Reflections on a Pair of Theorems by Budan and Fourier

ALKIVIADIS G. AKRITAS
University of Kansas
Lawrence, KS 66045

Isolation of the real roots of a polynomial equation is the process of finding real, disjoint intervals such that each contains exactly one real root and every real root is contained in some interval. This process is quite important because, as J. B. J. Fourier pointed out, it constitutes the first step toward the solution of general equations of degree greater than four, the second step being the approximation of roots to any desired degree of accuracy.

In the beginning of the 19th century F. D. Budan and J. B. J. Fourier presented two different (but equivalent) theorems which enable us to determine the maximum possible number of real roots that an equation has within a given interval.

Budan's theorem appeared in 1807 in the memoir "Nouvelle méthode pour la résolution des équations numériques" [**10**, p. 219], whereas Fourier's theorem was first published in 1820 in "Le Bulletin des sciences par la Société Philomatique de Paris," pp. 156, 181 [**10**, p. 223]. Due to the importance of these two theorems, there was a great controversy regarding priority rights. In his book (1859) "Biographies of distinguished scientific men," p. 383, F. Arago informs us that Fourier "deemed it necessary to have recourse to the certificates of early students of the Polytechnic School or Professors of the University" in order to prove that he had taught his theorem in 1796, 1797 and 1803 [**10**]. Based on Fourier's proposition, C. Sturm presented in 1829 an improved theorem whose application yields the exact number of real roots which a polynomial equation without multiple zeros has within a real interval; thus he solved the real root isolation problem. Since 1830 Sturm's method has been the only one widely known and used, and consequently Budan's theorem was pushed into oblivion. To our knowledge, Budan's theorem can be found only in [**16**] and [**6**] whereas Fourier's proposition appears in almost all texts on the theory of equations. We feel that Budan's theorem merits special attention because it constitutes the basis of Vincent's forgotten theorem of 1836 which, in turn, is the foundation of our method for the isolation of the real roots of an equation [**1**], a method which far surpasses Sturm's in efficiency [**2**], [**3**].

In the discussion which follows we present separately, and without proofs, the classical theorems by Fourier and Budan and we indicate how they lead to the corresponding real root isolation methods. Some empirical results are also presented for comparison.

Fourier's theorem

Fourier's theorem, first published in 1820, was also included in his *Analyse des Equations*, published posthumously by C. L. M. N. Navier in 1831. Found in almost all texts on the theory of equations, it is sometimes given under the name Budan-Fourier or even Budan [**9**], [**17**]. Hurwitz [**12**] presents it as a special case of a more general theorem and Obreschkoff [**13**, pp. 76–87] generalizes it for complex roots. The statement given below is the way it is rendered by Vincent [**16**, p. 342]. We must first define the notion of sign variation.

DEFINITION. We say that a sign variation exists between two nonzero numbers c_p and c_q $(p<q)$ of a finite or infinite sequence of real numbers $c_1, c_2, c_3, \ldots,$ if the following holds:

for $q=p+1$, c_p and c_q have opposite signs;
for $q \geqslant p+2$, the numbers $c_{p+1}, \ldots, c_{q-1}$ are all zero and c_p and c_q have opposite signs.

THEOREM 1 (Fourier 1820). *If in the sequence of the $m+1$ functions $P(x), P^{(1)}(x), \ldots, P^{(m)}(x)$ (where $P^{(i)}=$ the ith derivative), we replace x by any two real numbers p, q $(p<q)$ and if we represent the two resulting sequences of numbers by $\tilde{P}$ and $\tilde{Q}$, then*

(i) *the sequence* $\tilde{P}$ *cannot present fewer sign variations than the sequence* $\tilde{Q}$;

(ii) *the number of real roots of the equation* $P(x)=0$, *located between* p *and* q, *can never be more than the number of sign variations lost in passing from the substitution* $x=p$ *to the substitution* $x=q$;

(iii) *when the first number is less than the second, the difference is an even number.*

The sequence of the $m+1$ derivatives is called **Fourier's sequence.** In (iii) the "first number" means the number of the real roots of $P(x)=0$ located between p and q; the "second number," on the other hand, refers to the number of sign variations lost in passing from the substitution $x=p$ to the substitution $x=q$. Obviously, Fourier's theorem gives an upper bound on the number of real roots which the equation $P(x)=0$ (of degree m) has inside the interval (p,q).

