Differential Equations&*Mathematica*

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DE.03 Laplace Transform and Fourier Analysis Literacy Sheet

What you need to know when you're away from the machine.

□L.1)

Look at these calculations of

$\int_0^\infty E^{-s\,t}\, f[t]\, dt$

and the Laplace transform of f[t] for three sample f[t]'s:

```
Clear[f, s, t]
f[t_] = Sin[t];
{∫_0^∞ E^(-s t) f[t] dt, LaplaceTransform[f[t], t, s]}
```

$\{\text{If}[\text{Re}[s] > 0, \frac{1}{1+s^2}, \int_0^\infty E^{-s\,t}\,\text{Sin}[t]\,dt], \frac{1}{1+s^2}\}$

```
Clear[f, s, t, k]
f[t_] = t;
{∫_0^∞ E^(-s t) f[t] dt, LaplaceTransform[f[t], t, s]}
```

$\{\text{If}[\text{Re}[s] > 0, \frac{1}{s^2}, \int_0^\infty E^{-s\,t}\,t\,dt], \frac{1}{s^2}\}$

```
Clear[f, s, t, k]
f[t_] = DiracDelta[t - 5.4];
{∫_0^∞ E^(-s t) f[t] dt, LaplaceTransform[f[t], t, s]}
```

$\{E^{-5.4\,s}, E^{-5.4\,s}\}$

Explain why any Laplace literate person knows in advance that the calculations were guaranteed to come out the way they did.

□L.2)

Here is Mathematica's calculation of the the Laplace transform of
f[t] = 2.2 DiracDelta[t – 3.5] + 3.8 DiracDelta[t – 5.6]:

```
Clear[f, t, s]
f[t_] = 2.2 DiracDelta[t - 3.5] + 3.8 DiracDelta[t - 5.9];
LaplaceTransform[f[t], t, s]
```

$3.8\, E^{-5.9\,s} + 2.2\, E^{-3.5\,s}$

Use the definition

LaplaceTransform[f[t], t, s] = $\int_0^\infty E^{-s\,t}\, f[t]\, dt$

to say why you could have written this down without help from Mathematica.

Without machine help, write down the formula for the Laplace transform of

f[t] = DiracDelta[t – 1] + 2 DiracDelta[t – 2] + 3 DiracDelta[t – 3]

□L.3)

When you go with a function of t such as

f[t] = 3.8 $E^{-0.4\,t}$ Cos[4.7 t],

and calculate its Laplace transform F[s], you get:

```
Clear[F, f, t, s]
f[t_] = 3.8 E^(-0.4 t) Cos[4.7 t];
F[s_] = LaplaceTransform[f[t], t, s]
```

$\frac{3.8\,(0.4+s)}{22.09+(0.4+s)^2}$

You began with a function f[t] of t and learned that its Laplace transform F[s] is a function of s.
Is this a problem?

□L.4)

Here is a little table of Laplace transforms:

```
Clear[f, F, s, t, a, b];
{f[t_] = E^(a t), F[s_] = LaplaceTransform[f[t], t, s]}
Clear[f, F, s, t, a, b];
{f[t_] = Cos[a t], F[s_] = LaplaceTransform[f[t], t, s]}
Clear[f, F, s, t, a, b];
{f[t_] = Sin[a t], F[s_] = LaplaceTransform[f[t], t, s]}
Clear[f, F, s, t, a, b];
{f[t_] = E^(a t) Cos[b t],
 F[s_] = Together[ExpandAll[LaplaceTransform[f[t], t, s]]]}
Clear[f, F, s, t, a, b];
{f[t_] = E^(a t) Sin[b t],
 F[s_] = Together[ExpandAll[LaplaceTransform[f[t], t, s]]]}
```

