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New bounds for the Descartes method

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Abstract

We give a new bound for the number of recursive subdivisions in the Descartes method for polynomial real root isolation. Our proof uses Ostrowski's theory of normal power series from 1950 which has so far been overlooked in the literature. We combine Ostrowski's results with a theorem of Davenport from 1985 to obtain our bound. We also characterize normality of cubic polynomials by explicit conditions on their roots and derive a generalization of one of Ostrowski's theorems.

Keywords: Polynomial real root isolation; Descartes rule of signs; Modified Uspensky method; Recursion tree analysis; Normal polynomials; Root separation bounds; History of mathematics; Möbius transformations; Coefficient sign variations; Cylindrical algebraic decomposition

1. Introduction

Polynomial real root isolation is the task of computing disjoint intervals, each containing a single root, for all the real roots of a given univariate polynomial with real coefficients. Vincent (1836) showed that polynomial real root isolation can be performed using a test based on the Descartes Rule of Signs. The test evaluates a condition that implies that a given interval contains a single root, and another condition that implies that the interval does not contain any roots. If neither condition is satisfied, the interval is bisected and each subinterval is tested recursively. It is not obvious that Vincent's method terminates.

Collins and Akritas (1976) proposed a method with a much better worst-case computing time than Vincent's method. We will refer to the improved method as "Descartes method". A study

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by Johnson (1998) shows that the Descartes method typically outperforms Sturm's method and other methods for real root isolation. Johnson's findings are confirmed in experiments by Rouillier and Zimmermann (2001, Figures 2,3). Recent versions of the Descartes method use floating point arithmetic (Johnson and Krandick, 1997; Collins et al., 2002; Rouillier and Zimmermann, 2004), parallel computation (Decker and Krandick, 1999, 2001), or they minimize space requirements (Rouillier and Zimmermann, 2004). Lane and Riesenfeld (1981) describe a variation of the method that uses Bernstein bases.

We give a new bound (Theorem 28) for the number of recursive subdivisions in the Descartes method. The bound also applies when Bernstein bases are used. Our proof uses Ostrowski's theory (Ostrowski, 1950) of normal power series which has so far been overlooked in the literature. We combine Ostrowski's results with a theorem of Davenport (1985) to obtain our bound. We also characterize normality of cubic polynomials by explicit conditions on their roots and derive a generalization (Theorem 34) of one of Ostrowski's theorems.

The history of termination proofs starts with Vincent (1836). Alesina and Galuzzi (1998) present Vincent's original proof in modern mathematical language and provide extensive historical information on related earlier and later results. It seems that Vincent's method was forgotten until Uspensky (1948) modified Vincent's proof and bounded the number of recursive steps required by the method. Ostrowski (1950) used a result from his earlier work (Ostrowski, 1939) to improve Uspensky's bound. Ostrowski's contribution, though summarized in *Mathematical Reviews* (Marden, 1951), was completely overlooked in later literature until it became accessible through an electronic database (Alesina and Galuzzi, 1999). When Collins and Akritas (1976) improved Vincent's algorithm they based their analysis, later elaborated by Collins and Loos (1982), on Uspensky's work. Collins and Johnson (1989) improved the analysis significantly, but also their result is strictly weaker than Ostrowski's. Eventually, one of Ostrowski's theorems, the present Theorem 17, was independently rediscovered by Alesina and Galuzzi (1998, Corollary 8.2). The authors gave a concise and direct proof, but their approach cannot be used to prove the stronger Theorem 34 of this paper.

In Section 2 we review the Descartes method. In Section 3 we present Ostrowski's theory of normal power series and strengthen one of his results that links normality of polynomials and termination of the Descartes method (Theorem 11). We also present Ostrowski's sufficient condition on the roots of a polynomial to guarantee normality (Theorem 16). We use these results in Section 4 to prove Theorem 23 on the proximity of complex roots to those intervals on which the Descartes method recurs. In Section 5 we combine Theorem 23 with Davenport's root separation theorem to obtain new bounds for the recursion tree of the Descartes method. In Section 6 we use Theorem 11 to characterize the normal cubic polynomials by explicit conditions on their roots. We gauge the extent of the improvement by applying the Descartes method to 2.3 billion cubic polynomials. We use the new result to prove Theorem 34—thus strengthening Theorem 17.

2. Review of the Descartes method

Definition 1. Let $a = (a_0, \ldots, a_n)$ be a finite sequence of real numbers. The *number of sign variations* in a, $\mathrm{var}(a)$, is the number of pairs (i, j) with $0 \le i < j \le n$ and $a_i a_j < 0$ and $a_{i+1} = \cdots = a_{j-1} = 0$. Let A be the polynomial $a_0 + a_1 x + \cdots + a_n x^n$. The *number of coefficient sign variations* in A, $\mathrm{var}(A)$, is $\mathrm{var}(a)$.

Theorem 2 (*Descartes Rule of Signs*). *For any non-zero real polynomial the number of coefficient sign variations exceeds the number of positive real roots—counting multiplicities—by a non-negative, even integer.*

Proof. Let $A(x)$ be a non-zero real polynomial. If x^k is the highest power of x that divides A, the polynomial A/x^k has the same number of coefficient sign variations and positive real roots as A, and its constant term is non-zero. Hence, we may assume that the constant term of A is non-zero. Let a_0 be this constant term, let n be the degree of A, and let a_n be the leading coefficient. Let $v = \mathrm{var}(A)$, and let p be the number of positive real roots of A, counting multiplicities.

To show that v and p have the same parity we use an argument given by Conkwright (1941). Let $z_1, \ldots, z_n \in \mathbb{C}$ be the roots of A. Then

$$A(x) = a_n(x - z_1) \cdots (x - z_n), \tag{1}$$

and hence $a_0 = A(0) = (-1)^n a_n z_1 \ldots z_n$. Since the non-real roots occur in complex conjugate pairs, their product is positive. The product of the positive roots is likewise positive, no root is zero since a_0 is non-zero, and the product of the negative real roots has the sign $(-1)^{n-p}$. It follows that the sign of a_0/a_n is $(-1)^p$. Hence v and p have the same parity.

Gauss (1828) proves $v \geq p$ by showing that, for any non-zero real polynomial $B(x)$ and any positive real number a,

$$\mathrm{var}(B) < \mathrm{var}((x - a) \cdot B). \tag{2}$$

So, in Eq. (1), every positive root of A contributes at least one sign variation.

