Differential Equations&*Mathematica*

Produced by Bruce Carpenter Published by Math Everywhere, Inc.

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DE.05 First Order Differential Equations Literacy Sheet

□L.1)

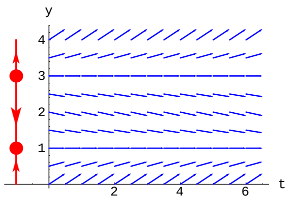
You are given this diffeq to analyze:

```
y′[t] == 0.2 (-3 + y[t]) (-1 + y[t])
y[0] == starter
```

Then you look at:

```
f[t, y]
0.2 (-3 + y) (-1 + y)
```

And you say to yourself, "One single phase line is enough."



Look again at the formula for f[t, y]:

```
f[t, y]
0.2 (-3 + y) (-1 + y)
```

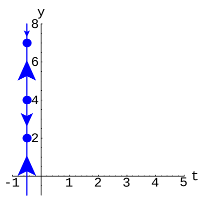
What is it about the formula for f[t, y] that made you know you could get by with just one phase line?
What information does the phase line give?

□L.2)

Here's an autonomous diffeq:

```
y′[t] == (2 - y[t]) (-7 + y[t]) (-4 + y[t])
y[0] == starter
```

And the phase line for it:



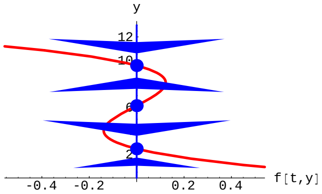
The phase line indicates four different types of solutions. Pencil in a sample of each.
For what starting values do you expect constant solutions?
Which dot on the phase line signals the place where you expect extreme sensitivity to errors in starter data on y[0]?

□L.3)

Here's a new autonomous diffeq containing some partially random coefficients:

```
y′[t] == 1.11611 (1 - 0.403063 y[t]) (1 - 0.162568 y[t]) (1 - 0.10456 y[t])
y[0] == starter
```

Look at this plot:



Points on the curve are of the form:

```
{f[t, y], y}
{1.11611 (1 - 0.403063 y) (1 - 0.162568 y) (1 - 0.10456 y), y}
```

But the real issues here are:
→ How are the arrowheads on the phase line related to the plot of {f[t, y], y}?
→ How are the dots on the phase line related to the plot of {f[t, y], y}?
→ Remembering that y′ = f[t, y], explain why it had to turn out this way.

□L.4)

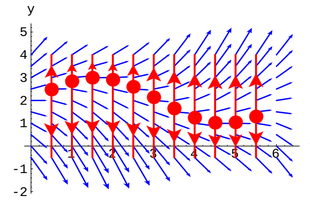
You are given this diffeq to analyze:

```
y′[t] == -2 - Sin[t] + y[t]
y[0] == starter
```

The first thing you do is look at:

```
f[t, y]
-2 + y - Sin[t]
```

And you say to yourself: "One single phase line is not enough. If I want to use phase lines, I'll want several."



Look again at the formula for f[t, y]:

```
f[t, y]
-2 + y - Sin[t]
```

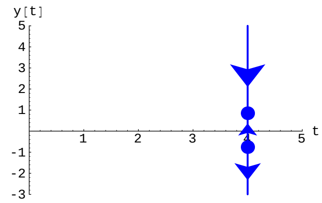
What is it about the formula for f[t, y] that made you want to use more than one phase line?
Is this diffeq autonomous?
Use the formula for f[t, y] to write down a formula for the curve on which all those dots all lie.

