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VINCENT'S FORGOTTEN THEOREM, ITS EXTENSION AND APPLICATION

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Abstract—Vincent's theorem of 1836, which was only recently discovered by the author of this article, is of extreme importance because it consitutes the basis of the fastest method existing for the isolation of the real roots of a polynomial equation (using exact integer arithmetic). In this paper this forgotten theorem is presented both in its original form and in an extended version, and is followed by a general discussion of its application.

1. INTRODUCTION

In the theory of equations it is well known that in the beginning of the 19th century the attention of the mathematicians had been focused on numerical methods for the solution of algebraic equations. During this period Fourier conceived the idea to proceed in two steps; that is, first to isolate the real roots and then to approximate them to any desired degree of accuracy.

Isolation of the real roots of a polynomial equation is the process of finding real, disjoint intervals such that each contains exactly one real root and every real root is contained in some interval. Since 1830 the only method widely known and used for this purpose is that of Sturm; it has been implemented in a computer algebra system—using exact integer arithmetic—and proven to be[1]

$$0(n^{13}L(|P|_\infty)^3),$$

where n is the degree of the square-free polynomial equation $P(x) = 0$ and $L(|P|_\infty)$ the length, in bits, of the maximum coefficient in absolute value.†

Quite recently, in Uspensky's *Theory of Equations* ([2] pp. 127–137) the author of this article discovered Vincent's forgotten theorem of 1836[3, 4], according to which, if a univariate polynomial equation with rational coefficients and without any multiple roots is successively transformed by transformations of the form $x = a_i + (1/\xi)$, for arbitrary, positive, integer elements a_i, one eventually obtains an equation with at most one sign variation in the sequence of its coefficients. As we will see, this theorem can also be used for the isolation of the real roots. However, as the reader observes, the statement of Vincent's theorem is incomplete because it does not provide a bound on the number of transformations of the form $x = a_i + (1/\xi)$, which have to be performed in order to obtain the equation with at most one sign variation. Such a bound is given, though, by the extended Vincent theorem, which was presented in a somewhat erroneous manner by Uspensky ([2], pp. 298–304),[5].

Two root isolation methods result from the above theorem, Vincent's and ours, corresponding to the two different ways of computing the a_i's[6, 7]. It has been shown[7] that Vincent's method behaves exponentially, whereas ours has the polynomial computing time bound

$$0(n^5L(|P|_\infty)^3),$$

which in fact is the best one achieved thus far using exact integer arithmetic[8].

In what follows, Vincent's forgotten theorem is presented both in its original form and in an extended version and is followed by a general discussion of its application.

†For a survey of computer algebra systems see *Proceedings of the Second Symposium on Symbolic and Algebraic Manipulation*, (Edited by S. R. Petrick), March, 1971, available from ACM.

2. VINCENT'S THEOREM AND ITS EXTENSION

Before we state Vincent's theorem we begin with the following:

Definition 1

We say that a *sign variation* exists between two numbers c_p and $c_p (p < q)$ of a finite or infinite sequence of real numbers

$$c_1, c_2, c_3, \dots,$$

if c_p and c_q are not zero and have opposite signs, and in case $q \geq p + 2$ (that is, c_q does not immediately follow c_p) the numbers $c_{p+1}, \dots, c_{q-1}$ are all zero.

THEOREM 1 (*Cardano–Descartes rule of signs*)

The number p of the positive roots of a polynomial equation with real coefficients

$$c_0 x^n + c_1 x^{n-1} + \cdots + c_{n-1} x + c_n = 0$$

is never greater than the number v of sign variations in the sequence of its coefficients $c_0, c_1, c_2, \dots, c_n$; if $v - p > 0$ then it is an even number.

The proof of the above theorem can be found in any text on the theory of equations. Subsequently we may say that a polynomial "has" or "presents" v sign variations, instead of using the lengthier terminology of Theorem 1.

