Differential Equations&*Mathematica*

Produced by Bruce Carpenter   Published by Math Everywhere, Inc.

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DE.08 Linearizations
Literacy Sheet

What you need to know when you're away from the machine.

□L.1)

Here is a system:

$x'[t] = -x[t]\,y[t] + 1$
$y'[t] = x[t]\,y[t] - y[t]$

a) Explain how you know that this system is non-linear.

b) Explain how you know that {x, y} = {1, 1} is an equilibrium point of this system.

c) What information do you seek when you linearize at this equillibrim point?

d) Linearize this system at {1, 1} and write down the resulting coefficient matrix.

e) The eigenvalues of a cleared matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

are given by

$$\frac{1}{2}\left(\text{Trace}[A] + \sqrt{\text{Trace}[A]^2 - 4\,\text{Det}[A]}\right)$$

and

$$\frac{1}{2}\left(\text{Trace}[A] - \sqrt{\text{Trace}[A]^2 - 4\,\text{Det}[A]}\right)$$

where

$\text{Trace}[A] = a + d$

and

$\text{Det}[A] = a\,d - b\,c.$

Use these formulas and a cheap calculator and apply them to the matrix you came up with in part d) above to determine whether the equillibrium point {1, 1} is an attractor, a repeller (or neither) of the given system.

□L.2)

a) What do you mean when you say that {a, b} is an equilibrium point of a given system

$x'[t] = m[x[t], y[t]]$
$y'[t] = n[x[t], y[t]]$?

b) Given that {a, b} is an equilibrium point of a given system

$x'[t] = m[x[t], y[t]]$
$y'[t] = n[x[t], y[t]]$,

come up with exact formulas for the solutions x[t] and y[t] with starter data x[0] = a and y[0] = b.

□L.3)

What good is the fact that near the equilibrium point of a given system, the linearization approximates the given system very, very well?

□L.4)

a) Here is a scaled plot of the flow of a certain system

$x'[t] = m[x[t], y[t]]$
$y'[t] = n[x[t], y[t]]$

centered on an equilibrium point:

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a x[t] + b y[t] == -4.3 Sin[x[t]] + 0.17 y[t]
c x[t] + d y[t] == 1.2 - 0.5 x[t] - 1.29 y[t]
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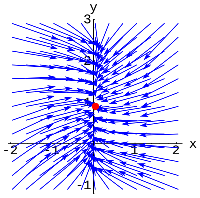
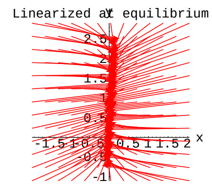
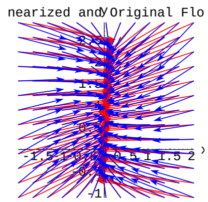


Is the equilibrium point a repeller or an attractor?

b) Here is the flow of the linearization of the system in part a):



Here are both flows together:



- Graphics -

How does this plot signal that the original system was not linear?

c) Take another look at the plot of both flows. What principle about a system and its linearization at an equilibrium point does this plot display?

d) You are given starter data and you go to the linearization to come up with approximate formulas for the solutions of the original nonlinear system.
If you are given starter data {x[0], y[0]} = {2, 2.5}, would you trust the formula you get from the linearization to be a
trustworthy approximations of the real thing? Why or why not?

If you are given starter data {x[0], y[0]} = {0.5, 2.5}, would you trust the formula you get from the linearization to be a
trustworthy approximations of the real thing? Why or why not?

□L.5)

The damped pendulum oscillator is modeled by the second order differential equation

$$y''[t] + \frac{b\,y'[t]}{L\,M} + \frac{g\,\text{Sin}[y[t]]}{L} = 0.$$

where

g is gravity, L is the length of the pendulum, M is the mass of the bob, and b is friction. Convert this system to a system of two first order differential equations, linearize it, and then convert the linearization back into a single second order differential equation.

□L.6)

Use your experience to help write a few words about when the pendulum oscillator is well approximated by its linearization and when it is not.

□L.7)

a) The coefficient matrix of the linearization of a certain system at an equilibrium point has eigenvalues

$0.3 + 1.3\,I$ and $0.3 - 1.3\,I$.