To show inequality (2) let $B = b_m x^m + \cdots + b_0$, let $a > 0$, and let $C = (x - a)B = c_{m+1}x^{m+1} + \cdots + c_0$. If $\mathrm{var}(B) > 0$ let (i, j) be an index pair that contributes to $\mathrm{var}(B)$. Then $0 \leq i < j \leq m$ and $b_i b_j < 0$ and either $j = i + 1$ or $b_{i+1} = 0$. If $\sigma : \mathbb{R} \longrightarrow \{-1, 0, 1\}$ denotes the sign function then

$$\sigma(c_{i+1}) = \sigma(b_i - a b_{i+1}) = \sigma(b_i).$$

So, if $(i_1, j_1), \ldots, (i_k, j_k)$ are all the index pairs that contribute to $\mathrm{var}(B)$, and if $0 \leq i_1 < j_1 \leq \cdots \leq i_k < j_k \leq m$, then

$$\mathrm{var}(c_{i_1+1}, \ldots, c_{i_k+1}, c_{m+1}) = \mathrm{var}(b_{i_1}, \ldots, b_{i_k}, b_m) = \mathrm{var}(B).$$

Now let i be the smallest index for which $b_i \neq 0$. Then $0 \leq i \leq i_1$ and $\sigma(c_i) = \sigma(-a b_i) = -\sigma(b_i) = -\sigma(b_{i_1}) = -\sigma(c_{i_1+1})$, and so

$$\mathrm{var}(C) \geq \mathrm{var}(c_i, c_{i_1+1}, \ldots, c_{i_k+1}, c_{m+1}) = 1 + \mathrm{var}(B).$$

If $\mathrm{var}(B) = 0$ then $\mathrm{var}(C) \geq \mathrm{var}(c_i, c_{m+1}) = \mathrm{var}(-a b_i, b_m) \geq 1$. □

Theorem 2 is named after Descartes although he merely stated that there can be as many positive real roots as there are coefficient sign variations (Descartes, 1954). Over time it became clear that there are at least as many sign variations as there are positive roots; according to Bartolozzi and Franci (1993), the assertion was first stated and proved by Gauss (1828). Some modern authors (Albert, 1943; Wang, 2004) seem to be unaware of Gauss's contribution.

Theorem 3. *Let A be a non-zero real polynomial. If $\mathrm{var}(A) = 0$ then A does not have any positive real root; if $\mathrm{var}(A) = 1$ then A has exactly one positive real root.*

Definition 4. Let S be a subring of $\mathbb{R}$ with $1 \in S$. We define three polynomial transformations $S[x] \longrightarrow S[x]$. Let $A = a_n x^n + \cdots + a_1 x + a_0$ be an element of $S[x]$.

(1) The *homothetic transformation* of A is the polynomial

$$H(A) = a_n x^n + 2a_{n-1}x^{n-1} + \cdots + 2^{n-1}a_1 x + 2^n a_0.$$

(2) The *Taylor shift by* 1 of A is the polynomial

$$T(A) = b_n x^n + \cdots + b_1 x + b_0$$

where $b_k = \sum_{j=k}^{n} \binom{j}{k} a_j$ for $k \in \{0, \ldots, n\}$.

(3) The *reciprocal transformation* of A is the polynomial

$$R(A) = a_0 x^n + \cdots + a_{n-1} x + a_n.$$

Note that $R(A) = 0$ if and only if $A = 0$, and that $x \mid A$ implies $R(A) = R(A/x)$.

The Descartes method can now be stated as Algorithm 1.

Algorithm 1 *(Descartes Method).* This version is specialized to root counting in $I = (0, 1)$. The algorithm can easily be modified to perform real root isolation.

```
int   roots_in_I (A ∈ S[x], A ≠ 0, A squarefree, S ⊂ ℝ subring, 1 ∈ S)
      d ← var(T R(A));
      if d ≤ 1 return d;
      B ← H(A); C ← T(B);
      if x | C  m ← 1; else m ← 0;   Note: m = 1 ⟺ A(1/2) = 0.
      return roots_in_I (B) + m + roots_in_I (C);
```

To show that Algorithm 1 is partially correct we relate the roots of transformed real polynomials to the roots of the untransformed polynomials. Since we want to use bijective mappings we add the point ∞ to $\mathbb{C}$.

Definition 5. Let $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere. We define three functions $\overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$.

$$h(z) = \begin{cases} z/2, & \text{if } z \in \mathbb{C}; \\ \infty, & \text{if } z = \infty. \end{cases}$$

$$t(z) = \begin{cases} z + 1, & \text{if } z \in \mathbb{C}; \\ \infty, & \text{if } z = \infty. \end{cases}$$

$$r(z) = \begin{cases} 1/z, & \text{if } z \in \mathbb{C} - \{0\}; \\ \infty, & \text{if } z = 0; \\ 0, & \text{if } z = \infty. \end{cases}$$

The functions h, t, and r are elements of the group of *Möbius transformations*. These are all functions $\overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ given by

$$z \longmapsto \frac{az + b}{cz + d} \tag{3}$$

with $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. Anderson (1999) explains how formula (3) handles division by 0 and evaluation at ∞. Also Carathéodory (1964) and Henrici (1974) discuss the properties of Möbius transformations.

Remark 6. Let $A \in \mathbb{R}[x]$, and let $n = \deg(A)$; we adopt the convention that $\deg(0) = 0$ and $\mathrm{ldcf}(0) = 0$. Then, for all $z \in \mathbb{C}$,

$$\begin{aligned} H(A)(z) &= 2^n A(h(z)), \\ T(A)(z) &= A(t(z)), \\ R(A)(z) &= \begin{cases} z^n A(r(z)), & \text{if } z \neq 0; \\ \mathrm{ldcf}(A), & \text{if } z = 0. \end{cases} \end{aligned}$$

So, for all $z \in \mathbb{C}$,

$$\begin{aligned} TH(A)(z) &= 2^n A((h \circ t)(z)), \\ TR(A)(z) &= \begin{cases} (t(z))^n A((r \circ t)(z)), & \text{if } z \neq -1; \\ \mathrm{ldcf}(A), & \text{if } z = -1. \end{cases} \end{aligned}$$

Remark 7. By Remark 6, the following statements hold for all polynomials $A \in \mathbb{R}[x]$.

1. The function h maps the roots of $H(A)$ one-to-one onto the roots of A; in particular, the roots of $H(A)$ in $(0, 1)$ correspond to the roots of A in $(0, 1/2)$.
2. The function t maps the roots of $T(A)$ one-to-one onto the roots of A.
3. The function r maps the non-zero roots of $R(A)$ one-to-one onto the non-zero roots of A; the roots of $R(A)$ are non-zero unless $A = 0$.
4. The function $h \circ t$ maps the roots of $TH(A)$ one-to-one onto the roots of A; in particular, the roots of $TH(A)$ in $(0, 1)$ correspond to the roots of A in $(1/2, 1)$.
5. The function $r \circ t$ maps those roots of $TR(A)$ that are different from -1 one-to-one onto the non-zero roots of A; the roots of $TR(A)$ are different from -1 unless $A = 0$. The positive real roots of $TR(A)$ correspond to the roots of A in $(0, 1)$.

Theorem 8. *Algorithm 1 is partially correct.*

Proof. Combine the observations (1), (4), and (5) of Remark 7 with Theorem 3. □

3. Ostrowski's theory

Definition 9. A power series

$$\sum_{k=-\infty}^{+\infty} a_k z^k$$

with non-negative real coefficients is *normal* (Ostrowski, 1939) if

(1) $a_k^2 \geq a_{k-1} a_{k+1}$ for all indices k, and
(2) $a_h > 0$ and $a_j > 0$ for indices $h < j$ implies $a_{h+1}, \ldots, a_{j-1} > 0$.