We remind the reader that the two main subjects of Fourier's life work were the theory of heat and the theory of the solution of numerical equations. Both of these subjects were carried forward by Sturm, who had personal and scientific relations with Fourier [**8**]. The manuscript of Fourier's treatise on the solution of numerical equations was by 1829 communicated to several persons including Sturm, who mentions explicitly what a great influence it had on his own work.

What Sturm did was to replace Fourier's sequence by

$$P(x), P^{(1)}(x), R_1(x),\dots,R_k(x)$$

which is called **Sturm's sequence** or **chain.** This new sequence is obtained by applying the Euclidean algorithm to the polynomials $P(x)$ and $P^{(1)}(x)$, and taking $R_i(x), i=1,\dots,k$ as the negative of the remainder polynomial; that is, the sequence is defined by the following relations:

$$\begin{aligned} P(x) &= P^{(1)}(x)Q_1(x)-R_1(x),\\ P^{(1)}(x) &= R_1(x)Q_2(x)-R_2(x),\\ &\vdots\\ R_{k-2}(x) &= R_{k-1}(x)Q_k(x)-R_k(x). \end{aligned}$$

The advantage of Sturm's sequence is that we can now obtain the exact number of real roots which the equation $P(x)=0$ has within a given interval. This is formally stated as follows:

THEOREM 2 (Sturm 1829). *If the equation* $P(x)=0$ *has only simple roots, then the number of its real roots in the interval* (p,q) *is equal to the difference*

$$v(p)-v(q),$$

where $v(\xi)$ *denotes the number of sign variations in Sturm's sequence for* $x=\xi$.

Sturm himself tells us [**8**] that the above theorem was merely a by-product of his extensive investigations on the subject of linear difference equations of the second order. The requirement that $P(x)=0$ has only simple roots is no restriction of the generality because we can first apply square-free factorization [**4**], [**15**] and then use Sturm's theorem.

Clearly Sturm's theorem can be used in the isolation of the real roots of an equation. The process itself is quite simple because all we have to do, once Sturm's sequence has been obtained, is to compute an absolute upper root bound b so that all the roots lie within the interval $(-b,b)$. We then subdivide this interval until in each subinterval there is at most one root; that is, Sturm's method is actually a bisection method. Quite recently, this method was implemented within a computer algebra system [**11**] using exact integer arithmetic algorithms and its computing time was thoroughly analyzed. (Computer algebra systems usually deal only with integer (rational) numbers, so that the user does not have to worry about round off and truncation errors. For a survey of such systems see [**14**].) It was shown that if $P(x)=0$ is an integral-coefficient univariate polynomial equation of degree $n>0$ without multiple roots, then the computing time of Sturm's method is

$$O\left(n^{13}L(|P|_\infty)^3\right)$$

where $L(|P|_\infty)$ is the length, in bits, of the maximum of the absolute values of the coefficients of P. This lengthy computing time shows Sturm's method leaves a lot to be desired; it has been determined that its slowness is due to the computation of the Sturm sequence.

Budan's theorem

Although Budan's theorem appeared much earlier than Fourier's, it seems to have been ignored; as far as we have been able to determine it does not appear in any of the standard texts on the theory of equations. The following statement of the theorem is from Vincent's paper [**16**, p. 342].

THEOREM 3 (Budan 1807). *If in an equation in* x, $P(x)=0$, *we make two transformations,* $x=p+x'$ *and* $x=q+x''$, *where* p *and* q *are real numbers such that* $p<q$, *then*

(i) *the transformed equation in* $x'=x-p$ *cannot have fewer sign variations than the transformed equation in* $x''=x-q$;
(ii) *the number of real roots of the equation* $P(x)=0$, *located between* p *and* q, *can never be more than the number of sign variations lost in passing from the transformed equation in* $x'=x-p$ *to the transformed equation in* $x''=x-q$;
(iii) *when the first number is less than the second, the difference is always an even number.*

Like Theorem 1, Budan's theorem also gives us an upper bound on the number of real roots of the equation $P(x)=0$ inside the interval (p,q). However, it only makes use of the transformations $x=p+x'$ and $x=q+x''$ and does not depend on any sequence of polynomials.

Theorems 1 and 3 are equivalent; this fact can be easily seen if in Fourier's sequence we replace x by any real number α. The $m+1$ resulting numbers are proportional to the corresponding coefficients of the transformed polynomial equation $P(x+\alpha)=0$, obtained by Taylor's expansion theorem.

Budan's theorem constitutes the basis of the following statement [**16**], [**3**].