A closer examination of Theorem 1 reveals that it is a rather weak proposition; it gives us the exact number of positive roots only in the following two special cases: (i) if there is no sign variation there is no positive root, and (ii) if there is one sign variation there is one positive root. As we will subsequently see, these two special cases are of great importance. Moreover, the converse of (i) is also true because we have:

LEMMA 1. (*Stodola* [9] p. 105)

If the polynomial equation

$$P(x) = c_0 x^n + c_1 x^{n-1} + \cdots + c_n = 0 \qquad (c_0 > 0)$$

with real coefficients c_k, $k = 0, 1, 2, \dots, n$, has only roots with negative real parts, then all its coefficients are positive, and hence, they present no sign variation.

Regarding the second special case of Theorem 1, we observe that the converse is not in general true as can be seen from the polynomial $x^3 - x^2 + 2x - 2 = (x - 1)(x - \sqrt{2}i)(x + \sqrt{2}i)$. However, under more restrictive conditions the desired proposition is true; formally this is stated as follows:

LEMMA 2. ([6] pp. 63–66)

Let $P(x) = 0$ be a polynomial equation of degree $n > 1$, without multiple roots, which has one positive real root $\xi \neq 0$ and $n - 1$ roots $\xi_1, \xi_2, \dots, \xi_{n-1}$ with negative real parts—the complex roots appearing in conjugate pairs—and which can be expressed in the form

$$\xi_j = -(1 + \alpha_j), \quad j = 1, 2, \dots, n - 1$$

with $|\alpha_j| < \epsilon_n$, where

$$\epsilon_n = \left(1 + \frac{1}{n}\right)^{(1/(n-1))} - 1.$$

Then $P(x)$, in its expanded form, presents exactly one sign variation.

Having thoroughly analyzed the two special cases of Theorem 1, we can now state Vincent's theorem which depends heavily upon them.

THEOREM 2 (*Vincent* 1836[4])

If in a polynomial equation with rational coefficients and without multiple roots one makes successive transformations of the form

$$x = a_1 + \frac{1}{x'},\quad x' = a_2 + \frac{1}{x''},\quad x'' = a_3 + \frac{1}{x'''},\ldots,$$

where each $a_1, a_2, a_3, \ldots$ is any positive integer, then the resulting, transformed equation has either zero or one sign variation. In the latter, the equation has a single positive real root represented by the continued fraction

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}$$

whereas in the former case there is no root.

The proof of this theorem can be found in Vincent's original paper[4]. Vincent himself states that Theorem 2 was hinted in 1827 by Fourier who never did give any proof of it, or if he did it was never found. As Uspensky notes ([2], p. 298) Vincent's theorem—which is based on an earlier theorem by Budan[10]—was so totally forgotten that even such a capital work as the *Enzyclopaedie der mathematischen Wissenschaften* ignores it. As far as we have been able to determine, Vincent's theorem is not mentioned by any authors with the exception of Uspensky[2] and Obreschkoff[9]. The author discovered it while reviewing methods for the isolation of the real roots of equations as presented by Uspensky.

The question naturally arises as to the maximum number of transformations of the form $x = a_i + (1/\xi)$, necessary to obtain the polynomial with at most one sign variation. Uspensky ([2], pp. 298–304) extended Vincent's theorem in order to obtain an answer to this question. His treatment though contains certain errors, in the statement and the proof, which were corrected in[2]. In what follows we give a new, corrected version of the extension of Vincent's theorem; for completeness we also add its proof, which is much shorter than the one by Uspensky ([2] pp. 298–304) due to the fact that we use Lemma 2.

THEOREM 3 (*Vincent–Uspensky–Akritas* [8])

Let $P(x) = 0$ be a polynomial equation of degree $n > 1$, with rational coefficients and without multiple roots, and let $\Delta > 0$ be the smallest distance between any two of its roots. Let m be the smallest index such that

$$F_{m-1}\frac{\Delta}{2} > 1 \quad\text{and}\quad F_{m-1}F_m\Delta > 1 + \frac{1}{\epsilon_n} \tag{1}$$

where F_k is the kth member of the Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

and

$$\epsilon_n = \left(1 + \frac{1}{n}\right)^{(1/(n-1))} - 1. \tag{2}$$

Then the transformation

$$x = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_m + \cfrac{1}{\xi}}}} \tag{3}$$

(which is equivalent to the series of successive transformations of the form $x = a_i + (1/\xi)$, $i = 1, 2, \ldots, m$) presented in the form of a continued fraction with arbitrary, positive, integral elements $a_1, a_2, \ldots a_m$, transforms the equation $P(x) = 0$ into the equation $\tilde{P}(\xi) = 0$, which has not more than one sign variation.