In 1950, Ostrowski linked the normality of a polynomial and the Descartes rule. He stated his result (Ostrowski, 1950, Lemma 1) for polynomials all of whose coefficients are positive. Generalizing slightly we show in Theorem 11 that it suffices to require that the leading coefficient be positive.

Definition 10. A polynomial with real coefficients is *positive* if its leading coefficient is positive.

Theorem 11. *A positive polynomial $A(x)$ is normal if and only if* $\mathrm{var}((x - \alpha)A(x)) = 1$ *for all positive real numbers α.*

Proof. (i) Let $A(x)$ be positive and normal, and let α be a positive real number. There is a non-negative integer m such that $A(x) = B(x) \cdot x^m$ where $B(x)$ is normal and all the coefficients of

$B(x)$ are positive. Let $B(x) = b_n x^n + \cdots + b_1 x + b_0$. Then

$$\frac{b_{n-1}}{b_n} \geq \frac{b_{n-2}}{b_{n-1}} \geq \cdots \geq \frac{b_0}{b_1}$$

and hence

$$\frac{b_{n-1}}{b_n} - \alpha \geq \frac{b_{n-2}}{b_{n-1}} - \alpha \geq \cdots \geq \frac{b_0}{b_1} - \alpha.$$

Since also $b_n > 0$ and $-\alpha b_0 < 0$, the polynomial

$$(x-\alpha)B(x) = b_n x^{n+1} + b_n \left(\frac{b_{n-1}}{b_n} - \alpha\right) x^n + \cdots + b_1 \left(\frac{b_0}{b_1} - \alpha\right) x - \alpha b_0$$

has exactly 1 coefficient sign variation. And so,

$$1 = \mathrm{var}((x-\alpha)B(x)) = \mathrm{var}((x-\alpha)B(x) \cdot x^m) = \mathrm{var}((x-\alpha)A(x)).$$

(ii) Conversely, let $A(x)$ be positive but not normal. There is a non-negative integer m such that $A = B(x) \cdot x^m$ where $B(x)$ has a non-zero constant term. Moreover, the polynomial $B(x)$ is positive and not normal—and hence non-constant. For any real number α let $C^{(\alpha)}(x) = (x-\alpha)B(x)$. Then $\mathrm{var}((x-\alpha)A(x)) = \mathrm{var}(C^{(\alpha)}(x))$, and it suffices to find a positive number α such that $\mathrm{var}(C^{(\alpha)}(x)) \neq 1$.

Let $B(x) = b_n x^n + \cdots + b_1 x + b_0$. Then $n \geq 1$ and $b_n > 0$ and $b_0 \neq 0$. Let $C^{(\alpha)}(x) = c^{(\alpha)}_{n+1} x^{n+1} + \cdots + c^{(\alpha)}_1 x + c^{(\alpha)}_0$. Then $c^{(\alpha)}_0 = -\alpha b_0$, $c^{(\alpha)}_k = b_{k-1} - \alpha b_k$ for $1 \leq k \leq n$, and $c^{(\alpha)}_{n+1} = b_n$.

If $\mathrm{var}(B(x)) \geq 2$ choose α so small that, for all k with $1 \leq k \leq n$, the signs of $c^{(\alpha)}_k$ and b_{k-1} are equal whenever $b_{k-1} \neq 0$; then $\mathrm{var}(C^{(\alpha)}(x)) \geq \mathrm{var}(B(x)) \geq 2$.

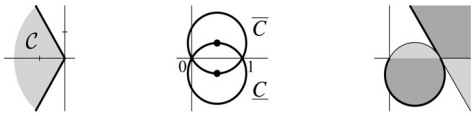
If $\mathrm{var}(B(x)) = 1$ the polynomial $B(x)$ has exactly one positive real root by the Descartes rule. So, for any $\alpha > 0$, the polynomial $C^{(\alpha)}(x)$ has two positive real roots, and, again by the Descartes rule, $\mathrm{var}(C^{(\alpha)}(x)) \geq 2$.

Finally, assume $\mathrm{var}(B(x)) = 0$. Then, since $b_n > 0$, all the coefficients of $B(x)$ are non-negative. If all the coefficients of $B(x)$ are positive, then, since $B(x)$ is not normal, there is an index k with $1 \leq k \leq n-1$ such that $0 < b_k/b_{k+1} < b_{k-1}/b_k$. Choose α such that $b_k/b_{k+1} < \alpha < b_{k-1}/b_k$. Now $\alpha > 0$ and $c^{(\alpha)}_{n+1} = b_n > 0$, $c^{(\alpha)}_{k+1} = b_k - \alpha b_{k+1} < 0$ and $c^{(\alpha)}_k = b_{k-1} - \alpha b_k > 0$, and hence $\mathrm{var}(C^{(\alpha)}(x)) \geq 2$. If not all the coefficients of $B(x)$ are positive, there is a zero-coefficient. Let b_k be the zero-coefficient with the highest index; then $c^{(\alpha)}_{k+1} < 0$ for any positive α. Since $b_0 > 0$ there is an index $j < k$ such that $b_{j+1} = 0$ and $b_j > 0$; then $c^{(\alpha)}_{j+1} > 0$. Now $c^{(\alpha)}_0 < 0$ implies $\mathrm{var}(C^{(\alpha)}(x)) \geq 2$ also in this case. □

By Theorem 11, the Descartes rule will reveal the existence of a single positive root of a positive polynomial if the other roots $\alpha_1, \ldots, \alpha_{n-1}$ are such that $(x-\alpha_1)\cdots(x-\alpha_{n-1})$ is a normal polynomial.

Theorem 12. *A positive linear polynomial is normal if and only if its root is negative or zero.*

Proof. Let A be a positive linear polynomial, and let $\alpha \in \mathbb{R}$ be its root. Then there is a positive real number a such that $A(x) = a(x-\alpha) = ax - a\alpha$. Now A is normal if and only if $-a\alpha \geq 0$, that is, if and only if $\alpha \leq 0$. □



(a) (b) (c)

Fig. 1. (a) A positive quadratic polynomial is normal if and only if its roots are in the cone $\mathcal{C}$. (b) If a polynomial A has a simple root in the interval $(0, 1)$ and no other real or non-real roots in $\underline{C} \cup \overline{C}$ then $\mathrm{var}(TR(A)) = 1$. (c) The image of $\underline{C}$ under r.

Definition 13. Let

$$\mathcal{C} = \left\{ a + ib \,\middle|\, a \le 0 \text{ and } |b| \le |a|\sqrt{3} \right\}.$$

For an illustration see Fig. 1(a); the cone contains its bordering rays and the vertex 0.

Theorem 14. *A positive quadratic polynomial is normal if and only if its roots are elements of the cone $\mathcal{C}$.*

Proof. Let A be a positive quadratic polynomial, and let $c > 0$ be its leading coefficient.

