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An Unknown Theorem for the Isolation of the Roots of Polynomials

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A little known theorem concerning the isolation of roots of polynomial equations, published in 1836 by the French mathematician A. J. H. Vincent is discussed. This theorem is of great importance because one (of the two) isolation method derived from it turns out to be the fastest existing thus far—if exact integer arithmetic algorithms are used. Certain computational results which offer an empirical comparison of the classical methods are also presented.

1. INTRODUCTION

It is well known that in the beginning of the 19th century the attention of the mathematicians had already focused on numerical methods for the solution of the general equation of degree greater than four. During this period Fourier conceived the idea to proceed in two steps; that is, first to isolate the roots and then to approximate them to any desired degree of accuracy. Approximation is a special topic in itself [13] and will not be discussed in this paper; moreover, we will be concerned only with real roots.

Isolation of the real roots of a polynomial equation is the process of finding real, disjoint intervals such that each contains exactly one real root and every real root is contained in some interval. In order to accomplish this Sturm's method is the only one widely known and used; it was developed in 1829 and is based on a theorem by Fourier, which is found in the literature under the name Budan-Fourier, or, even, Budan!! [4], [11], [18]. However, in 1834, another French mathematician, Alexandre Joseph Hidulphe Vincent [12], [15], also published a "note" (of thirty pages) in the *Mémoires de la Societé royale de Lille* concerning the isolation (and approximation) of the real roots of polynomial equations with numerical coefficients. The same memorandum appeared two years later, with a few additions, under the title "Note sur la résolution des équations numériques" in the October issue, 1836, of the *Journal de Mathématiques Pures et Appliquées* [17]. According to a footnote, the article was reprinted "for the benefit of the professors". The main theorem in Vincent's paper is based on Budan's proposition which is ignored by most of the existing literature, mainly due to its equivalence with the one by Fourier [11], [14], [16], [18]. Nevertheless, the article and the method described therein were consigned to oblivion for more than a century, although it seems that several people had dealt with variations of this method.

We may attempt to explain the fact that Vincent's theorem was forgotten by noting the careful manner in which he pays tribute to Sturm and notes the "beauty" and usefulness of Sturm's celebrated theorem on the location of the roots of the equations [6]. In 1834, the same year in which Vincent first published his paper, Sturm published his work on second order differential equations, known today as the Sturm-Liouville theory, for which he received the "Grand Prix des Sciences Mathématiques" from the Académie des Sciences. Two years later, when Vincent's paper was reprinted, Sturm was elected in the Académie des Sciences. It is, therefore, not surprising that Sturm's method outshone all others. There is, however, another possible reason that discouraged people from using Vincent's theorem—and his method. As we mentioned above, Vincent's theorem is based on Budan's proposition (1807), with the help of which we can obtain, performing the substitutions $x = x' + p$ and $x = x'' + q$, an upper bound on the number of the real roots that an equation has within the interval (p, q) [4]. Using Budan's theorem, Vincent performs transformations of the form $x = y + 1$ (or, equivalently, $x \leftarrow x + 1$); however, in order to obtain the coefficients of the transformed equation, he uses Taylor's expansion, a somewhat inefficient and cumbersome procedure. It was only in the middle of our century that Uspensky simplified Vincent's method considerably by using the Ruffini-Horner method in order to obtain the above mentioned coefficients [9].

So far as we have been able to determine, Vincent's theorem is not mentioned by any author with the exception of Uspensky [16] and Obreschkoff [14]. Uspensky notes that even such a capital work as the *Enzyclopaedie der mathematischen Wissenschaften* ignores it. This little known theorem though, is the basis of two contrasting methods for the isolation of the real roots of polynomial equations; the first, due to Vincent, behaves exponentially, whereas, the second method, due to the first author [2], has the best theoretical computing time bound achieved thus far. Notice that the methods discussed in this paper have been implemented in software systems for computerized algebra, using exact integer arithmetic algorithms.

In what follows we will present Vincent's theorem and the propositions on which it is based. A short description of the two root isolation methods will be given, together with some empirical results for comparison.