Proof. In order to prove the theorem, it suffices to show that, after the m successive transformations of the form $x = a_i + (1/\xi)$, the real parts of all complex roots, as well as all real roots except for at most one, become negative.

Indeed, let (P_k/Q_k) be the kth convergent to the continued fraction

$$a_1 + \frac{1}{a_2 +} \frac{1}{a_3 +} \cdots$$

From the law of convergents we have:

$$P_{k+1} = a_{k+1}P_k + P_{k-1},$$
$$Q_{k+1} = a_{k+1}Q_k + Q_{k-1}.$$

Since $Q_1 = 1$ and $Q_2 = a_2 \geqq 1$, it follows that $Q_k \geqq F_k$. Further, the relation (3) can be expressed in the form

$$x = \frac{P_m\xi + P_{m-1}}{Q_m\xi + Q_{m-1}},$$

from which it follows that

$$\xi = -\frac{P_{m-1} - Q_{m-1}x}{P_m - Q_m x}. \tag{4}$$

Clearly, if x_0 is any root of the equation $P(x) = 0$, the quantity ξ_0, determined by (4), is the corresponding root of the transformed equation $\tilde{P}(\xi) = 0$.

(a) Assume that x_0 is a complex root of $P(x) = 0$; that is $x_0 = a \pm ib$, $b \neq 0$. In this case the real part of the corresponding root ξ_0 is

$$\text{r.p.}(\xi_0) = -\frac{(P_{m-1} - Q_{m-1}a)(P_m - Q_m a) + Q_{m-1}Q_m b^2}{(P_m - Q_m a)^2 + Q_m^2 b^2}. \tag{5}$$

This is certainly negative if

$$(P_{m-1} - Q_{m-1}a)(P_m - Q_m a) \geqq 0.$$

If, on the contrary

$$(P_{m-1} - Q_{m-1}a)(P_m - Q_m a) < 0,$$

then clearly the value of a is contained between the two consecutive convergents

$$\frac{P_{m-1}}{Q_{m-1}}, \frac{P_m}{Q_m},$$

whose difference in absolute value is

$$\frac{1}{Q_{m-1}Q_m}.$$

Hence,

$$\left|\frac{P_{m-1}}{Q_{m-1}} - a\right| < \frac{1}{Q_{m-1}Q_m} \quad \text{and} \quad \left|\frac{P_m}{Q_m} - a\right| < \frac{1}{Q_{m-1}Q_m},$$

from which it follows that

$$|(P_{m-1} - Q_{m-1}a)(P_m - Q_m a)| < \frac{1}{Q_{m-1}Q_m} \leqq 1. \tag{6}$$

From (5) and (6) we conclude that the r.p. (ξ_0) will be negative if

$$Q_{m-1}Q_m b^2 > 1.$$

To prove that this is true in our case, first observe that, since Δ is the minimum distance between any two roots of $P(x) = 0$, we have

$$|(a + ib) - (a - ib)| = |2ib| = 2|b| \geqq \Delta,$$

from which we obtain $|b| \geqq (\Delta/2)$; morover, we known that $Q_m \geqq Q_{m-1} \geqq F_{m-1}$, and, from (1), $F_{m-1}\,(\Delta/2) > 1$. Then clearly $F_{m-1}|b| > 1$, which implies $Q_{m-1}|b| > 1$ and $Q_m|b| > 1$. From the last two inequalities we obtain $Q_{m-1}Q_m b^2 > 1$, proving thus, that the r.p. $(\xi_0) < 0$; this is obviously true for all complex roots of the transformed equation $\tilde{P}(\xi) = 0$.