$\{E^{a\,t}, \frac{1}{-a+s}\}$

$\{\text{Cos}[a\,t], \frac{s}{a^2+s^2}\}$

$\{\text{Sin}[a\,t], \frac{a}{a^2+s^2}\}$

$\{E^{a\,t}\,\text{Cos}[b\,t], \frac{-a+s}{a^2+b^2-2\,a\,s+s^2}\}$

$\{E^{a\,t}\,\text{Sin}[b\,t], \frac{b}{a^2+b^2-2\,a\,s+s^2}\}$

Use this table to come up with the function f[t] whose Laplace transform is:

i) F[s] = $\frac{3\,s}{s^2+4} + \frac{5}{s^2+9}$,

ii) F[s] = $\frac{s-3}{13-6\,s+s^2}$,

iii) F[s] = $\frac{s-5}{13-6\,s+s^2}$,

iv) F[s] = $\frac{3}{s^2+5\,s+6} = \frac{3}{(s+2)\,(s+3)}$.

For iv), you might want to look at:

```
Clear[f, g, h, t, s, x, y]
h[t_] = ∫_0^t f[t - x] g[x] dx;
LaplaceTransform[h[t], t, s]
```

LaplaceTransform[f[t], t, s] LaplaceTransform[g[t], t, s]

□L.4)

How do you express the Laplace Transform of y′[t] in terms of the Laplace transform of y[t]?
How do you express the Laplace Transform of y′′[t] in terms of the Laplace transform of y[t]?

□L.5)

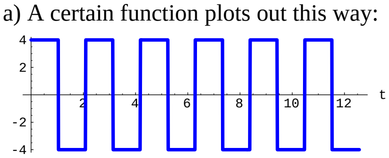
Given

$y''[t] + 4\,y[t] = 0.5$

with y[0] = 2 and y′[0] = –1,
Write down a formula for the Laplace transform of the solution y[t].
Use the table above to write down a formula for y[t].

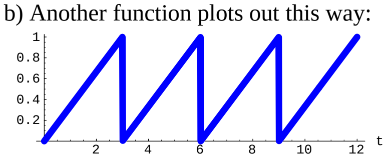
□L.6)

a) A certain function plots out this way:



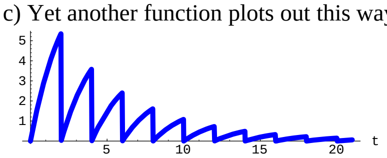
Does this plot make you want to say that the plotted function is periodic? If you say that this function is periodic, then what's your best estimate of the period of the function?

b) Another function plots out this way:



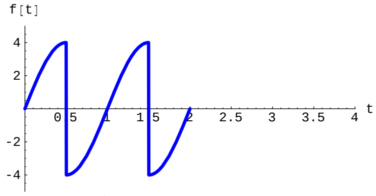
Does this plot make you want to say that the plotted function is periodic? If you say that this function is periodic, then what's your best estimate of the period of the function?

c) Yet another function plots out this way:



Does this plot make you want to say that the plotted function is periodic? If you say that this function is periodic, then what's your best estimate of the period of the function?

d) Here's the plot of a certain function on [0, 2]:

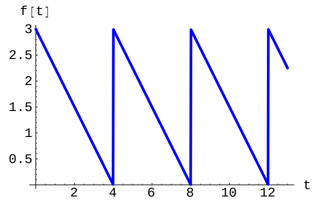


Given that f[t] is periodic with period L = 2, pencil in your sketch of how the plot of f[t] looks on [0, 4].

□L.7)

a) Look at this plot of

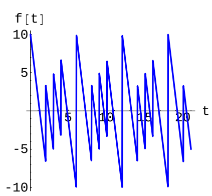
$f[t] = 3\,(\text{Ceiling}[\frac{t}{4}] - \frac{t}{4})$:



This function is periodic.
Identify the period L of this function.

b) Look at this plot of

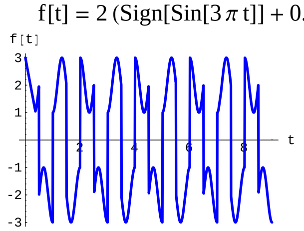
$f[t] = 10\,(\text{Ceiling}[\frac{t}{3}] + \text{Floor}[\frac{t}{2}] - \frac{5t}{6})$:



This function is periodic.
Identify the period L of this function.

c) Look at this plot of

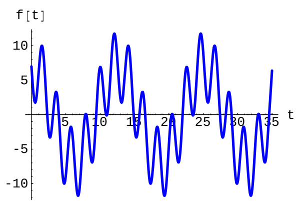
$f[t] = 2\,(\text{Sign}[\text{Sin}[3\pi t]] + 0.5\,\text{Cos}[2\pi t])$:



This function is periodic.
Identify the period L of this function.

d) Look at this plot of

$f[t] = 7\,\text{Cos}[t/2] - 5\,\text{Sin}[3t]$:

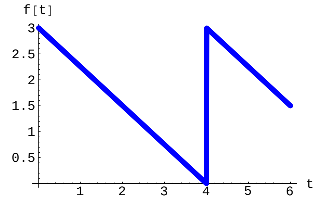


This function is periodic.
Identify the period L of this function.

□L.8)

That nitwit Calculus Cal starts to plot this periodic function:

```
Clear[f, t]
f[t_] = 3 (Ceiling[t/4] - t/4);
Plot[f[t], {t, 0, 6}, PlotStyle → {{Thickness[0.02], Blue}},
 AxesLabel → {"t", "f[t]"}, AspectRatio → 1/GoldenRatio];
```



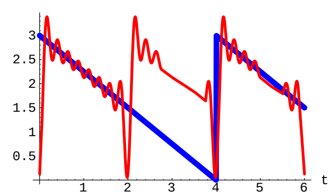
Intent on getting a decent fast Fourier fit with as little thought as possible, Cal copies and pastes some code:

```
L = 2;
n = 9;
Clear[t, realfastfit]
realfastfit[t_] = Chop[ComplexExpand[FastFourierfit[f, L, n, t]]]
```

```
2.125 - 0.25 Cos[π t] - 0.25 Cos[2 π t] - 0.25 Cos[3 π t] - 0.25 Cos[4 π t] -
 0.25 Cos[5 π t] - 0.25 Cos[6 π t] - 0.25 Cos[7 π t] - 0.25 Cos[8 π t] +
 0.472607 Sin[π t] + 0.228956 Sin[2 π t] + 0.144338 Sin[3 π t] +
 0.0993128 Sin[4 π t] + 0.069925 Sin[5 π t] + 0.0481125 Sin[6 π t] +
 0.0303309 Sin[7 π t] + 0.0146939 Sin[8 π t]
```

Smiling in satisfaction, Cal plots f[t] and his fast Fourier fit:

```
Plot[{f[t], realfastfit[t]}, {t, 0, 6},
  PlotStyle -> {{Thickness[0.02], Blue}, {Thickness[0.01], Red}},
  AxesLabel -> {"t", ""},
  AspectRatio -> 1/GoldenRatio];
```

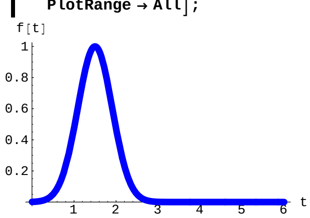


Cal is dumbfounded (as always).
Step in and tell Cal what he did wrong and then tell him how to fix it.

□L.9)

Here is a function f[t] plotted on [0, 3]:

```
Clear[f, t];
f[t_] = E^(-3 (t-1.5)^2);
fplot = Plot[f[t], {t, 0, 6}, PlotStyle → {{Thickness[0.025], Blue}},
  AspectRatio → 1/GoldenRatio, AxesLabel → {"t", "f[t]"},
  PlotRange → All];
```