THEOREM 4. *Let* $P(x)=0$ *be a polynomial equation of degree* $n>1$, *with rational coefficients and without multiple roots, and let* $\Delta>0$ *be the smallest distance between any two of its roots. Let* m *be the smallest index such that*

$$\frac{1}{2}F_{m-1}\Delta>1 \text{ and } F_{m-1}F_m\Delta>1+\frac{1}{\varepsilon_n},$$

where F_k *is the kth member of the Fibonacci sequence*

$$1,1,2,3,5,8,13,21,\ldots,$$

and

$$\varepsilon_n=\left(1+\frac{1}{n}\right)^{\frac{1}{n-1}}-1.$$

Let $a_1,a_2,\ldots,a_m$ *be arbitrary positive integers. Then the transformation*

$$x=a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cfrac{\ddots}{\qquad+\cfrac{1}{a_m+\cfrac{1}{y}}}}} \tag{1}$$

(which is equivalent to the series of successive transformations of the form $x = a_i + 1/\xi$, $i = 1,2,\ldots,m$) transforms the equation $P(x) = 0$ into the equation $\tilde{P}(y) = 0$, which has not more than one sign variation.

This theorem is an extended version of the one originally presented by Vincent [**16**], [**4**]. The latter was first hinted by Fourier and, in his paper, Vincent indicates his surprise that Fourier did not try to go further and prove the proposition that was the main subject of Vincent's article. He states, however, the belief that such a proof may exist in other manuscripts which were not published because of the untimely death of Navier.

Theorem 4 can also be used in the isolation of the real roots of an equation. To see roughly why it is true and also how it is applied, observe the following:

(i) The continued fraction transformation (1) can be also written as

$$x = \frac{P_m y + P_{m-1}}{Q_m y + Q_{m-1}}, \tag{2}$$

where P_k/Q_k is the kth convergent to the continued fraction

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}$$

and, as we recall,

$$P_{k+1} = a_{k+1}P_k + P_{k-1},$$
$$Q_{k+1} = a_{k+1}Q_k + Q_{k-1}.$$

(ii) The distance between two consecutive convergents is

$$\left|\frac{P_{m-1}}{Q_{m-1}} - \frac{P_m}{Q_m}\right| = \frac{1}{Q_{m-1}Q_m}.$$

It can be proven that the smallest values of the Q_i occur when all of the $a_i = 1$. Then $Q_m = F_m$, the mth Fibonacci number. This explains why there is a relation between the Fibonacci numbers and the distance Δ in Theorem 4.

(iii) Let $\tilde{P}(y) = 0$ be the equation obtained from $P(x) = 0$ after a transformation of the form (2). Observe that (2) maps the interval $0 < y < \infty$ onto the x-interval whose unordered endpoints are the consecutive convergents P_{m-1}/Q_{m-1} and P_m/Q_m. If this x-interval has length less than Δ, then it contains at most one root of $P(x) = 0$, and the corresponding equation $\tilde{P}(y) = 0$ has at most one root in $(0, \infty)$.

(iv) If $\tilde{y}$ was this positive root, then the corresponding root $\tilde{x}$ of $P(x) = 0$ could be easily obtained from (2). We only know though, that $\tilde{y}$ lies in the interval $(0, \infty)$; therefore, substituting y in (2) once by 0 and once by ∞, we obtain for the positive root $\tilde{x}$ its isolating interval whose unordered endpoints are P_{m-1}/Q_{m-1} and P_m/Q_m. To each positive root there corresponds a different continued fraction; at most m partial quotients have to be computed for the isolation of any positive root. (Negative roots can be isolated if we replace x by $-x$ in the original equation.)

REMARK. It is clear that if we knew the value of Δ, we could compute m from the inequalities of Theorem 4. Then, without any tests, we could obtain $\tilde{P}(y) = 0$. However, in our algorithmic procedure (to be described below), we do not initially know Δ. Thus we need the stronger conclusion that $\tilde{P}(y) = 0$ has at most one sign variation in order to have an effective test for root isolation. This is what requires the additional complexities in our theorem. For details see [**3**].

From the above discussion it is obvious that the calculation of the partial quotients $a_1, a_2, \ldots, a_m$ (for each positive root) constitutes the real root isolation procedure. (From Budan's theorem we know that the value of a particular partial quotient a_i has been computed if $P(x+a_i)=0$ has more sign variations in the sequence of its coefficients than $P(x+a_i+1)=0$.) There are two methods, Vincent's and ours, corresponding to the two different ways in which the computation of the a_i's may be performed. As we will see, the difference between these two methods can be thought of as being analogous to the difference between the integrals of Riemann and Lebesgue. That is, it is well known that the sum $1+1+1+1+1$ can be computed in the following two ways: (a) $1+1=2$, $2+1=3$, $3+1=4$, $4+1=5$ (Riemann) and (b) $5\cdot 1=5$ (Lebesgue).