If the roots of A are complex conjugates $a + ib$ and $a - ib$ with real numbers a, b then $A(x) = c(x - (a+ib))(x - (a-ib))$. Now $A(x) = cx^2 - 2acx + c\left(a^2 + b^2\right)$ is normal if and only if $-2ac \ge 0$ and $c(a^2 + b^2) \ge 0$ and $(-2ac)^2 \ge c \cdot c(a^2 + b^2)$, that is, if and only if $a \le 0$ and $4a^2 \ge a^2 + b^2$, or, equivalently, if and only if $a \pm ib \in \mathcal{C}$.

Otherwise, the roots of A are real numbers α and β, and we have $A(x) = c(x-\alpha)(x-\beta) = cx^2 - c(\alpha+\beta)x + c\alpha\beta$. Now A is normal if and only if $-c(\alpha+\beta) \ge 0$ and $c\alpha\beta \ge 0$ and $(-c(\alpha+\beta))^2 \ge c \cdot c\alpha\beta$, that is, if and only if $\alpha + \beta \le 0$ and $\alpha\beta \ge 0$ and $(\alpha+\beta)^2 \ge \alpha\beta$, or, equivalently, if and only if $\alpha, \beta \le 0$. □

In Section 6 we will characterize normal cubic polynomials. The "if"-direction of Theorems 12 and 14 can be generalized to polynomials of any degree using an earlier result of Ostrowski. Ostrowski (1939) showed that the product of two normal series, if it exists, is normal. Later, Ostrowski (1950) gave a simpler proof for the case of polynomials.

Theorem 15. *The product of two normal polynomials is normal.*

Proof. Let $A = \sum_{h=0}^{m} a_h x^h$ and $B = \sum_{j=0}^{n} b_j x^j$ be normal polynomials. Any normal polynomial can be written as $P \cdot x^k$ where k is a non-negative integer and P is a normal polynomial and all the coefficients of P are positive. Hence it suffices to consider the case where all the coefficients of A and B are positive.

Let $C = A \cdot B = \sum_{k=0}^{m+n} c_k x^k$. Write $c_k = \sum_h a_h b_{k-h}$ where h and k range over the set of all integers and all a_h with $h \notin \{0, \ldots, m\}$, all b_j with $j \notin \{0, \ldots, n\}$, and all c_k with $k \notin \{0, \ldots, m+n\}$ are taken as zero. Clearly, all the coefficients of C are positive; it remains to show that $c_k^2 - c_{k-1}c_{k+1} \ge 0$ for all k.

Using the following partition of the set of summation indices

$$\left\{ (h, j) \in \mathbb{Z}^2 \middle| h > j \right\} = \left\{ (j+1, h-1) \in \mathbb{Z}^2 \middle| h \le j \right\} \cup \left\{ (h, h-1) \in \mathbb{Z}^2 \right\}$$

we obtain, for any index k,

$$
\begin{aligned}
&c_k^2 - c_{k-1}c_{k+1}\\
&\quad= \sum_{h\le j} a_h a_j b_{k-h} b_{k-j} + \sum_{h>j} a_h a_j b_{k-h} b_{k-j}\\
&\qquad - \sum_{h\le j} a_h a_j b_{k-h+1} b_{k-j-1} - \sum_{h>j} a_h a_j b_{k-h+1} b_{k-j-1}\\
&\quad= \sum_{h\le j} a_h a_j b_{k-h} b_{k-j} + \sum_{h\le j} a_{j+1} a_{h-1} b_{k-j-1} b_{k-h+1} + \sum_{h} a_h a_{h-1} b_{k-h} b_{k-h+1}\\
&\qquad - \sum_{h\le j} a_h a_j b_{k-h+1} b_{k-j-1} - \sum_{h\le j} a_{j+1} a_{h-1} b_{k-j} b_{k-h} - \sum_{h} a_h a_{h-1} b_{k-h+1} b_{k-h}\\
&\quad= \sum_{h\le j} (a_h a_j - a_{h-1} a_{j+1})(b_{k-j} b_{k-h} - b_{k-j-1} b_{k-h+1}),
\end{aligned}
$$

that is,

$$
c_k^2 - c_{k-1}c_{k+1} = \sum_{h\le j} (a_h a_j - a_{h-1} a_{j+1})(b_{k-j} b_{k-h} - b_{k-j-1} b_{k-h+1}). \tag{4}
$$

Since A is normal and $a_0, \ldots, a_m$ are positive, one has

$$
\frac{a_{m-1}}{a_m} \ge \frac{a_{m-2}}{a_{m-1}} \ge \cdots \ge \frac{a_0}{a_1},
$$

and hence $a_h a_j - a_{h-1} a_{j+1} \ge 0$ for all $h \le j$; the analogous statement holds for the coefficients of B. Hence each summand on the right-hand side of Eq. (4) is non-negative, and thus $c_k^2 - c_{k-1}c_{k+1} \ge 0$ for all k. □

Theorem 16. *If the roots of a positive polynomial are in the cone $\mathcal{C}$ then the polynomial is normal.*

Proof. Let A be a positive polynomial all of whose roots are elements of the cone $\mathcal{C}$. The complete factorization of A over the field of real numbers is a product of linear and quadratic factors. We may assume that all these factors are positive. Since all the roots are in the cone $\mathcal{C}$, Theorems 12 and 14 apply, and each factor is normal. Thus, by Theorem 15, the polynomial A is normal. □

Of all the theorems in this section, we will invoke only Theorem 17 in Sections 4 and 5.

Theorem 17. *If the roots of a non-zero polynomial $A(x)$ are in the cone $\mathcal{C}$ then* $\operatorname{var}((x-\alpha)A(x)) = 1$ *for all positive real numbers α.*

Proof. Let A be a non-zero polynomial and such that all of its roots are elements of the cone $\mathcal{C}$. If A is positive then A is normal by Theorem 16, and hence Theorem 11 implies $\operatorname{var}((x-\alpha)A(x)) = 1$ for all positive α. If A is not positive then $-A$ is positive and the roots of $-A$ are elements of the cone $\mathcal{C}$. Hence, as before, $\operatorname{var}((x-\alpha)(-A)(x)) = 1$, but $\operatorname{var}((x-\alpha)(-A)(x)) = \operatorname{var}((x-\alpha)A(x))$. □

4. Three circles

By Theorem 17, Algorithm 1 will stop calling itself when it encounters a polynomial $TR(A)$ that has exactly one positive root and whose other roots are elements of the cone $\mathcal{C}$. We want to state this condition in terms of the roots of the polynomial A. Since A is non-zero, Remark 7(5)

implies that the function $r \circ t$ maps the roots of $TR(A)$ one-to-one onto the non-zero roots of A. But much more is true since $r \circ t$ is a Möbius transformation.