2. Vincent's Theorem and its Extension

We begin with some preliminaries. As we know, most methods for the isolation of the roots of polynomials with numerical coefficients rely on the following rule:

Cardano-Descartes rule of signs [8]

Consider the polynomial

$$P(x) = c_n x^n + c_{n-1} x^{n-1} + \ldots + c_1 x + c_0$$

with real coefficients. If v is the number of sign variations in the sequence of coefficients $c_n, c_{n-1}, \ldots, c_0$ (zero coefficients are simply omitted) and p is the number of positive roots of $P(x)$, then

$$v - p = 2\lambda,$$

where $\lambda \geqslant 0$ is an integer.

Notice that the above rule gives the exact number of roots only if there is either one or no sign variation. In the first case there is one positive root, whereas, in the second case there is no root. This observation will be used in Vincent's theorem.

As we mentioned in the Introduction, Vincent's theorem is based on a proposition by Budan which, to our knowledge, can be found only in [17]. (All the books on the theory of equations we have seen, [11], [14], [16], [18], simply mention Fourier's theorem and refer to it as Budan-Fourier or even Budan!! [11], [18].) Vincent renders Budan's theorem as follows:

Theorem 1 (Budan 1807). *If in an equation in* x, $P(x) = 0$, *we make two transformations* $x = p + x'$ *and* $x = q + x''$, *where* p *and* q *are real numbers such that* $p < q$, *then*

(*i*) *the transformed equation in* $x' = x - p$ *cannot have fewer sign variations than the transformed equation in* $x'' = x - q$;

(*ii*) *the number of real roots of the equation* $P(x) = 0$ *located between* p *and* q *can never be more than the number of variations lost in passing from the transformed equation in* $x' = x - p$ *to the transformed equation in* $x'' = x - q$;

(*iii*) *when the first number is less than the second, the difference is always an even number.*

At this point we would like to ask the reader to look up Fourier's theorem in any text on the theory of equations, and to compare it with Theorem 1; although they are equivalent, their statements are completely different. Using Budan's theorem, Vincent carries out consecutive transformations until the transformed equation presents one or no sign variation, in which case the number of roots can be determined unambiguously. We have the following:

THEOREM 2 (VINCENT 1836). *If in a polynomial equation with rational coefficients and without multiple roots one makes successive transformations of the form*

$$x = a_1 + \frac{1}{x'},\ x' = a_2 + \frac{1}{x''},\ x'' = a_3 + \frac{1}{x'''},\ \ldots$$

where each $a_1, a_2, a_3, \ldots$ *is any positive integer, then the resulting transformed equation has either zero or one sign variation. In the latter, the equation has a single positive root represented by the continued fraction*

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}$$

whereas in the former case there is no root.

The proof of this theorem can be found in Vincent's original paper. The negative roots are investigated by replacing x by $-x$ in the original polynomial and by investigating the positive roots of the transformed polynomial. Vincent himself states ([17], p. 342) that Theorem 2 was hinted in 1827 by Fourier, who never did give any proof of it (or if he did, it was never found).

The dependence of Vincent's theorem on the one by Budan is easily seen if each transformation of the form $x = a_i + \frac{1}{y}$ is replaced by the equivalent pair of transformations $x = a_i + y'$, $y' = \frac{1}{y}$. (Observe, also, that the inversion, $y' = \frac{1}{y}$, is easily performed on a given polynomial by simply inverting the order of its coefficients.)

Intuitively speaking, the purpose of the series of successive transformations of the form $x = a_i + \frac{1}{y}$, performed on the equation $P(x) = 0$, is to force one of its positive real roots in the internal $(0, 1)$ and all other in $(1, \infty)$ or vice versa—excluding, of course, the case when 1 is a root. In the first case, the subsequent substitution $x \leftarrow \frac{1}{1+x}$ will result in an equation with only one real root in $(0, \infty)$, whereas in the second case the same is achieved with the subsequent substitution $x \leftarrow 1 + x$. The question naturally arises as to the maximum number of transformations of the form $x = a_i + \frac{1}{y}$, necessary to obtain this polynomial with at most one sign variation. Uspensky ([16] pp. 298–204) extended Vincent's theorem in order to obtain an answer to this question. His treatment, though, contains certain errors, in the statement and the proof, which were corrected in [3]. In what follows we give a new, corrected version of the extension of Vincent's theorem. (The proof can be found in [3], [16].)