□L.5)

Here's a new diffeq:

```
y′[t] == Cos[t] (-0.85 + y[t]) (-Sin[t] + y[t])
y[0] == starter
```

Look at this plot of the phase line that corresponds to t = 4.0:



Imagine that plots of solutions y[t] of this diffeq come in from the left and pass through the plotted phase line.
Some of them are going up as they cross over this line. Others are going down.
Now look at this calculation:

```
f[4.0, y]
-0.653644 (-0.85 + y) (0.756802 + y)
```

And give values for a and b so that
If

$y[4.0] > b$,

then the plot of y[t] is going down as it crosses this line.
If

$a < y[4.0] < b$,

then the plot of y[t] is going up as it crosses this line.
If

$y[4.0] < a$,

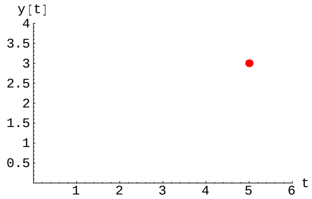
then the plot of y[t] is going down as it crosses this line.

□L.6)

You are given this diffeq to analyze:

```
y′[t] == Sin[t] (-t + y[t])
y[0] == starter
```

Look at this plot which plots the point {t, y} = {5.0, 3.0}:



Imagine that the plot of a solution y[t] of this diffeq comes in from the left and passes through the plotted point.

Look at this quick calculation:

```
f[5.0, 3.0]
1.91785
```

Use this quick calculation to determine whether:

a) The plot of y[t] comes in from the left and down to this point. Or:

b) The plot of y[t] comes in from the left and up to this point.

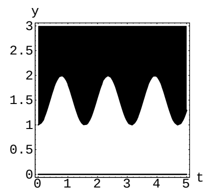
□L.7)

Here's a non-autonomous diffeq:

$$y'[t] == y[t]\left(1 - \frac{y[t]}{1+\mathrm{Sin}[2t]^2}\right)$$

$$y[0] == \text{starter}$$

Look at this plot



The black region signals {t,y}'s for which y'= f[t,y]<0.
The white region signals {t,y}'s for which y'= f[t,y]>0.

Is this a bifurcation plot?

Imagine that some solution curves are advancing from left to right through the plotted region.

What do these solution curves do when they are in the white region?

What do these solution curves do when they are in the black region?

Where do you expect the tops of the crests and the bottoms of the dips of some of the solution curves to be?

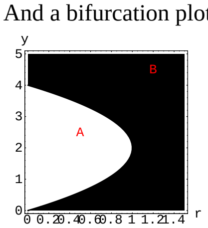
□L.8)

Here's single diffeq containing a parameter r:

$$y'[t] == -r + \left(1 - \frac{y[t]}{4}\right) y[t]$$

$$y[0] == \text{starter}$$

And a bifurcation plot:



The black region signals {r,y}'s for which y'= f[t,y]<0.
The white region signals {r,y}'s for which y'= f[t,y]>0.

Use this plot to estimate what happens to solutions y[t] of the diffeq that:

i) correspond to 0 < y[0] < 5 and r > 1.

ii) correspond to 1 < y[0] < 3 and r = 0.4.

iii) correspond to 4 < y[0] < 5 and r = 0.4.

Estimate reasonable values for

r, a and b

so that in the long term solutions get close to 3

provided a < y[0] < b.

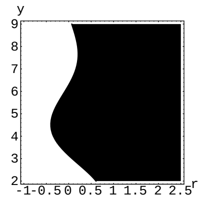
□L.9)

Here's a squirrelly first order diffeq containing a parameter r:

$$y'[t] == -r + E^{-\frac{y[t]}{5}}\,\mathrm{Sin}[y[t]]$$

$$y[0] == \text{starter}$$

And a bifuraction plot:



The black region signals {r,y}'s for which y'= f[t,y]<0.
The white region signals {r,y}'s for which y'= f[t,y]>0.

Two bifurcation points are visible in this plot.

Mark them with your pencil.

Shade in combinations {r, y[0]} where you expect to find extreme sensitivity to slight errors in starter data.