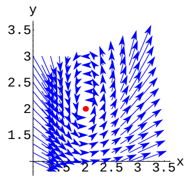
Is this enough to tell you that the equilibrium point is definitely a repeller?

b) The coefficient matrix of the linearization of a certain system at an equilibrium point has eigenvalues

$-0.3 + 1.3\,I$ and $-0.3 - 1.3\,I$.
Is this enough to tell you that the equilibrium point is definitely a attractor?

c) The coefficient matrix of the linearization of a certain system at an equilibrium point has eigenvalues
$0.4\,I$ and $-0.4\,I$.
Is this enough to tell you that trajectories of this system oscillate around the equilibrium point on closed curves?

d) Here is the flow of a nonlinear system in the vicinity of an equilibrium point:



This makes it clear that the equilibrium point is a repeller. But the eigenvalues of the linearization at the equilibrium point are
$0.4\,I$ and $-0.4\,I$.
So the linearization predicts neither a repeller nor an attractor. Does this make you want to change your answers to any of the previous parts? Why or why not?

□L.8)

What are Lyapunov's rules?

□L.9)

You are dealing with a gradient system
$x'[t] = m[x[t], y[t]]$
$y'[t] = n[x[t], y[t]]$
This means that there is a function f[x, y] with
$m[x, y] = \partial_x f[x, y]$
and
$n[x, y] = \partial_y f[x, y]$;
i.e.
$\{m[x, y], n[x, y]\} = \nabla f[x, y]$ (gradient).
Remember that the $\nabla f[x, y]$points in the direction of greatest initial increase of f[x, y] as {x, y} leaves {a, b}.

a) When you linearize a gradient system at an equilibrium point {xx, yy} and find that the eigenvalues of the resulting matrix are both negative, which do you expect:
i) f[xx, yy] > f[x, y] for {x, y} nearby {xx, yy}; i.e. the equilibrium point is a local maximizer of f[x, y]:
ii) f[xx, yy] < f[x, y] for {x, y} nearby {xx, yy}; i.e. the equilibrium point is a local minimizer f[x, y];
iii) f[xx, yy] < f[x, y] for some {x, y} nearby {xx, yy} and f[xx, yy] > f[x, y] for some other {x, y} nearby {xx, yy}; i.e. the equilibrium point is a saddle point of f[x, y].

b) When you linearize a gradient system at an equilibrium point {xx, yy} and find that the eigenvalues of the resulting matrix are both positive, which do you expect:
i) f[xx, yy] > f[x, y] for {x, y} nearby {xx, yy}; i.e. the equilibrium point is a local maximizer of f[x, y]:
ii) f[xx, yy] < f[x, y] for {x, y} nearby {xx, yy}; i.e. the equilibrium point is a local minimizer f[x, y];
iii) f[xx, yy] < f[x, y] for some {x, y} nearby {xx, yy} and f[xx, yy] > f[x, y] for some other {x, y} nearby {xx, yy}; i.e. the equilibrium point is a saddle point of f[x, y].

c) When you linearize a gradient system at an equilibrium point {xx, yy} and find that one of the eigenvalues of the resulting matrix is positive and the other eigenvalue is negative, which do you expect:
i) f[xx, yy] > f[x, y] for {x, y} nearby {xx, yy}; i.e. the equilibrium point is a local maximizer of f[x, y]:
ii) f[xx, yy] < f[x, y] for {x, y} nearby {xx, yy}; i.e. the equilibrium point is a local minimizer f[x, y];
iii) f[xx, yy] < f[x, y] for some {x, y} nearby {xx, yy} and f[xx, yy] > f[x, y] for some other {x, y} nearby {xx, yy}; i.e. the equilibrium point is a saddle point of f[x, y].

□L.10)

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Most math books give you formulas to plug and chug with,
but math books that math profs call "cookbooks" are those
that don't do much of a job explaining where the formulas come from.
Many differential equations books fall into the "cook book" category.
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You have a friend who was looking at a math cook book and thought that the cook book said that to check the nature of an equilibrium point {p, q} of a given system
$x'[t] = m[x[t], y[t]]$
$y'[t] = n[x[t], y[t]]$
you look at the eigenvalues of the matrix

$$\begin{pmatrix} \partial_x m[x, y] & \partial_y m[x, y] \\ \partial_x n[x, y] & \partial_y n[x, y] \end{pmatrix} /. \{x \to p,\ y \to q\}$$

Was your friend right?