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NOTE ON VINCENT'S THEOREM*

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1. In 1834 Vincent gave a new method for separating the real roots of an algebraic equation, which was based on the following extremely remarkable theorem[1]:

If the equation

$$f(x) = A_0 x^n + \cdots + A_n = 0, \qquad A_0 \neq 0$$

with real coefficients and without multiple roots is transformed by the successive transformations

$$x = x_1 = a_1 + \frac{1}{x_2}, \qquad x_2 = a_2 + \frac{1}{x_3}, \qquad \cdots, \qquad x_{\nu-1} = a_{\nu-1} + \frac{1}{x_\nu}$$

where $a_1, a_2, \cdots, a_{\nu-1}$ *are* $\geqq 1$, *into the equation*

$$f_\nu(x_\nu) = A_0^{(\nu)} x_\nu^n + \cdots + A_n^{(\nu)} = 0,$$

then for a ν_0 *the polynomial* $f_\nu(x_\nu)$ *with* $\nu > \nu_0$ *presents at the most one variation of signs.*[2]

2. Vincent's theorem and its proof contain no estimate of ν_0. Such an estimate was given one year ago by Uspensky,[3] who found a bound of ν_0 depending only on the smallest distance Δ between two roots of $f(x)$. Uspensky makes use of Fibonacci's series N_μ:

$$1, 1, 2, 3, 5, 8 \cdots$$

determined by

$$N_1 = 1, \qquad N_2 = 1, \qquad N_3 = N_1 + N_2, \cdots, N_\mu = N_{\mu-1} + N_{\mu-2}, \cdots$$

and asserts that if for an integer m:

$$\Delta N_{m-1} > \tfrac{1}{2}, \qquad \Delta N_{m-1} N_m > 1 + \frac{1}{\epsilon_n}, \qquad \epsilon_n = \left(1 + \frac{1}{n}\right)^{1/(n-1)} - 1,$$

we can put $\nu_0 = m$.[4]

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[1] Vincent, *Mémoire sur la résolution des équations numériques*. Mém. Soc. R. des Sc. de Lille (1834), pp. 1–34; *Note sur la résolution des équations numériques*, J. des Math. p. et appl., vol. 1 (1836), pp. 341–372. The result was for the first time published in the 6th edition of Bourdon's Algèbre.

[2] Vincent supposes unnecessarily that $a_0, a_1, \cdots, a_{\nu-1}$ are all > 1.

[3] In *Theory of Equations*, 1948, pp. 298–303.

[4] As a matter of fact Uspensky must use the first inequality with 2 instead of $\frac{1}{2}$. Further he assumes unnecessarily that $a_1, a_2, \cdots, a_{\nu-1}$ are positive *integers*.

3. In what follows we will show that this result can be improved. Uspensky's first condition can be dropped altogether, while his second condition can be replaced by

$$\Delta N_m N_{m-1} \geqq \sqrt{3}.[5]$$

The use of Fibonacci's series N_μ in this connection is based on the fact that N_μ is a lower bound for the denominator Q_μ of the μ^{th} convergent to the continued fraction

$$a_1 + \frac{1}{a_2} + \cdots .$$

This follows at once from the law of convergents

$$Q_{\mu+1} = a_{\mu+1}Q_\mu + Q_{\mu-1}.$$

The complete enunciation of our result is given in Section 7 and the proof is contained in Sections 8, 9. The proof is based on three lemmas given in Sections 4–6, which appear to be of interest in themselves.

4. LEMMA 1. *Let the coefficients a_ν of the polynomial*

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$$

be positive. Then the necessary and sufficient condition that the sequence of coefficients in the product $(x - \alpha)f(x)$ presents precisely one variation of signs for each positive α is that the $n - 1$ inequalities

$$(1) \qquad a_\nu^2 \geqq a_{\nu-1}a_{\nu+1} \qquad (\nu = 1, 2, \cdots, n-1)$$

are satisfied.[6]

PROOF. If (1) is satisfied, it follows that $\dfrac{a_\nu}{a_{\nu-1}} \geqq \dfrac{a_{\nu+1}}{a_\nu}$ $(\nu = 1, 2, \cdots, n-1)$, and therefore for any positive α:

$$(2) \qquad \frac{a_1}{a_0} - \alpha \geqq \cdots \geqq \frac{a_\nu}{a_{\nu-1}} - \alpha \geqq \frac{a_{\nu+1}}{a_\nu} - \alpha \cdots \geqq \frac{a_n}{a_{n-1}} - \alpha.$$

On the other hand we have identically

$$(3) \qquad \begin{aligned} f(x)(x-\alpha) = {} & a_0x^{n+1} + a_0\left(\frac{a_1}{a_0} - \alpha\right)x^n + a_1\left(\frac{a_2}{a_1} - \alpha\right)x^{n-1} + \cdots \\ & + a_\nu\left(\frac{a_{\nu+1}}{a_\nu} - \alpha\right)x^{n-\nu} + \cdots + a_{n-1}\left(\frac{a_n}{a_{n-1}} - \alpha\right)x - \alpha a_n. \end{aligned}$$

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[5] The limit $1 + 1/\epsilon_n$ given by Uspensky increases with growing n monotonically to ∞ and is $\geqq 1 + 1/\epsilon_2 = 3$. Our limit $\sqrt{3}$ is therefore more advantageous than that of Uspensky for all $n \geqq 2$.

