] < 0$.

Given that $f[x]$ is another function with

$$f'[x] = g[x],$$

which way is the plot of $f[x]$ going on each of the intervals you listed?

G.11) Diffeqs and tangent vectors*

□G.11.a.i)

Here's the exponential diffeq:

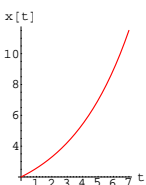
```
Clear[x, t, r];
diffeq = x'[t] == r x[t]
x'[t] == r x[t]
```

Here's a formula of the solution $x[t]$ of this diffeq with $r = 0.25$ and $x[0] = 2$.

```
r = 0.25;
starter = 2;
Clear[x, t]
x[t_] = starter Er t
2 E0.25 t
```

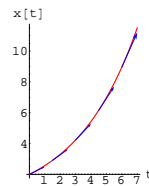
And a plot of the solution:

```
solutionplot = Plot[x[t], {t, 0, 7}, AxesLabel -> {"t", "x[t]"},
  PlotStyle -> {{Thickness[0.01], Red}}];
```



Here are some pointers with their tails tacked at some points on the curve.

```
pointers = Table[Arrow[{1, r x[t]},
  Tail -> {t, x[t]}, VectorColor -> Blue], {t, 0, 6, 1.5}];
(plot = Show[solutionplot, pointers];)
```

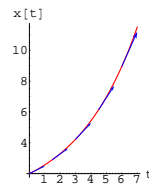


What do you say the relationship between the solution curve and those vectors is?

□G.11.a.ii)

Take another look the last plot again:

```
Show[plot];
```



The plotted vectors are

$$\{1, 0.25 x[t]\}$$

with tails at $\{t, x[t]\}$ for a selection of t 's.

The slope of each plotted vector is

$$0.25 x[t].$$

The plotted curve is a solution of

$$x'[t] = 0.25 x[t].$$

Use these two facts to try to explain why the plot came out the way it did.

□G.11.b.i)

Here's the exponential diffeq:

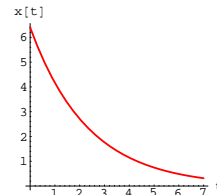
```
Clear[x, t, r];
diffeq = x'[t] == r x[t]
x'[t] == r x[t]
```

Here's a formula of the solution $x[t]$ of this diffeq with $r = -0.43$ and $x[0] = 6.4$.

```
r = -0.43;
starter = 6.4;
x[t_] = starter Er t
6.4 E-0.43 t
```

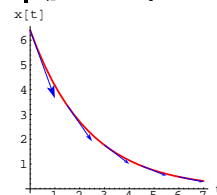
And a plot of the solution:

```
solutionplot = Plot[x[t], {t, 0, 7}, AxesLabel -> {"t", "x[t]"},
  PlotStyle -> {{Thickness[0.01], Red}}];
```



Here are some pointers with their tails tacked at some points on the curve.

```
pointers = Table[Arrow[{1, r x[t]},
  Tail -> {t, x[t]}, VectorColor -> Blue], {t, 0, 6, 1.5}];
(plot = Show[solutionplot, pointers];)
```

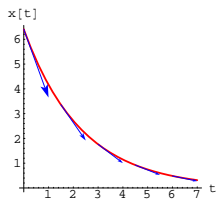


What do you say the relationship between the solution curve and those vectors is?

□G.11.b.ii)

Take another look at the plot again:

```
Show[plot];
```



The plotted vectors are

$$\{1, -0.43 x[t]\}$$

with tails at $\{t, x[t]\}$ for a selection of t 's.

The slope of each plotted vector is

$$-0.43 x[t].$$

The plotted curve is a solution of

$$x'[t] = 0.43 x[t].$$

Use these two facts to try to explain why the plot came out the way it did.

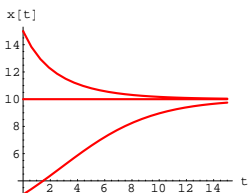
□G.11.c.i)

Here's a logistic differential equation:

```
Clear[x, t];
diffeq = x'[t] == 0.3 x[t] (1 - x[t]/10)
x'[t] == 0.3 (1 - x[t]/10) x[t]
```