(b) Assume now that x_0 is a real root of $P(x) = 0$. Suppose first that for all real roots x_i,

$$(P_{m-1} - Q_{m-1}x_i)(P_m - Q_m x_i) > 0.$$

From (4) it follows that all real roots of the transformed equation $\tilde{P}(\xi) = 0$ will be negative; moreover, we know from (a), that all the complex roots of $\tilde{P}(\xi) = 0$ have negative real parts. Consequently, due to Lemma 1, $\tilde{P}(\xi)$ presents no sign variation. Suppose, now, that for some real root x_0

$$(P_{m-1} - Q_{m-1}x_0)(P_m - Q_m x_0) \leqq 0. \tag{7}$$

Then, clearly, x_0 is contained between the two consecutive convergents

$$\frac{P_{m-1}}{Q_{m-1}}, \; \frac{P_m}{Q_m}$$

and hence,

$$\left|\frac{P_m}{Q_m} - x_0\right| \leqq \frac{1}{Q_{m-1}Q_m}.$$

Let x_k, $k \neq 0$, be any other root, real or complex, of $P(x) = 0$, and ξ_k the corresponding root of the transformed equation. Then, keeping in mind that

$$P_m Q_{m-1} - P_{m-1}Q_m = (-1)^m,$$

it follows from (4) that

$$\xi_k + \frac{Q_{m-1}}{Q_m} = \frac{(-1)^m}{Q_m(P_m - Q_m x_k)}$$

or

$$\xi_k = -\frac{Q_{m-1}}{Q_m}\left[1 - \frac{(-1)^m}{Q_{m-1}Q_m\left(\frac{P_m}{Q_m} - x_k\right)}\right] = -\frac{Q_{m-1}}{Q_m}(1+\alpha_k),$$

where

$$\alpha_k = \frac{(-1)^{m-1}}{Q_{m-1}Q_m\left(\frac{P_m}{Q_m} - x_k\right)}.$$

Now,

$$\left|\frac{P_m}{Q_m} - x_k\right| = \left|\frac{P_m}{Q_m} - x + x - x_k\right| \geqq |x - x_k| - \left|\frac{P_m}{Q_m} - x\right| \geqq \Delta - \frac{1}{Q_{m-1}Q_m} > 0,$$

and consequently

$$|\alpha_k| \leqq \frac{1}{Q_{m-1}Q_m\Delta - 1} \leqq \frac{1}{F_{m-1}F_m\Delta - 1};$$

from the last expression and the second inequality of (1) we deduce that

$$|\alpha_k| < \epsilon_n.$$

Thus, the roots ξ_k, $k = 1, 2, \ldots, n-1$, of the transformed equation, corresponding to the roots x_k, $k = 1, 2, \ldots, n-1$, of the equation $P(x) = 0$, which are all different from x_0, are of the form

$$\xi_k = -\frac{Q_{m-1}}{Q_m}(1+\alpha_k), \quad |\alpha_k| < \epsilon_n; \tag{8}$$

that is, the roots of the transformed equation have negative real parts and are clustered together around -1. If we make the substiutions

$$\xi = \frac{Q_{m-1}}{Q_m}u, \quad \xi_k = \frac{Q_{m-1}}{Q_m}\bar{\xi}_k, \quad k = 0, 1, \ldots, n-1,$$

where,

$$\bar{\xi}_0 > 0 \quad \text{and} \quad \bar{\xi}_k = -(1+\alpha_k), \quad k = 1, 2, \ldots, n-1,$$

the transformed polynominal $\tilde{P}(\xi)$ can be written in the form

$$\tilde{P}(\xi) = \left(\frac{Q_{m-1}}{Q_m}\right)^n \bar{P}(u) = c\left(\frac{Q_{m-1}}{Q_m}\right)^n (u - \bar{\xi}_0)(u - \bar{\xi}_1)\ldots(u - \bar{\xi}_{n-1}).$$