□L.10)

Here are five diffeqs:

$$y1'[t] == -\mathrm{Sin}[t] + 3.8\,(-2 + y[t])\,(-1 + y[t])$$
$$y1[0] == \text{starter}$$
$$y2'[t] == 3.8\,E^{-0.9\,y[t]}$$
$$y2[0] == \text{starter}$$
$$y3'[t] == -r + 3.8\,(-2 + y[t])\,(-1 + y[t])$$
$$y3[0] == \text{starter}$$
$$y4'[t] == r - t + 3.8\,\mathrm{Sin}[y[t]]$$
$$y4[0] == \text{starter}$$
$$y5'[t] == -\mathrm{Sin}[y[t]] + 3.8\,(-2 + y[t])\,(-1 + y[t])$$
$$y5[0] == \text{starter}$$

Which of these diffeqs are autonomous?

Which of the autonomous diffeqs contains an extra parameter?

Which of the non-autonomous diffeqs contains an extra parameter?

□L.11)

A body falling in a vacuum is subject only to the force of acceleration. This means that the velocity v[t] of a body falling in a vacuum solves the simple diffeq

v′[t] = 9.8 (meters per second per second).

When air resistance is incorporated, it is often assumed that the air resistance is proportional to the velocity itself. This leads to the model:

$$v'[t] == 9.8 - r\,v[t]$$
$$v[0] == 0$$

Come up with the r that makes the terminal velocity equal to 19.6 meters per second.

(You are not asked to solve the diffeq.)

□L.12)

Here's a little diffeq:

$$y'[t] == 1 + \frac{y[t]}{5}$$
$$y[0] == 0.5$$

You separate the variables and integrate to get a formula for the solution:

$$\{\{y[t] \to -5. + 5.5\; 2.71828^{0.2\,t}\}\}$$

Analyze the formula and determine whether the solution escapes to ∞ in finite time or in infinite time.

□L.13)

Here's a little diffeq:

$$y'[t] == 1 + y[t]^2$$
$$y[0] == 0$$

You separate the variables and integrate to get a formula for the solution:

$$\{\{y[t] \to \mathrm{Tan}[t]\}\}$$

Analyze the formula and determine whether the solution escapes to ∞ in finite time or in infinite time.

□L.14)

Here's the exponential diffeq:

```
y′[t] == 0.3 y[t]
y[0] == 2
```

You know in advance that the formula for the solution is

$y[t] = 2\,E^{0.3\,t}$.

Reproduce this formula through the technique of separating and integrating.

□L.15)

Here's a linear diffeq:

```
y′[t] == E^-t - 2 y[t]
y[0] == 0
```

This is the same as

$y'[t] + 2\,y[t] = E^{-t}$ with $y[0] = 2$.

A formula for the solution is:

$\{\{y[t] \to E^{-2\,t}\,(-1 + E^{t})\}\}$

Reproduce this formula by hitting with an integrating factor and then integrating out.

□L.16)

Here is the diffeq of the leaking bucket:

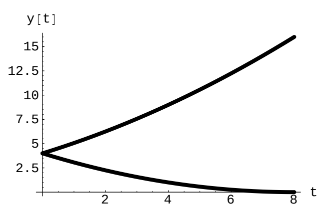
$y'[t] == -0.5\,\sqrt{y[t]}$
$y[0] == 4$

You ask *Mathematica* for the formulas for the solution and you get:

```
N[DSolve[bucketdiffeq, y[t], t], 4]
```

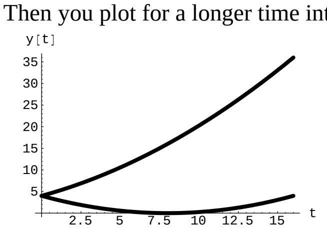
$\{\{y[t] \to 0.0625\,(64. - 16.\,t + t^{2.})\},\ \{y[t] \to 0.0625\,(64. + 16.\,t + t^{2.})\}\}$

You plot both results:



One of the plots is correct; one is bogus. Which plot is correct?

Then you plot for a longer time interval:



Is either plot entirely correct? Why?
Pencil in the correct plot.
How does the danger zone begin to account for this madness?