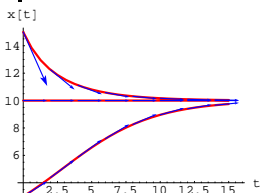
Here are three solution curves for this equation.

```
Clear[x1, x2, x3];
x1[t_] = x[t] /. NDSolve[{diffeq, x[0] == 3}, x[t], {t, 0, 15}][[1]];
x2[t_] = x[t] /. NDSolve[{diffeq, x[0] == 10}, x[t], {t, 0, 15}][[1]];
x3[t_] = x[t] /. NDSolve[{diffeq, x[0] == 15}, x[t], {t, 0, 15}][[1]];
solutionplot = Plot[{x1[t], x2[t], x3[t]}, {t, 0, 15},
  AxesLabel -> {"t", "x[t]"}, PlotStyle -> {{Thickness[0.01], Red},
  {Thickness[0.01], Red}, {Thickness[0.01], Red}},
  PlotRange -> All];
```



Here are some pointers with tails tacked to the solution curves.

```
scale = 1.75;
pointers1 = Table[Arrow[{1, 0.3 x1[t] (1 - x1[t]/10)}, Tail -> {t, x1[t]},
  VectorColor -> Blue, ScaleFactor -> scale], {t, 0, 14, 2}];
pointers2 = Table[Arrow[{1, 0.3 x2[t] (1 - x2[t]/10)}, Tail -> {t, x2[t]},
  VectorColor -> Blue, ScaleFactor -> scale], {t, 0, 14, 2}];
pointers3 = Table[Arrow[{1, 0.3 x3[t] (1 - x3[t]/10)}, Tail -> {t, x3[t]},
  VectorColor -> Blue, ScaleFactor -> scale], {t, 0, 14, 2}];
plot = Show[solutionplot, pointers1, pointers2, pointers3];
```

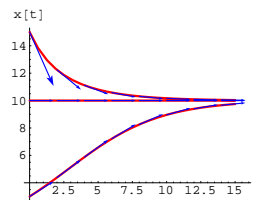


What do you say is the relationship between the solution curves and those vectors?

□G.11.c.ii)

Take another look the plot again:

```
Show[plot];
```



The plotted vectors are

$$1.75 \{1, 0.3 x[t] (1 - \frac{x[t]}{10})\}$$

with tails at $\{t, x[t]\}$ for a selection of t 's.

The slope of each plotted vector is

$$0.3 x[t] (1 - \frac{x[t]}{10}).$$

The plotted curves are solutions of

$$x'[t] = 0.3 x[t] (1 - \frac{x[t]}{10})$$

Use these two facts to try to explain why the plot came out the way it did.

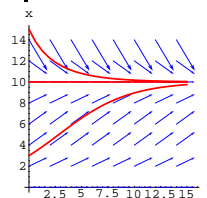
□G.11.c.iii)

Here is the plot of the three solutions of

$$x'[t] == 0.3 x[t] (1 - \frac{x[t]}{10})$$

from part i) above shown with a selection of new pointers:

```
scale = 1.75;
flow = Table[Arrow[scale {1, 0.3 x (1 - x/10)},
  Tail -> {t, x}, VectorColor -> Blue], {x, 0, 14, 2}, {t, 0, 14, 2}];
setup = Show[flow, solutionplot, Axes -> True, AxesLabel -> {"t", "x"}];
```



What do these pointers suggest about solutions of

$$x'[t] = 0.3 x[t] (1 - \frac{x[t]}{10})$$

with $0 < x[0] < 10$?

What do these pointers suggest about solutions of

$$x'[t] = 0.3 x[t] (1 - \frac{x[t]}{10})$$

with $x[0] = 10$?

What do these pointers suggest about solutions of

$$x'[t] = 0.3 x[t] (1 - \frac{x[t]}{10})$$

with $x[0] > 10$?

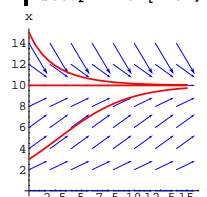
□G.11.c.iv)

Here is the plot of the three solutions of

$$x'[t] = 0.3 x[t] (1 - \frac{x[t]}{10})$$

from part i) above shown with the pointers from part iii):

```
flow = Table[Arrow[scale {1, 0.3 x (1 - x/10)},
  Tail -> {t, x}, VectorColor -> Blue], {x, 0, 14, 2}, {t, 0, 14, 2}];
setup = Show[flow, solutionplot, Axes -> True, AxesLabel -> {"t", "x"}];
```



What do these pointers suggest about the behavior of the solution of

$$x'[t] = 0.3 x[t] (1 - \frac{x[t]}{10})$$

whose plot passes through $\{t, x\} = \{5, 4\}$?

What do these pointers suggest about the behavior of the solution of

$$x'[t] = 0.3 x[t] (1 - \frac{x[t]}{10})$$

whose plot passes through $\{t, x\} = \{5, 12\}$?