Remark 18. Anderson (1999) reviews some properties of Möbius transformations. These transformations are homeomorphisms of the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ that map circles in $\overline{\mathbb{C}}$ to circles. In particular, circles and lines in $\mathbb{C}$ are mapped to circles and lines. To identify the image of a given circle or line K under a given Möbius transformation it suffices to select three distinct points on K, to compute their images under the transformation, and to determine the unique circle or line L that contains those images. The sets $\mathbb{C} - K$ and $\mathbb{C} - L$ each have exactly two connected components. Each component of $\mathbb{C} - K$ is mapped to a different component of $\mathbb{C} - L$ since Möbius transformations are homeomorphisms of $\overline{\mathbb{C}}$. By applying the transformation to a single point in $\mathbb{C} - K$ one can determine the image of each component of $\mathbb{C} - K$.

Definition 19. We define three circular disks.

$$\begin{aligned} \underline{C} &= \left\{ z \in \mathbb{C} \;\middle|\; \left| z - \left(1/2 - i\sqrt{3}/6\right)\right| < \sqrt{3}/3 \right\}, \\ \overline{C} &= \left\{ z \in \mathbb{C} \;\middle|\; \left| z - \left(1/2 + i\sqrt{3}/6\right)\right| < \sqrt{3}/3 \right\}, \\ C &= \left\{ z \in \mathbb{C} \;\middle|\; |z - 1/2| < 1/2 \right\}. \end{aligned}$$

Remark 20. The Möbius transformation $r \circ t$ maps the cone $\mathcal{C}$ one-to-one onto $\overline{\mathbb{C}} - (\underline{C} \cup \overline{C})$ and the half-plane $\{z \in \mathbb{C} \mid \mathrm{Re}(z) \le 0\}$ one-to-one onto $\mathbb{C} - C$. Both statements can be verified using the method described in Remark 18.

Fig. 1(a) shows the cone $\mathcal{C}$. Fig. 1(b) shows the boundaries of the open disks $\underline{C}$ and $\overline{C}$. Fig. 1(c) shows how the Möbius transformation r operates on the boundary of $\underline{C}$. If z traverses the boundary of $\underline{C}$ clockwise from 1 towards 0, the reciprocal $r(z)$ traverses the ray $\{1-s+\sqrt{3}si \mid s \ge 0\}$ upwards starting at 1. Similarly, if z traverses the boundary of $\overline{C}$ counterclockwise from 1 towards 0, the reciprocal $r(z)$ traverses the ray $\{1 - s - \sqrt{3}si \mid s \ge 0\}$ downwards starting at 1. The point $z = 0$ is mapped to $r(0) = \infty \notin \mathbb{C}$. Thus the figure illustrates how the function $t^{-1} \circ r = (r \circ t)^{-1}$ maps $\overline{\mathbb{C}} - (\underline{C} \cup \overline{C})$ one-to-one onto $\mathcal{C}$.

Theorem 21 (*Two-Circle Theorem*). *Let A be a real polynomial with a single, simple root in the interval $(0, 1)$ and no other real or non-real roots in the open disks $\underline{C}$ and $\overline{C}$. Then* $\mathrm{var}(TR(A)) = 1$.

Proof. Let A be as described. Then $A \neq 0$ and, by Remark 7(5), the roots of $B = TR(A)$ are all different from -1. Therefore, the function $(r \circ t)^{-1}$ maps the non-zero roots of A one-to-one onto the roots of B. Hence, B has a single, simple root in $(r \circ t)^{-1}((0, 1)) = (0, \infty)$, and its other roots are in $(r \circ t)^{-1}(\overline{\mathbb{C}} - (\underline{C} \cup \overline{C}))$ which equals $\mathcal{C}$ by Remark 20. Now Theorem 17 yields $\mathrm{var}(B) = 1$. □

The two-circle condition is not necessary for the termination of the Descartes method. Indeed, the polynomial $A = 32x^3 - 16x^2 + 2x - 1$ has the single, simple root $1/2$ in the interval $(0, 1)$, the pair of complex conjugate roots $\pm i/4$ inside the open disks $\underline{C}$ and $\overline{C}$, and $\mathrm{var}(TR(A)) = 1$.

Our two-circle theorem improves upon a two-circle theorem of Collins and Johnson (1989). They use the disks

$$D_1 = \{z \in \mathbb{C} \mid |z| < 1\} \quad \text{and} \quad D_2 = \{z \in \mathbb{C} \mid |z - 1| < 1\}$$

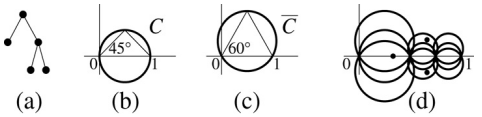


Fig. 2. (a) Recursion tree for $A = 27648x^3 - 46080x^2 + 25251x - 4321$. (b), (c) Triangles with circumscribing disks C, $\overline{C}$. (d) Circumscribing disks for the intervals at the leaf nodes of the tree in (a). Also shown are $1/3$ and $2/3 \pm i \cdot 5/32$, the roots of A.

instead of $\underline{C}$ and $\overline{C}$. But $\underline{C} \cup \overline{C}$ is a proper subset of $D_1 \cup D_2$, and the area of $\underline{C} \cup \overline{C}$ is exactly one-third of the area of $D_1 \cup D_2$. Indeed, the Möbius transformation

$$z \longmapsto \frac{i\sqrt{3}}{3}z + \left(\frac{1}{2} - i\frac{\sqrt{3}}{6}\right)$$

maps $D_1 \cup D_2$ onto $\underline{C} \cup \overline{C}$.

The following well-known theorem completes our converse of Theorem 3.

Theorem 22. *If a polynomial A does not have any roots in the open disk C then* $\operatorname{var}(TR(A)) = 0$.

Proof. Let A be as described. Then $A \neq 0$ and, by Remark 7(5), the roots of $B = TR(A)$ are all different from -1. Therefore, the function $(r \circ t)^{-1}$ maps the non-zero roots of A one-to-one onto the roots of B. But since the roots of A are all in $\mathbb{C} - C$, the roots of B have non-positive real parts by Remark 20. Hence, in the decomposition of B into a product of a constant and monic linear and quadratic factors, every linear factor is of the form $x - \alpha$ where $\alpha \leq 0$, and every quadratic factor is of the form $(x - (a + ib))(x - (a - ib)) = x^2 - 2ax + (a^2 + b^2)$ where $a \leq 0$. Since all the non-zero coefficients of all the linear and quadratic factors of B have the same sign, the non-zero coefficients of B all have the same sign. □

When we bound the recursion depth of the Descartes method we will use Theorem 23 which summarizes the preceding results.

