ON THE BUDAN-FOURIER CONTROVERSY

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In this short note we deal with the theorems by Budan and Fourier regarding the solution of polynomial equations with rational coefficients [1].

Budan's theorem appeared in 1807 in the memoir:

> "Nouvelle méthode pour la résolution des équations numériques",

whereas, Fourier's theorem was first published in 1820 in:

> "Le Bulletin des sciences par la Société Philomatique de Paris" p. 156, 181.

Although, as we will see, these two theorems differ in their statement, they are equivalent and they both give an upper bound on the number of the real roots that an equation has within a real interval (multiplicities counted).

Due to the importance of the two theorems mentioned above, there was a great controversy regarding priority rights. In his book (1859)

> "Biographies of Distinguished Scientific Men " p. 383,

F. Arago informs us that Fourier "deemed it necessary to have recourse to the certificates of early students of the Polytechnic School or Professors of the University" in order to prove that he had taught his theorem in 1796, 1797, and in 1803.

In what follows we indicate that these two theorems lead to two completely different methods for the isolation of the real roots of an equation, and, hence, each one deserves its own place in the History of Mathematics. We only present the statement of Budan's theorem since it hardly appears in the literature. On the contrary, Fourier's theorem can be found in every text on the theory of equations and hence the following observations suffice:

F_1: Fourier's theorem appears under the names Budan-Fourier [3] or even Budan!! [6]

F_2: For a given equation $P(x) = 0$, Fourier's theorem makes use of the Fourier sequence, i.e., $P(x), P^{(1)}(x), P^{(2)}(x), \ldots, P^{(m)}(x)$.

Based on Fourier's theorem Sturm, in 1829, presented his own proposition with the help of which we can obtain the exact number of the real roots of an equation inside a given interval. What Sturm did was to simply substitute Fourier's sequence with what we nowadays call Sturm's sequence or chain. This new theorem results in Sturm's method for the isolation of the real roots of an equation, a method widely known and used since 1830. Moreover, it should be noted that Sturm's method basically subdivides the given interval until the root isolation is achieved; that is, in this case we are dealing with a bisection method. (More details on Sturm's method can be found in books on the theory of equations or in texts on numerical analysis.)

Let us now see how Budan's theorem leads to a new method for the isolation of the roots of a polynomial equation, a method which uses continued fractions.

Budan's theorem (1807). If in an equation in x, $P(x) = 0$, we make two transformations, $x = p + x'$ and $x = q + x''$, where p and q are real numbers such that $p < q$, then

(i) the transformed equation in $x' = x - p$ cannot have fewer sign variations than the transformed equation in $x'' = x - q$;

(ii) the number of real roots of the equation $P(x) = 0$, located between p and q, can never be more than the number of sign variations lost in passing from the transformed equation in $x' = x - p$ to the transformed equation in $x'' = x - q$;

(iii) when the first number is less than the second, the difference is always an even number.

NOTE: In (iii) the "first number" implies the real roots located between p and q; the "second number", on the other hand, refers to the number of sign variations lost in passing from the transformed equation in $x' = x - p$ to the transformed equation in $x'' = x - q$.

The reader should notice that the last theorem differs from the one by Fourier in that it uses translations instead of Fourier's sequence. In addition, Budan's theorem is the basis of Vincent's forgotten theorem of 1836, which is rendered as follows [5]:

Vincent's Theorem (1836). If in a polynomial equation with rational coefficients and without multiple roots one makes successive transformations of the form

$$x = a_1 + \frac{1}{x'}, x' = a_2 + \frac{1}{x''}, x'' = a_3 + \frac{1}{x'''}, \ldots$$

where each $a_1, a_2, a_3, \ldots$ is any positive integer, then the resulting transformed equation has either zero or one sign variation. In the latter the equation has a single positive real root represented by the continued fraction

$$a_1 + \frac{1}{a_2} + \frac{1}{a_3} + \cdots$$

whereas, in the former case there is no root.

Vincent's theorem can be used in order to isolate the real roots of an equation. Actually, there are two root isolation methods corresponding to the two different ways in which the partial quotients a_i may be computed.

The first method is due to Vincent himself and dates back to 1836. However, it is exponential in nature and hence of little practical importance. (At this point we would like to mention that Uspensky [4] in his treatment of Vincent's method, seems to ignore Budan's theorem, and after each translation $x \leftarrow 1 + x$, he performs the unnecessary transformation $x \leftarrow \frac{1}{1+x}$, something which Vincent avoids.)

The second method (which is the fastest existing [2] when exact integer arithmetic is used) is due to the author of this paper and is based on the observation that each partial quotient a_i is the lower bound on the value of the positive roots of an equation. Note, that our method is the only one with polynomial computing time bound which uses <u>continued fractions</u> (for more details see [2]).

The following figure summarizes what we have said so far.

<u>Resumé</u>

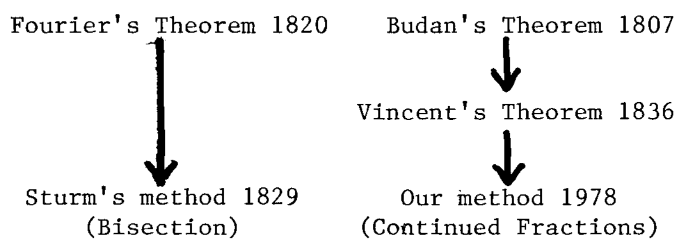

Figure 1

We see, therefore, that based on Budan's theorem we obtain a new method for the isolation of the real roots of an equation, a method which is completely different from Sturm's.

<u>References</u>

[1] Akritas, A.G., On the Budan-Fourier Controversy. Abstracts of papers presented to the American Mathematical Society, Vol. 1, No. 5, 443, 1980.

[2] Akritas, A.G., The fastest exact algorithms for the isolation of the real roots of a polynomial equation. Computing, Vol. 24, 299-313, 1980.

[3] Obreschkoff, N., Verteilung und Berechnung der Nullstellen reeller Polynome. VEB Deutscher Verlag der Wissenschaften, Berlin, 1963.

[4] Uspensky, J.V., Theory of equations, McGraw-Hill Co., New York, 1948.

[5] Vincent, A.J.H., Sur la résolution des équations numériques, Journal des Mathématiques Pures et Appliquées, Vol. 1, 341-372, 1836.

[6] Weisner, L., Introduction to the theory of equations. The McMillan Co., New York, 1938.