THEOREM 3 (VINCENT-USPENSKY-AKRITAS). *Let $P(x) = 0$ be a polynomial equation of degree $n > 1$, with rational coefficients and without multiple roots, and let $\Delta > 0$ be the smallest distance between any two of its roots. Let m be the smallest index such that*

$$F_{m-1}\frac{\Delta}{2} > 1 \text{ and } F_{m-1}F_m\Delta > 1 + \frac{1}{\epsilon_n},$$

where F_k is the k-th member of the Fibonnacci sequence

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ \ldots$$

and

$$\epsilon_n = \left(1 + \frac{1}{n}\right)^{1/(n-1)} - 1.$$

Then the transformation

$$x = a_1 + \cfrac{1}{a_2 + \cfrac{}{\ddots + \cfrac{1}{a_m + \cfrac{1}{\xi}}}} \tag{1}$$

(which is equivalent to the series of successive transformations of the form $x = a_i + \frac{1}{\xi}$, $i = 1, 2, \ldots, m$) presented in the form of a continued fraction with arbitrary, positive, integral elements $a_1, a_2, \ldots, a_m$, transforms the equation $P(x) = 0$ into the equation $\widetilde{P}(\xi) = 0$, which has not more than one sign variation in the sequence of its coefficients.

3. APPLICATIONS OF VINCENT'S THEOREM

Theorem 3 can be used in order to isolate the real roots of a polynomial equation. The fact that it holds only for equations without multiple roots does not restrict the generality, because in the opposite case all we have to do is to express $P(x)$ in the form $P = \prod_{i=1}^{e} S_i^i$, where each of the S_i's has only single roots ([16], pp. 65–69). Each of these single roots is of multiplicity i for the polynomial $P(x)$ and thus we see that the above theorem can be applied on the S_i's. So in the rest of this discussion it is assumed that $P(x) = 0$ is without multiple roots.

From the statement of Theorem 3 we know that a transformation of the form (1), with arbitrary, positive integer elements $a_1, a_2, \ldots, a_m$ transforms $P(x) = 0$ into an equation $\widetilde{P}(\xi) = 0$, which has at most one sign variation; this transformation can be also written as

$$x = \frac{P_m\xi + P_{m-1}}{Q_m\xi + Q_{m-1}}, \tag{2}$$

where $\frac{P_k}{Q_k}$ is the kth convergent to the continued fraction

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}$$

Since the elements $a_1, a_2, \ldots, a_m$ are arbitrary there is obviously an infinite number of transformations of the form (1). However, with the help of Budan's theorem we can easily determine those that are of interest to us: namely, there is a finite number of them (equal to the number of the positive roots of $P(x) = 0$) which lead to an equation with exactly one sign variation. Suppose that $\widetilde{P}(\xi) = 0$ is one of these equations; then from the Cardano-Descartes rule of signs we know that it has one root in the interval $(0, \infty)$. If $\hat{\xi}$ was the positive root, then the corresponding root $\hat{x}$ of $P(x) = 0$ could be easily obtained from (2). We only know though that $\hat{\xi}$ lies in the interval $(0, \infty)$; therefore, substituting ξ in (2) once by 0 and once by ∞ we obtain for the positive root $\hat{x}$ its isolating interval whose unordered end-points are $\frac{P_{m-1}}{Q_{m-1}}$ and $\frac{P_m}{Q_m}$. In this fashion we can isolate all the positive roots of $P(x) = 0$. If we subsequently replace x by $-x$ in the original equation, the negative roots become positive and, hence, they too can be isolated in the way mentioned above. Thus we see that we have a procedure for isolating all the real roots of $P(x) = 0$.