Theorem 23. *Let A be a real polynomial with* $\operatorname{var}(TR(A)) \geq 2$. *Then either the open disk C contains at least two roots of A, or the interval $(0, 1)$ contains exactly one real root and the union of the open disks $\underline{C}$ and $\overline{C}$ contains a pair of complex conjugate roots.*

Proof. If A has no root in C then $\operatorname{var}(TR(A)) = 0$ by Theorem 22. Thus, A has at least one root in C. If this is the only root in C, the root is real and it is, in fact, the only real root in the interval $(0, 1)$. Then $\underline{C} \cup \overline{C}$ must contain a pair of complex conjugate roots because otherwise $\operatorname{var}(TR(A)) = 1$ by Theorem 21. □

5. Bounds for the recursion tree

For any input polynomial A the recursion tree of Algorithm 1 is a full binary tree; Fig. 2 shows an example. With every node of the tree we associate a pair (B, I) consisting of a polynomial B and an interval I. With the root of the tree we associate the pair $(A, (0, 1))$. If an internal node is associated with the pair (B, I) we associate one child with the pair (B_L, I_L) where $B_L = H(B)$ and I_L is the open left half of I, and we associate the other child with the pair (B_R, I_R) where $B_R = TH(B)$ and I_R is the open right half of I.

Remark 24. By Remark 7(1) and (4), the function h maps the roots of B_L in $(0, 1)$ onto the roots of B in I_L, and the function $h \circ t$ maps the roots of B_R in $(0, 1)$ onto the roots of B in I_R. Thus,

there is a sequence of elements of $\{h, t\}$ whose composition m maps the roots of B in $(0, 1)$ onto the roots of the input polynomial A in I. When m maps the interval $(0, 1)$ onto the interval I it transforms at the same time the disks C, $\underline{C}$ and $\overline{C}$ of Section 4. These disks are the circumscribing disks of isosceles triangles with base $(0, 1)$ and base angles $45°$, $-60°$ and $60°$, respectively, as shown in Fig. 2. But h, t and, hence, m are Möbius transformations and thus preserve angles (Anderson, 1999). Moreover, the transformations h, t and, hence, m map straight lines in $\mathbb{C}$ onto straight lines in $\mathbb{C}$ and circles in $\mathbb{C}$ onto circles in $\mathbb{C}$. Therefore, the images $m(C)$, $m(\underline{C})$ and $m(\overline{C})$ are the circumscribing disks of the isosceles triangles with base I and base angles $45°$, $-60°$ and $60°$, respectively. Fig. 2 shows the disks that are considered at the leaf nodes of a particular recursion tree.

The depth of the recursion tree can be bounded using the root separation theorem of Mahler (1964). To obtain a bound that also covers the width of the tree we use a generalization by Davenport (1985) of Mahler's theorem in a form due to Johnson (1998).

Definition 25. Let $A = a_n x^n + \cdots + a_1 x + a_0$ be a non-zero polynomial of degree n with complex coefficients and the complex roots $\alpha_1, \ldots, \alpha_n$. The *Euclidean norm* of A is $|A|_2 = (a_n^2 + \cdots + a_0^2)^{1/2}$, the *measure* of A is $M(A) = |a_n| \cdot \prod_{i=1}^{n} \max(1, |\alpha_i|)$, and the *discriminant* of A is $D(A) = a_n^{2n-2} \prod_{i<j} (\alpha_i - \alpha_j)^2$.

Remark 26. A theorem of Landau (1905) implies $M(A) \leq |A|_2$. The inequality was independently rediscovered more than once. Ostrowski (1961) summarizes its history and proves a generalization. Mignotte (1974, 1982) gives a short elementary proof. The discriminant $D(A)$ is known to be a polynomial in the coefficients of A (van der Waerden, 1949); hence $D(A) \geq 1$ if A is a squarefree integer polynomial.

Theorem 27. *Let A be a non-zero complex polynomial of degree n with the roots $\alpha_1, \ldots, \alpha_n$. Let k be an integer, $1 \leq k \leq n$, and let $(\beta_1, \ldots, \beta_k)$ be a sequence of roots of A such that*

$$\beta_i \notin \{\alpha_1, \ldots, \alpha_i\} \quad \text{and} \quad |\beta_i| \leq |\alpha_i| \quad \text{for all } i \in \{1, \ldots, k\}.$$

Then

$$\prod_{i=1}^{k} |\alpha_i - \beta_i| \geq 3^{k/2} D(A)^{1/2} M(A)^{-n+1} n^{-k-n/2}.$$

Proof. Johnson (1998). □

Theorem 28. *Let A be a non-zero real polynomial of degree n, measure M, and discriminant D. Let the integers $h \geq 0$ and $k \geq 1$ be such that k is the number of internal nodes of depth h in the recursion tree of Algorithm 1 with input A where* depth *is the distance from the root. Then*

(1) $k \leq n$, *and*
(2) $2^{(1-h)k} > 3^k D^{1/2} M^{-n+1} n^{-k-n/2}$.

Proof. Let $I_1 < \cdots < I_k$ be the open subintervals of $(0, 1)$ that are associated with the internal nodes of depth h, and let $A_1, \ldots, A_k$ be the corresponding polynomials. The intervals have width 2^{-h}. For every index $i \in \{1, \ldots, k\}$ let C_i, $\underline{C_i}$ and $\overline{C_i}$ be the circumscribing disks of the isosceles triangles with base I_i and base angles $45°$, $-60°$ and $60°$, respectively. By Remark 24 the roots of A_i in the disks C, $\underline{C}$ and $\overline{C}$, correspond, respectively, to the roots of A in the disks C_i, $\underline{C_i}$ and $\overline{C_i}$. But the polynomials A_i are at internal nodes of the recursion tree, so $\mathrm{var}(TR(A_i)) \geq 2$, and

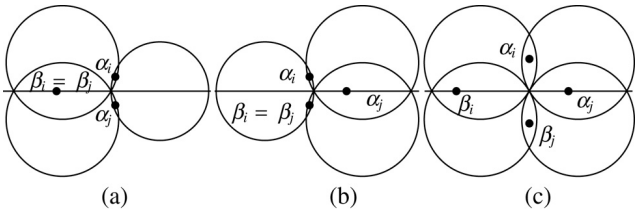


Fig. 3. Adjacent intervals with coinciding roots. Here, $j = i + 1$. (a) $|R_i| = 3$ and $|R_j| = 2$. Note that $|\beta_i| \le |\alpha_i|$ and $|\beta_j| \le |\alpha_j|$ and $\alpha_i, \beta_i \in \overline{C}_i$ and $\alpha_j, \beta_j \in \underline{C}_i$. (b) $|R_i| = 2$ and $|R_j| = 3$. (c) $|R_i| = 3$ and $|R_j| = 3$. In $\overline{C}_i$ the root with the smaller modulus is labeled β_i and the other root α_i; likewise for $\underline{C}_j$, β_j and α_j.

hence, by Theorem 23, either C_i contains at least two roots of A, or I_i contains exactly one real root of A and $\underline{C}_i \cup \overline{C}_i$ contains a pair of complex conjugate roots of A.

Assertion (1) holds since every disk C_i contains at least one root of A, and the disks $C_1, \ldots, C_k$ are pairwise disjoint.