Since $\bar{P}(u)$ satisfies all the assumptions of Lemma 2, it presents exactly one sign variation, and, obviously, the same is true for the transformed polynomial $\tilde{P}(\xi)$. The last thing to consider now is the case when (7) holds as an equality; that is

$$(P_{m-1} - Q_{m-1}x_0)(P_m - Q_m x_0) = 0.$$

If $P_{m-1} - Q_{m-1}x_0 = 0$ then we see, from (4), that $\xi_0 = 0$, and clearly the transformed equation $\tilde{P}(\xi) = 0$ has no sign variation (Lemma 1). In the case $P_m - Q_m x_0 = 0$ we have $\xi_0 = \infty$ and the transformed equation reduces to degree $n-1$. Since again all the roots have negative real parts, we conclude, from Lemma 1, that $\tilde{P}(\xi) = 0$ presents no sign variation. Thus we have proved the theorem completely.//

From the above theorem we clearly see that m is the desired bound on the number of transformations of the form $x = a_i + (1/\xi)$ which have to be performed in order to obtain the equation with at most one sign variation in the sequence of its coefficients.

3. GENERAL DISCUSSION

We first show that the generality of Theorem 3 is in no way restricted by the assumption that the polynomial equation $P(x) = 0$ should not have multiple roots. (For convenience we consider $P(x)$ to be a primitive polynomial; that is the greatest common divisor (g.c.d.) of its coefficients is 1.) The following theorem will be used.

THEOREM 4

Let G be a Gaussian ring (or unique factorization domain) of characteristic zero and P a primitive, nonconstant polynomial in $G[x]$. Let $P = P_1^{e_1} \ldots P_n^{e_n}$ be the unique factorization of P into irreducible factors and P' its derivative. Then g.c.d. $(P, P') = P_1^{e_1-1} \ldots P_n^{e_n-1}$.

The proof of this theorem is quite obvious and is left as an exercise for the reader. Note that the integral domain I of the integers is a Gaussian ring.

Let now P be an integral, primitive, univariate polynomial of positive degree and let $P = P_1^{e_1} \ldots P_n^{e_n}$ be the unique factorization of P into irreducible factors P_i, where for all i, $e_i > 0$. Let $e = \max(e_1, \ldots, e_n)$ and for $1 \le i \le e$ define $J_i = \{j: e_j = i\}$ and

$$S_i = \prod_{j \in J_i} P_j.$$

There follows that $P = \prod_{i=1}^{e} S_i^i$, where some of the S_i's may be 1. This is called the *square-free factorization of P*. (A polynomial S is called *square-free* in the case where there is no polynomial Q of positive degree such that Q^2 divides S.) Each of the square-free factors S_i, $1 \le i \le e$ has simple roots, which are of multiplicity i for the polynomial P. Therefore, if we wish to isolate the real roots of $P(x) = 0$ in the case where there are multiple roots present, all we have to do is to obtain the square-free factors of P and then apply Theorem 3 to each one of them.

The square-free factors—of the polynomial P mentioned above—are obtained with the help of Theorem 4. Indeed, notice that

$$R = \text{g.c.d.}(P, P') = \prod_{i=1}^{n} P_i^{e_i-1} = \prod_{i=2}^{e} S_i^{i-1}$$

The greatest square-free divisor of P is

$$T = \frac{P}{R} = \prod_{i=1}^{n} P_i = \prod_{i=2}^{e} S_i.$$

and hence

$$V = \text{g.c.d.}(R, T) = \prod_{i=2}^{e} S_i.$$

As a result of the above we have

$$S_1 = \frac{T}{V}.$$

Repeating the process with R in place of P we can compute S_2 and eventually obtain all the square-free factors of P. The algorithm for the above process is quite obvious and a detailed description of it can be found elsewhere ([6], pp. 30–31); it has been shown that its computing time bound is

$$0(n^6 + n^4 L(|P|_1)^2),$$

where $|P|_1$ is the sum-norm.