The calculation of the quantities $a_1, a_2, \ldots, a_m$—for the transformations of the form (1) which lead to an equation with exactly one sign variation—constitutes the polynomial real root isolation procedure. Two methods actually result, Vincent's and the one developed by the first author, corresponding to the two different ways in which the computation of the a_i's may be performed; the difference between these two methods can be thought of as being analogous to the difference between the integrals of Riemann and Lebesgue (think of the addition).

Vincent's method basically consists of computing a particular a_i by a series of unit incrementations; that is, $a_i \leftarrow a_i + 1$, which corresponds to the substitution $x \leftarrow x + 1$. This "brute force" approach results in a method with an exponential behavior; that is, for big values of the a_i's this method may take a long time (even years in a computer) in order to isolate the real roots of an equation. Therefore, Vincent's method is of little practical importance. Examples of this approach can be found in Vincent's paper

[17], and in Uspensky's book ([16]) pp. 129–137). The reader should notice that in the preface of his book Uspensky claims that he himself invented this method. A simple comparison with Vincent's paper though makes clear that what can be considered a contribution on Uspensky's part is only the fact that he used the Ruffini-Horner method in order to perform the transformations $x \leftarrow x + 1$, whereas, Vincent used Taylor's expansion theorem [7]. Moreover, Uspensky seems to ignore Budan's theorem and, while computing a particular a_i, he performs, after each transformation $x \leftarrow x + 1$, the unnecessary transformation $x \leftarrow \frac{1}{x+1}$, something which Vincent avoids. The exponential nature of Vincent's method motivated Collins and Akritas to develop a new method with polynomial computing time bound [10]; this method, however, is based on a modified version of Vincent's theorem and does not take advantage of the continued fractions [5].

On the contrary, the method developed by the first author is an aesthetically pleasing interpretation of Theorem 3. Basically it consists of immediately computing a particular a_i as the lower bound b on the values of the positive roots of a polynomial; that is, $a_i \leftarrow b$ which corresponds to the substitution $x \leftarrow x + b$ performed on the particular polynomial under consideration. It is obvious that this method is independent of how big the values of the a_i's are. (An unsuccessful treatment of the big values of the a_i's can be found in Uspensky's book ([16] p. 136). In this discussion it is assumed that $b = \lfloor \alpha_s \rfloor$ where α_s is the smallest positive root.) Since the substitutions $x \leftarrow x + 1$ and $x \leftarrow x + b$ can be performed in about the same time [7], we can easily see that our method results in enormous savings of computing time. It turns out that our method is the fastest existing for the isolation of the real roots of a polynomial equation, when exact integer arithmetic algorithms are used.

4. Empirical Results and Conclusions

In what follows we present two tables with the computation times in seconds. They were obtained by using the SAC-1 computer algebra system on the IBM S/370 Model 165 computer located at the Triangle Universities Computation Center (North Carolina) [1]. Table 1 compares Sturm's method with the one developed by the first author for randomly generated polynomials of degrees 5–20; clearly, for this class of polynomials Sturm's method is completely out of the race. Table 2, on the contrary, compares Vincent's method with ours for polynomials of degree 5 with randomly generated roots i.e. each polynomial is the product of 5 linear terms. In this case the exponential nature of Vincent's method is obvious.

We see, therefore, that based on the forgotten theorem of an unknown French mathematician, contemporary of Sturm, we have been able to develop a new method for the isolation of the real roots of polynomial

TABLE 1

POLYNOMIALS WITH RANDOMLY GENERATED COEFFICIENTS

Degree	Sturm	Method Developed by the First Author
5	2.05	.26
10	33.28	.46
15	156.40	.94
20	524.42	2.36

TABLE 2

POLYNOMIALS OF DEGREE 5 WITH RANDOMLY GENERATED ROOTS

Roots are in the interval	Vincent	Method Developed by the First Author
$(0, 10^2)$	.45	.16
$(0, 10^3)$	1.61	.71
$(0, 10^4)$	16.43	2.01
$(0, 10^5)$	175.62	4.81

equations; this method by far surpasses not only Sturm's but also all others recently developed using exact integer arithmetic algorithms.

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