Assertion (2) holds if A has a multiple root since $D = 0$ in that case. If all roots are simple, define, for every index $i \in \{1, \ldots, k\}$, a set R_i of roots of A in $\underline{C}_i \cup \overline{C}_i$. If C_i contains at least two roots of A, let $R_i = \{s, t\}$ where s and t are either two arbitrary distinct real roots in I_i or two arbitrary non-real complex conjugate roots in C_i; otherwise, let $R_i = \{r, s, t\}$ where r is the unique real root in I_i, and s and t are two arbitrary non-real complex conjugate roots in $\underline{C}_i \cup \overline{C}_i$. For notational convenience let $R_0 = R_{k+1} = \emptyset$. Note that, for all distinct indices $i, j \in \{1, \ldots, k\}$, the intersection of R_i and R_j is either empty or it consists of two non-real complex conjugate roots and $j = i - 1$ or $j = i + 1$. Moreover, if $R_i \cap R_{i+1} \neq \emptyset$ then $R_{i-1} \cap R_i = \emptyset$ and $R_{i+1} \cap R_{i+2} = \emptyset$. So, for all indices $i \in \{1, \ldots, k\}$, the set R_i is either disjoint from all sets R_j, $j \neq i$, or there is exactly one set R_j such that $j \neq i$ and $R_i \cap R_j \neq \emptyset$.

Let $i \in \{1, \ldots, k\}$. If R_i is disjoint from all sets R_j, $j \neq i$, select two distinct elements from R_i that are both in C_i or both in $\underline{C}_i$ or both in $\overline{C}_i$, and label them α_i and β_i so that $|\beta_i| \le |\alpha_i|$. If there is exactly one set R_j such that $j \neq i$ and $R_i \cap R_j \neq \emptyset$ then select $\alpha_i, \beta_i, \alpha_j, \beta_j \in R_i \cup R_j$ as described in Fig. 3 for the case $j = i + 1$. Since $R_i \cap R_j \neq \emptyset$, at least one of the sets R_i and R_j has 3 elements, and the figure shows how the roots are selected depending on whether only R_i has 3 elements or only R_j or both R_i and R_j.

By construction, the selected roots $\alpha_1, \ldots, \alpha_k$ and $\beta_1, \ldots, \beta_k$ not only satisfy $\beta_i \notin \{\alpha_1, \ldots, \alpha_i\}$ and $|\beta_i| \le |\alpha_i|$ for all $i \in \{1, \ldots, k\}$ but also, for all $i \in \{1, \ldots, k\}$, both roots α_i and β_i are in one of the disks C_i, $\underline{C}_i$, $\overline{C}_i$, or, if $i > 1$, in the disk $\overline{C}_{i-1}$, so $|\alpha_i - \beta_i| < 2^{1-h}/\sqrt{3}$. Now Theorem 27 implies

$$2^{(1-h)k} 3^{-k/2} > \prod_{i=1}^{k} |\alpha_i - \beta_i| \ge 3^{k/2} D^{1/2} M^{-n+1} n^{-k-n/2}. \quad \square$$

Theorem 29. *Let A be a non-zero squarefree integer polynomial of degree $n \ge 2$ with Euclidean norm d. Let h and k be as in Theorem 28, and let $\log = \log_2$. Then*

(1) $k \le n$, *and*
(2) $(h-1)k < (n-1)\log d + (k + n/2)\log n - k \log 3$, *and*
(3) $h \le (n-1)\log d + (n/2 + 1)\log n - \log 3$.

Proof. Assertion (1) holds due to assertion (1) of Theorem 28. To show assertion (2), consider assertion (2) of Theorem 28, apply Remark 26, take logarithms, and multiply by -1. To show

assertion (3), consider assertion (2) and collect all terms involving k on one side to obtain $k(h - 1 - \log n + \log 3) < (n-1)\log d + n/2 \log n$. If $h - 1 - \log n + \log 3 < 0$ then assertion (3) clearly holds. If, on the other hand, $h - 1 - \log n + \log 3 \geq 0$ then $k \geq 1$ implies $h - 1 - \log n + \log 3 < (n-1)\log d + n/2 \log n$, and hence assertion (3) holds also in this case. □

Remark 30. Theorem 29 is stronger than an earlier result by Krandick (1995, Satz 47), and the proof is shorter. The theorem implies the dominance relations $hk \preceq n \log(nd)$ and $h \preceq n \log(nd)$ which can be used in an asymptotic computing time analysis of Algorithm 1 when the ring S of coefficients is $\mathbb{Z}$; the notation $\preceq$ is due to Collins (1974).

6. Normal cubics

By Theorem 16 any positive polynomial whose roots are in the cone $\mathcal{C}$ is normal. By Theorems 12 and 14 the converse holds for linear and quadratic polynomials. For cubic polynomials, however, the converse is false. Indeed, the normal polynomial $x^3 + 5x^2 + 16x + 30$ has roots $-1 \pm 3i \notin \mathcal{C}$. Theorems 31 and 32 together completely characterize the normal cubic polynomials.

Theorem 31. *Let A be a positive polynomial all of whose roots are real. Then A is normal if and only if the roots are all non-positive.*

Proof. If the roots of A are all non-positive then Theorem 16 implies that A is normal. Otherwise, A has a positive root. In this case, $\operatorname{var}((x-1)A(x)) > 1$ by Theorem 2, and A is not normal by Theorem 11. □

Theorem 32. *Let A be a positive cubic polynomial whose roots are a and $b \pm ic$ where a, b, c are real numbers. Then A is normal if and only if*

$$a \leq 0 \quad \textit{and} \tag{5}$$

$$b \leq 0 \quad \textit{and} \tag{6}$$

$$c^2 - 3b^2 - 2ab - a^2 \leq 0 \quad \textit{and} \tag{7}$$

$$c^4 + 2b^2c^2 + 2abc^2 - a^2c^2 + b^4 + 2ab^3 + 3a^2b^2 \geq 0. \tag{8}$$

Proof. We may assume that A is monic since A is normal if and only if $A/\operatorname{ldcf}(A)$ is normal. Hence,

$$A = (x - a) \cdot (x - (b + ic)) \cdot (x - (b - ic))$$

and thus

$$A = x^3 + a_2x^2 + a_1x + a_0$$

where

$$a_2 = -a - 2b,$$
$$a_1 = 2ab + b^2 + c^2,$$
$$a_0 = -ab^2 - ac^2.$$

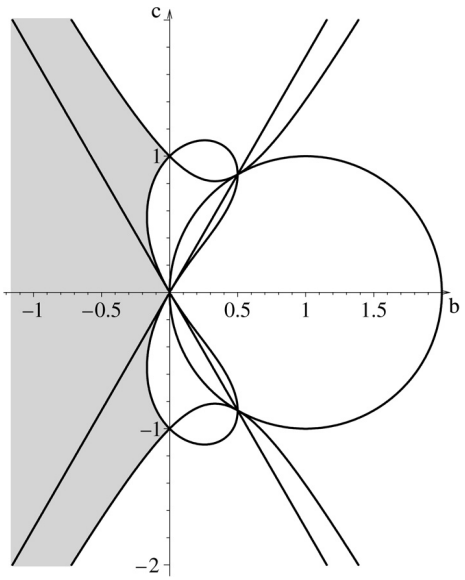


Fig. 4. For $a = -1$ the points (b, c) that satisfy (5)–(8) are precisely the points in the left half-plane (6) between the two branches of the hyperbola (7) and outside of the figure "8" (8). For $a = 0$ the solution set coincides with the cone $\mathcal{C}$ which is delimited by the curve $c^2 - 3b^2 = 0$. The solutions of inequality (10) are precisely the points outside the circle.