We can now focus our attention on how Theorem 3 is used in order to isolate the real roots of an integral polynomial equation $P(x) = 0$, which does not have multiple roots. From the statement of the theorem we know that a transformation of the form (3), with arbitrary, positive, integer elements $a_1, a_2, \ldots, a_m$ transforms $P(x) = 0$ into an equation $\tilde{P}(\xi) = 0$, which has *at most* one sign variation; this transformation can be also written as

$$x = \frac{P_m\xi + P_{m-1}}{Q_m\xi + Q_{m-1}}, \tag{9}$$

where the P_i's and Q_i's ($i = m-1, m$) are defined in the beginning of the proof of Theorem 3. Since the elements $a_1, a_2, \ldots a_m$ are arbitrary, there is obviously an infinite number of transformations of the form (3). However, with the help of Budan's[10] theorem we can easily determine those that are of interest to us; namely, there is a finite number of them (equal to the number of positive roots of $P(x) = 0$) which lead to an equation with *exactly* one sign variation. Suppose that $\tilde{P}(\xi) = 0$ is one of these equations; then from the Cardano–Descartes rule of signs we know that it has one root in the interval $(0, \infty)$. If $\hat{\xi}$ was this positive root, then the corresponding root $\hat{x}$ of $P(x) = 0$ could be easily obtained from (9). However, we only know that $\hat{\xi}$ lies in the interval $(0, \infty)$. Therefore, substituting ξ in (9) once by 0 and once by ∞ we obtain for the positive root $\hat{x}$ its isolating interval, whose unordered endpoints are (P_{m-1}/Q_{m-1}), and (P_m/Q_m). In this fashion we can isolate all the positive roots of $P(x) = 0$. If we subsequently replace x by $-x$ in the original equation, the negative roots will become positive and hence they, too, can be isolated in the way mentioned above. Thus we see that we have a procedure for isolating all the real roots of $P(x) = 0$.

As we mentioned in the Introduction the calculation of the quantities $a_1, a_2, \ldots, a_m$—for the transformations of the form (3) which lead to an equation with exactly one sign variation—constitutes the polynomial real root isolation procedure. Two methods actually result, Vincent's and ours, corresponding to the two different ways in which the computation of the a_i's may be performed; the difference between these two methods can be thought of as being analogous to the difference between the integrals of Riemann and Lebesgue (think of the addition).

Vincent's method basically consists of computing a particular a_i by a series of unit incrementations, i.e. $a_1 \leftarrow a_i + 1$, which corresponds to the substitution $x \leftarrow x + 1$. This "brute force" approach results in a method with an exponential behavior (in the length of the biggest coefficient in absolute value) and hence of little practical importance. Examples of this approach can be found in[2] and [4].

On the contrary, we think that our method is an aesthetically pleasing interpretation of Theorem 3; basically it consists of immediately computing a particular a_i as the lower bound b on the values of the positive roots of a polynomial, i.e. $a_i \leftarrow b$, which corresponds to the substitution $x \leftarrow x + b$ (performed on the particular polynomial under consideration). (An unsuccessful treatment of the big values of the a_i's can be found in ([2], p. 136).) Since the substitutions $x \leftarrow x + 1$ and $x \leftarrow x + b$ can be performed in about the same time[11], we easily see that our method results in enormous savings in computing time. We have implemented our method in a computer algebra system and have been able to show that its computing time bound is

$$0(n^5 L(|P|_\infty)^3),$$

which is the best one achieved thus far; empirical results also verify the superiority of our method over all others existing[8]. In order to obtain this computing time bound we needed—

among other things—a lower bound for Δ, the smallest distance between any two roots. This is given by the following:

THEOREM 5 (Mahler [12])

If $P(x)$ is an integral, univariate polynomial of degree $n \geq 2$ then

$$\Delta \geq \sqrt{(3)}\, .\, n^{(-(n+2)/2)} .\, |P|_1^{-(n-1)}$$

where $|P|_1$ is the sum-norm and $\Delta = \min_{1 \leq i < j \leq k} |\alpha_i - \alpha_j|$, if $\alpha_1, \alpha_2, \ldots, \alpha_k$ are the k distinct roots of $P(x)$; in case $k = 1, \Delta = \infty$.

More details regarding the computing time analysis of our method can be found in [6] and [8]. It should be pointed out that the algorithms described in these references use exact integer arithmetic, which is equivalent to exact rational arithmetic. Our method has not yet been implemented using machine numbers, and, therefore, we cannot say anything about its behavior in such an environment or about rounding errors.

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