[6] In counting the number of variations of signs we observe the usual convention that all zeros are to be omitted.

And it follows from (2) that the sequence of coefficients in this product has exactly one variation of signs.

Suppose that for certain k, $0 < k < n$, the corresponding inequality (1) is not satisfied. Then we have

$$a_k^2 < a_{k-1}\,a_{k+1}, \qquad \frac{a_k}{a_{k-1}} < \frac{a_{k+1}}{a_k}.$$

If α is a number from the open interval $\left(\frac{a_k}{a_{k-1}}, \frac{a_{k+1}}{a_k}\right)$, we have

$$\frac{a_k}{a_{k-1}} - \alpha < 0 < \frac{a_{k+1}}{a_k} - \alpha$$

and we see that the sequence of coefficients in (3) presents at least three variations of signs, namely, one between x^{n+1} and x^{n-k+1}, one between x^{n-k+1} and x^{n-k}, and one between x^{n-k} and $x^0 = 1$. Lemma 1 is thus proved.[7]

5. Lemma 2. *Let*

$$f(x) = \sum_{\nu=0}^{n} a_\nu x^{n-\nu}, \qquad g(x) = \sum_{\mu=0}^{m} b_\mu x^{m-\mu}$$

be two polynomials with positive coefficients a_ν, b_μ satisfying the conditions

$$\text{(4)} \qquad \begin{cases} a_\nu^2 - a_{\nu-1}a_{\nu+1} \geqq 0 & (\nu = 1, 2, \cdots, n-1) \\ b_\mu^2 - b_{\mu-1}b_{\mu+1} \geqq 0 & (\mu = 1, 2, \cdots, m-1) \end{cases}$$

Then the product

$$f(x)g(x) = \sum_{\kappa=0}^{n+m} c_\kappa x^{n+m-\kappa}$$

has the analogous property

$$\text{(5)} \qquad c_\kappa^2 - c_{\kappa-1}c_{\kappa+1} \geqq 0 \quad (\kappa = 1, 2, \cdots, n+m-1).$$

Proof. This lemma is contained in a more general result concerning infinite series and published in 1939.[8] Its proof follows at once from the identity

$$\text{(6)} \qquad c_\kappa^2 - c_{\kappa-1}c_{\kappa+1} = \sum_{\nu \geqq \lambda} (a_\nu a_\lambda - a_{\nu+1}a_{\lambda-1})(b_{\kappa-\nu}b_{\kappa-\lambda} - b_{\kappa-\nu-1}b_{\kappa-\lambda+1}).$$

In this identity all a_ν with indices outside the range $<0, n>$ and all b_μ with indices outside the range $<0, m>$ are to be taken as zero. Each of the polynomials $f(x)$, $g(x)$ can be linear. In this case its coefficients are only subject to the condition of being positive.

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[7] This lemma can also be proved by using certain results given by D. André in *Annales Scientifiques de l'Ecole Normale Supérieure* (2) vol. 12 (1883), supplément, pp. 33–44.

[8] A. Ostrowski, *Note sur les produits de séries normales*, Bulletin de la Société royale des Sciences de Liége (1939), pp. 458–468.

6. LEMMA 3. *Suppose that the roots* $-x_\nu$ *of the real polynomial*

(7) $$f(x) = \sum_{\nu=0}^{n} a_\nu x^{n-\nu} = a_0 \prod_{\nu=1}^{n} (x + x_\nu), \qquad a_0 > 0$$

are contained in the sector with the angle 120° *having the negative x-axis as bisector:*

(8) $$| \arg x_\nu | \leqq \frac{\pi}{3}.$$

Then all a_ν *are positive, and for any* $\alpha > 0$ *the product* $(x - \alpha)f(x)$ *presents exactly one variation of signs.*

PROOF. For real x_ν the corresponding linear factors $x + x_\nu$ are linear polynomials with non-negative coefficients. Let $-\xi \pm i\eta$ be a pair of conjugate roots among the x_ν. Then the product of the corresponding linear factors is

$$x^2 + 2x\xi + \xi^2 + \eta^2$$

and satisfies the inequality

$$(2\xi)^2 \geqq (\xi^2 + \eta^2)$$

since by hypothesis $| \eta/\xi | \leqq \sqrt{3} = \tan \pi/3$. We see that $f(x)$ is the product of polynomials with positive coefficients which satisfy the conditions of Lemma 2. Therefore, conditions (1) of Lemma 1 are satisfied and our assertion follows from this lemma.