By definition, A is normal if and only if all of the following hold.

$$a_2 \geq 0, \tag{9}$$
$$a_1 \geq 0, \tag{10}$$
$$a_0 \geq 0, \tag{11}$$
$$a_2^2 \geq a_1, \tag{12}$$
$$a_1^2 \geq a_2 a_0, \tag{13}$$
$$a_2 = 0 \Rightarrow a_1 = a_0 = 0, \tag{14}$$
$$a_1 = 0 \Rightarrow a_0 = 0. \tag{15}$$

Implication (15) is redundant since it follows from (13), (9) and (11). Also the implication $(a_2 = 0 \Rightarrow a_1 = 0)$ in (14) is redundant since it follows from (12) and (10). We note the pairwise equivalence of (5) and (11), (7) and (12), and (8) and (13). We will show that the conjunction of (5)–(8) is equivalent to the conjunction of (9)–(15).

Assume (5)–(8). Clearly, (5) and (6) imply (9) and (10). The pairwise equivalences yield (11), (12) and (13). The implication $(a_2 = 0 \Rightarrow a_0 = 0)$ in (14) holds since $a_2 = 0$ together with (5) and (6) implies $a = 0$.

Assume now (9)–(15). The pairwise equivalences yield (5), (7), and (8). To complete the proof we have to show (6). By (5) we have $a \leq 0$. If $a = 0$ then (6) follows from (9), so we may assume $a < 0$. Next observe that if (a, b, c) satisfies (9)–(15) then, for any $t > 0$, (ta, tb, tc) satisfies (9)–(15). So we may assume $a = -1$. Now (9) implies that $b \leq 1/2$, and we need to show that $b \leq 0$. Fig. 4 illustrates the situation. If $b = 1/2$ then, by (9), $a_2 = 0$, hence, by (14), $a_0 = 0$, and thus $a = 0$, a contradiction. So, $b < 1/2$ and we need to show $b \leq 0$. Multiplying (7) and (10), and combining the result with (8) we obtain the inequalities

$$(c^2 - 3b^2 + 2b - 1)(-2b + b^2 + c^2) \le 0$$
$$\le c^4 + 2b^2c^2 + 2abc^2 - a^2c^2 + b^4 + 2ab^3 + 3a^2b^2.$$

Collecting all the terms on the left-hand side and factoring yields

$$-2b(2b-1)((b-1)^2 + c^2) \le 0,$$

so $0 < b < 1/2$ is impossible, and we have $b \le 0$ as desired. □

Fig. 4 supports the notion that Theorem 32 recognizes more normal cubics than Theorem 16. In an attempt to quantify the improvement we perform extensive experiments that use Algorithm 1.

Definition 33. The *max-norm* of a complex polynomial $A = a_n x^n + \cdots + a_1 x + a_0$ is $|A|_\infty = \max(|a_n|, \ldots, |a_0|)$.

Let m be a positive integer. The set of all normal cubic integer polynomials of max-norm m can be efficiently enumerated. For each such polynomial A,

$$A = a_3x^3 + a_2x^2 + a_1x + a_0,$$

we want to decide whether all of its roots are in the cone $\mathcal{C}$. Since A is cubic, either A has one real root and two non-real complex conjugate roots, or all the roots of A are real. In particular, if A has a multiple root then all the roots of A are real. Since all the coefficients of A are non-negative, all the real roots of A are non-positive and, hence, in $\mathcal{C}$. Using polynomial factorization and Algorithm 1 we thus reduce the decision problem to the case where A is irreducible and has a single real root $\alpha \in \mathcal{C}$. The other roots of A are the roots of the polynomial

$$B = A(x)/(x - \alpha) = a_3x^2 + (a_3\alpha + a_2)x + (a_3\alpha^2 + a_2\alpha + a_1).$$

By Theorem 14, these roots are in $\mathcal{C}$ if and only if B is normal. We decide the latter by performing arithmetic in $\mathbb{Z}[\alpha]$ on the coefficients of B.

The computing time of the decision method can be reduced by a factor of about 3.5 by using floating point computations instead of exact arithmetic. Indeed, we use the floating point interval arithmetic techniques described by Collins et al. (2002), and we fall back to exact arithmetic just in case the floating point results are inconclusive. In our experiments we represent α by an isolating interval of width 2^{-40}, and we use standard double precision arithmetic (IEEE, 1985). For all our inputs, the floating point method is inconclusive only in case the roots of B lie on the boundary of $\mathcal{C}$; this situation occurs when B is normal and $(a_3\alpha + a_2)^2 = a_3 \cdot (a_3\alpha^2 + a_2\alpha + a_1)$.

Table 1 shows that only about 57% of the $2,353,361,850$ normal cubic polynomials we examined have all of their roots in the cone $\mathcal{C}$. It seems reasonable to expect smaller ratios when the experiment is carried out for polynomials of higher degrees. The table also shows that we had to use exact arithmetic for relatively few polynomials.

We can now generalize Theorem 17.

Theorem 34. *Let $A(x)$ be a non-zero polynomial such that $A(x) = B(x) \cdot C(x)$ where all the roots of $B(x)$ are in the cone $\mathcal{C}$ and $C(x)$ is a product of cubic polynomials each of whose roots are as described in Theorem 32 then*

$$\mathrm{var}((x - \alpha)A(x)) = 1 \qquad \textit{for all real } \alpha > 0.$$

Proof. Theorems 11, 15, 17 and 32. □

Table 1
For any positive integer m, let $N(m)$ be the number of normal cubic integer polynomials with max-norm m, and let $C(m)$ be the number of those normal cubic integer polynomials of max-norm m that have all roots in the cone $\mathcal{C}$

m	$N(m)$	$C(m)$	$C(m)/N(m)$	Boundary
100	780 708	445 288	.57036	122
200	6 232 898	3 558 002	.57084	277
300	21 019 770	12 004 290	.57110	453
400	49 814 320	28 450 698	.57113	640
500	97 252 440	55 564 678	.57134	807
600	168 075 834	96 011 988	.57124	996
700	266 842 438	152 459 384	.57135	1140
800	398 334 336	227 573 618	.57131	1355
900	567 119 096	324 020 078	.57134	1766
1000	777 890 010	444 469 060	.57138	1695

The ratios $C(m)/N(m)$ are rounded to five decimal digits. The last column lists the number of polynomials that have non-real roots on the boundary of $\mathcal{C}$.

It is easy to state higher-degree analogues of Theorem 32. The analogous theorems result in additional improvements of Theorem 17, but it is not clear how the improvements can be used to obtain better general bounds for the Descartes method.

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