Vincent's method basically consists of computing a particular a_i by a series of unit incrementations $a_i \leftarrow a_i+1$ (replace a_i by a_i+1), with each one of which we have to perform the translation $\tilde{P}(x) \leftarrow \tilde{P}(x+1)$ (for some polynomial equation $\tilde{P}(x)=0$) and check for a change in the number of sign variations. This "brute force" approach results in a method with exponential behavior and hence is of little practical importance. As an example, let us isolate the roots of the polynomial equation

$$P(x)=(x-\alpha)(x-\beta)=0$$

where $\alpha=5\cdot 10^9+\varepsilon$ and $\beta=\alpha+1$. Consider $a_1^{(\alpha)}$, the first partial quotient for α, which is $5\cdot 10^9$. Using Vincent's method we set $a_1^{(\alpha)} \leftarrow 1$, $\tilde{P}(x) \leftarrow P(x)$ and compute $\tilde{P}(x) \leftarrow \tilde{P}(x+1)$. Since the number of sign variations in the sequence of coefficients of the transformed polynomial $\tilde{P}(x)$ has not changed, we set $a_1^{(\alpha)} \leftarrow a_1^{(\alpha)}+1$ and compute a new $\tilde{P}(x) \leftarrow \tilde{P}(x+1)$, checking again the number of sign variations. This process is repeated $5\cdot 10^9$ times and, on the fastest computer available, it would take about six years! (Note, however, that Vincent's method can be quite efficient when the values of the partial quotients are small; for examples see [**15**].)

Our method, on the contrary, basically consists of computing a particular a_i as the lower bound b on the values of the positive roots of a polynomial equation. (It is assumed that $b=\lfloor \alpha_s \rfloor$ (the floor function or greatest integer function), where α_s is the smallest positive root.) This is achieved with the help of

CAUCHY'S RULE. *Let $P(x)=x^n+c_{n-1}x^{n-1}+\cdots+c_1x+c_0=0$ be a polynomial equation of degree n with integral coefficients, at least one of which is negative. If λ is the number of negative coefficients of $P(x)$, then*

$$b=\max_{\substack{1\leq k\leq n\\ c_{n-k}<0}} |\lambda c_{n-k}|^{1/k}$$

is an upper bound on the values of the positive roots of $P(x)=0$.

Proof. From the way b is defined we conclude that

$$b^k \geq \lambda |c_{n-k}|$$

for each k such that $c_{n-k}<0$; for these k's the last inequality can also be written as

$$b^n \geq \lambda |c_{n-k}| b^{n-k}.$$

Summing over all the appropriate k's we obtain

$$\lambda b^n \geq \lambda \sum_{\substack{k=1\\ c_{n-k}<0}}^{n} |c_{n-k}| b^{n-k}$$

or

$$b^n \geq \sum_{\substack{k=1\\ c_{n-k}<0}}^{n} |c_{n-k}| b^{n-k}.$$

From the last inequality we conclude that if we substitute b for x in $P(x)=0$, the first term, i.e.,

b^n, will be greater than or equal to the sum of the absolute values of all the negative coefficients. Therefore, $P(x) > 0$ for all $x > b$.

Observe that computing the lower bound b of $P(x) = 0$ is equivalent to computing the upper bound on the values of the positive roots of $P(1/x) = 0$. It might be thought that Cauchy's rule requires a great amount of computation, since it seems that the calculation of kth roots is needed. This, however, is not true because instead of computing each kth root we compute, very efficiently, the smallest integer $m(k)$ such that

$$|\lambda c_{n-k}|^{1/k} \leqslant 2^{m(k)},$$

and then we set $b = 2^{K+1}$, where K is the maximum of the $m(k)$'s. For details see [**5**].

Once we have computed $a_i \leftarrow b, b \geqslant 1$, we need to perform only one translation, namely, $\tilde{P}(x) \leftarrow \tilde{P}(x+b)$ which takes the same amount of time as $\tilde{P}(x) \leftarrow \tilde{P}(x+1)$ [**7**]; therefore, with our method we have enormous savings of computing time, and the previous example is solved in a matter of a few seconds. In what follows we present a recursive definition of our method as found in [**3**]:

Let

$$P(x) = 0 \tag{3}$$

be a polynomial equation without multiple roots and with v sign variations in the sequence of its integer coefficients.