THEOREM. *Let* $f(x)$ *be a real polynomial of* n^{th} *degree with n distinct roots* $x_1, x_2, \cdots, x_n$, *and put*

(9) $$\Delta = \operatorname*{Min}_{\nu \neq \mu} | x_\nu - x_\mu |.$$

Let $a_1, a_2, \cdots$ *be an arbitrary sequence of positive numbers and for each* $\nu = 1, 2, \cdots$

$$r_\nu = a_1 + \cfrac{1}{a_2 + \cfrac{}{\ddots + \cfrac{1}{a_\nu}}} = \frac{P_\nu}{Q_\nu}, \qquad Q_\nu > 0$$

the ν^{th} *convergent of the corresponding continued fraction, where* P_ν *and* Q_ν *have the usual meaning. If for an index* $m \geqq 2$, *we have*

(11) $$\Delta Q_m Q_{m-1} \geqq \sqrt{3},$$

the polynomial

(12) $$(Q_m \xi + Q_{m-1})^n f\left(\frac{P_m \xi + P_{m-1}}{Q_m \xi + Q_{m-1}}\right) = F(\xi)$$

presents at most one variation of signs.

7. PROOF. If $a \pm ib$ is a pair of conjugate roots of $f(x)$, then the corresponding roots of $F(\xi)$ are given by $\alpha \pm i\beta$ where

$$\alpha + i\beta = -\frac{P_{m-1} - (a + ib)Q_{m-1}}{P_m - (a + ib)Q_m}, \tag{13}$$

and therefore

$$-\alpha = \frac{Q_{m-1}Q_m}{(P_m - aQ_m)^2 + b^2Q_m^2}[b^2 + (r_m - a)(r_{m-1} - a)] \tag{14}$$

$$\beta = \frac{b(P_mQ_{m-1} - P_{m-1}Q_m)}{(P_m - aQ_m)^2 + b^2Q_m^2} = \frac{\pm b}{(P_m - aQ_m)^2 + b^2Q_m^2}. \tag{15}$$

Consider the bracketed expression in (14). If the product $(r_m - a)(r_{m-1} - a) \leqq 0$, then a is situated between r_m and r_{m-1}, and it follows from the inequality relating the geometric and arithmetic means that

$$|(r_m - a)(r_{m-1} - a)| \leqq \frac{(r_m - r_{m-1})^2}{4}. \tag{16}$$

On the other hand it follows from the properties of the convergents of continued fractions and from (11) that

$$|r_{m-1} - r_m| = \frac{1}{Q_mQ_{m-1}} \leqq \frac{\Delta}{\sqrt{3}} \tag{17}$$

and therefore

$$|(r_m - a)(r_{m-1} - a)| \leqq \frac{\Delta^2}{12}. \tag{18}$$

Furthermore, since by definition of Δ

$$2|b| = |a + ib - (a - ib)| \geqq \Delta, \tag{19}$$

(18) implies that $|(r_m - a)(r_{m-1} - a)| \leqq \frac{b^2}{3}$,

$$b^2 + (r_m - a)(r_{m-1} - a) \geqq \frac{2b^2}{3} \tag{20}$$

and this inequality is, of course, also true if $(r_m - a)(r_{m-1} - a)$ is positive. Therefore, we have $-\alpha > 0$ and

$$\left|\frac{\beta}{\alpha}\right| = \frac{|b|}{Q_{m-1}Q_m[b^2 + (r_m - a)(r_{m-1} - a)]}.$$

Hence, in virtue of (11), (19) and (20)

$$\left|\frac{\beta}{\alpha}\right| = \frac{|b|}{Q_{m-1}Q_m[b^2 + (r_m - a)(r_{m-1} - a)]} \leqq \frac{\Delta|b|\cdot 3}{\sqrt{3}\cdot 2b^2} \leqq \sqrt{3}, \tag{21}$$

and we see that the complex roots of $F(\xi)$ satisfy the conditions of Lemma 3.

8. On the other hand, if x_0 is a *real* root of $f(x)$, we obtain for the corresponding root ξ_0 of $F(\xi)$ from (14) with $b = 0$

$$-\xi_0 = \frac{Q_{m-1} Q_m (r_m - x_0)(r_{m-1} - x_0)}{(P_m - x_0 Q_m)^2}, \tag{22}$$

provided $x_0 \neq r_m$. This can only be negative if x_0 lies between r_m and r_{m-1}. But this interval cannot contain more than one root of $f(x)$ as by (17) its length $|r_m - r_{m-1}|$ is $<\Delta$. Therefore, only one real root of $F(\xi)$ can be positive.

The same conclusion holds if one of the roots of $f(x)$ is $= r_m$ since then all other real roots remain outside this interval.

If now $F(\xi)$ has no positive root, it presents by Lemma 3 no variations of signs. If on the other hand $F(\xi)$ has a positive root ξ_0, then we have

$$F(\xi) = (\xi - \xi_0) F^*(\xi),$$

where the roots of $F^*(\xi)$ satisfy the conditions of Lemma 3. Then it follows from this lemma that $F(\xi)$ presents exactly one variation of signs.

Our theorem is proved.

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