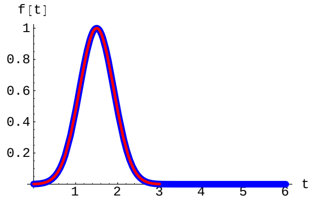
Here's a Fourier fit of f[t] on [0, 3]:

```
L = 3;
n = 7;
Clear[approxf];
approxf[t_] = Chop[ComplexExpand[FastFourierfit[f, L, n, t]]]
```

$$0.341002 - 0.473545\,\text{Cos}[\frac{2\pi t}{3}] + 0.157901\,\text{Cos}[\frac{4\pi t}{3}] - 0.0255914\,\text{Cos}[2\pi t] + 0.001813\,\text{Cos}[\frac{8\pi t}{3}] - 0.0002135\,\text{Cos}[\frac{10\pi t}{3}] - 0.000130251\,\text{Cos}[4\pi t]$$

Now see the plot of approxf[t] on [0, L] along with the plot of f[t] on [0, 6]:

```
approxfplot = Plot[approxf[t], {t, 0, 3},
  PlotStyle -> {{Thickness[0.01], Red}}, DisplayFunction -> Identity];
 Show[fplot, approxfplot];
```

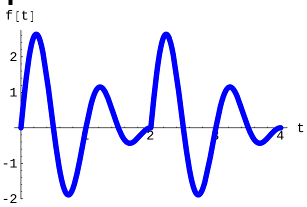


Your job is to pencil in the plot of approxf[t] on [3, 6].

□L.10)

a) Here is a function f[t]:

```
Clear[f, t]
f[t_] = 3 Sin[2 π t] (Ceiling[t/2] - t/2);
fplot = Plot[f[t], {t, 0, 4},
  PlotStyle → {{Thickness[0.02], Blue}}, AspectRatio → 1/GoldenRatio,
  AxesLabel → {"t", "f[t]"}, PlotRange → All];
```

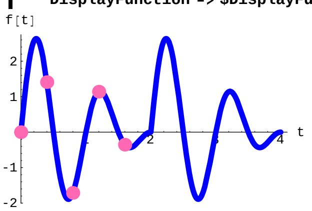


This function is periodic with period L = 2.
Pick 5 data points off the first period:

```
L = 2;
n = 3;
fdata = Table[N[{t, f[t]}], {t, 0, L - L / (2 n - 1), L / (2 n - 1)}];
```

```
fdataplot = ListPlot[fdata, PlotStyle -> {HotPink, PointSize[0.05]},
DisplayFunction -> Identity];

both = Show[fplot, fdataplot,
  DisplayFunction -> $DisplayFunction];
```



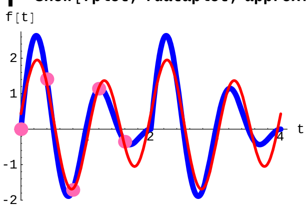
Stay with L = 2 and n = 3 and look at:

```
Clear[approxf]
approxf[t_] = Chop[ComplexExpand[FastFourierfit[f, L, n, t]]]
```

0.144338 + 0.433013 Cos[π t] - 0.144338 Cos[2 π t] + 1.5 Sin[2 π t]

Now see the plot of approxf[t] along with the other plots:

```
approxfplot = Plot[approxf[t], {t, 0, 4},
  PlotStyle -> {{Thickness[0.01], Red}}, DisplayFunction -> Identity];
 Show[fplot, fdataplot, approxfplot];
```

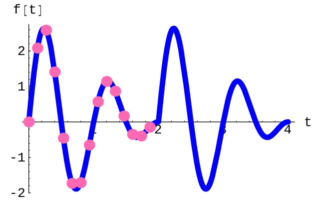


Use this plot to explain how the fast Fourier fit works.

b) Here's what happens when you raise n:

```
L = 2;
n = 8;
fdata = Table[N[{t, f[t]}], {t, 0, L - L / (2 n - 1), L / (2 n - 1)}];
fdataplot = ListPlot[fdata, PlotStyle -> {HotPink, PointSize[0.04]},
DisplayFunction -> Identity];

both = Show[fplot, fdataplot,
  DisplayFunction -> $DisplayFunction];
```



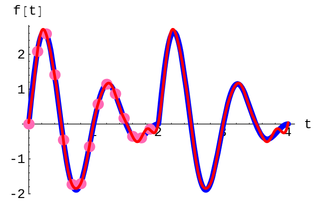
Here is the corresponding Fourier fit:

```
Clear[approxf]
approxf[t_] = Chop[ComplexExpand[FastFourierfit[f, L, n, t]]]
```

0.226333 + 0.61162 Cos[π t] + 0.09375 Cos[2 π t] - 0.408671 Cos[3 π t] - 0.1875 Cos[4 π t] - 0.121659 Cos[5 π t] - 0.09375 Cos[6 π t] - 0.0812898 Cos[7 π t] + 1.5 Sin[2 π t]

Now see the plot of approxf[t] along with the other plots:

```
approxfplot = Plot[approxf[t], {t, 0, 4},
  PlotStyle -> {{Thickness[0.01], Red}}, DisplayFunction -> Identity];
 Show[fplot, fdataplot, approxfplot];
```



Use this plot as a basis to explain this:
If you use a lot of data points, then you can expect an excellent Fourier fit.

□L.11)

You are given a periodic forcing function f[t] and an forced oscillator diffeq

$$y''[t] + b\, y'[t] + c\, y[t] = f[t]$$

with y[0] = p and y′[0] = q given. But try as you might, you cannot get Mathematica to calculate either the Laplace transform of f[t] or the convolution integral involving f[t].

Instead, you get a good Fourier approximation of f[t] and use the Laplace transform (or convolution integral method) to get an exact formula for the solution of

$$y''[t] + b\, y'[t] + c\, y[t] = \text{good Fourier approximation of } f[t]$$

with y[0] = p and y′[0] = q.

Comment on this statement:
Although the solution you got is not the exact solution to the original problem, it is the exact solution to a problem virtually indistinguishable from the original problem.

□L.12)

What are the main ideas behind the technique of combining Fourier fit and the Laplace transform to come up with good approximate formulas for periodically forced oscillators?

□L.13)

Here is a forcing function and a pretty good Fourier fit:

```
Clear[f, t]
f[t_] = 4 Sign[Cos[1.6 t]];
```

You can get a period of f[t] by solving

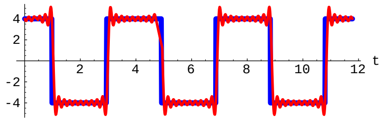
```
Solve[1.6 t == 2 π, t]
```

{{t → 3.92699}}

And a Fourier Fit:

```
L = 3.92699;
n = 20;
realfit[t_] = Chop[N[ComplexExpand[FastFourierfit[f, L, n, t]]]]
```

5.08248 Cos[1.6 t] - 1.66612 Cos[4.8 t] + 0.965685 Cos[8. t] -
0.652741 Cos[11.2 t] + 0.46834 Cos[14.4 t] - 0.341632 Cos[17.6 t] +
0.24512 Cos[20.8 t] - 0.165685 Cos[24. t] + 0.0960315 Cos[27.2 t] -
0.0314807 Cos[30.4 t] + 0.4 Sin[1.6 t] - 0.4 Sin[4.8 t] +
0.4 Sin[8. t] - 0.4 Sin[11.2 t] + 0.4 Sin[14.4 t] - 0.4 Sin[17.6 t] +
0.4 Sin[20.8 t] - 0.4 Sin[24. t] + 0.4 Sin[27.2 t] - 0.4 Sin[30.4 t]



Go with this forcing function and use what you see to report on which of the following undamped oscillators you expect to be in danger of going into resonance of near resonance (beating):

y′′[t] + 24.6 y′[t] = f[t],
y′′[t] + 2.5 y[t] = f[t],
y′′[t] + 64.0 y[t] = f[t],
y′′[t] + 123 y[t] = f[t],

and

y′′[t] + 5 y[t] = f[t].