If $v = 0$ *or* $v = 1$: From the Cardano-Descartes rule of signs we know that $v = 0$ implies that (3) has no positive roots, whereas $v = 1$ indicates that (3) has exactly one positive root, in which case $(0, \infty)$ is its isolating interval; in either case, no transformation of (3) is necessary, and the method terminates.

If $v > 1$: In this case (3) has to be further investigated. We first compute the lower bound b on the values of the positive roots and then we obtain the translated equation $P_b(x) = P(x+b) = 0$, which also has v sign variations provided $P(b) \neq 0$ (if $P(b) = 0$, we have found an integer root of the original equation and v is decreased). The equation $P_b(x) = 0$ is now transformed by the substitutions $x \leftarrow x+1$ and $x \leftarrow 1/(x+1)$, and the procedure is applied again twice, once with $P_b(1/(x+1)) = 0$ in place of (3) and once with $P_b(x+1) = 0$.

We have implemented our method in a computer algebra system (for a detailed description of the algorithms see [**2**]) and have been able to show that its computing time bound is

$$O\left(n^5 L(|P|_\infty)^3\right),$$

which is the fastest obtained so far when exact integer arithmetic algorithms are used.

TABLES 1 and 2 show the observed computing times for the methods of Sturm, Vincent, and ours for certain classes of polynomials. All times are in seconds and were obtained using the SAC-1 computer algebra system on the IBM S/370 computer, located at the Triangle Universities Computation Center (North Carolina), where a subroutine CCLOCK is available which reads the computer clock [**3**]. TABLE 1 clearly indicates that, for this class of polynomials, Sturm's method is completely out of the race, whereas TABLE 2 makes clear the exponential nature of Vincent's method.

Polynomials with Randomly Generated Coefficients		
	Computation Time	
Degree	Sturm	Our Method
5	2.05	.26
10	33.28	.48
15	156.40	.94
20	524.42	2.36

TABLE 1

Polynomials of Degree 5 with Randomly Generated Roots		
	Computation Time	
Roots are in the Interval	Vincent	Our Method
$(0, 10^2)$	.45	.16
$(0, 10^3)$	1.61	.71
$(0, 10^4)$	16.43	2.01
$(0, 10^5)$	175.62	4.81

TABLE 2

References

[1] A. G. Akritas, A new method for polynomial real root isolation, Proceedings of the 16th Annual Southeast Regional ACM Conference, Atlanta, GA (1978) 39–43.

[2] ______, The fastest exact algorithms for the isolation of the real roots of a polynomial equation, Computing, 24 (1980) 299–313.

[3] ______, An implementation of Vincent's theorem, Numer. Math., 36 (1980) 53–62.

[4] ______, Vincent's forgotten theorem, its extension and application, Internat. J. Comput. Math. with Applications, 7 (1981) 309–317.

[5] ______, Exact algorithms for the implementation of Cauchy's rule, Internat. J. Comput. Math., 9 (1981) 323–333.

[6] A. G. Akritas and S. D. Danielopoulos, On the forgotten theorem of Mr. Vincent, Historia Math., 5 (1978) 427–435.

[7] ______, On the complexity of algorithms for the translation of polynomials, Computing, 24 (1980) 51–60.

[8] M. Bôcher, The published and unpublished works of Charles Sturm on algebraic and differential equations, Bull. Amer. Math. Soc., 18 (1911) 1–18.

[9] L. E. Dickson, First Course in the Theory of Equations, Wiley, New York, 1922.

[10] F. Cajori, A history of the arithmetical methods of approximation to the roots of numerical equations of one unknown quantity, Colorado College Publications, General Series No. 51, Science Series vol. XII, no. 7 (1910) 171–215, Colorado Springs, CO.

[11] L. E. Heindel, Integer arithmetic algorithms for polynomial real zero determination, J. Assoc. Comput. Mach., 18 (1971) 533–548.

[12] A. Hurwitz, Über den Satz von Budan-Fourier, Math. Ann., 71 (1912) 584–591.

[13] N. Obreschkoff, Verteilung und Berechnung der Nullstellen reeller Polynome, VEB Deutscher Verlag der Wissenschaften, Berlin, 1963.

[14] S. R. Petricle, ed., Proceedings of the 2nd symposium on symbolic and algebraic manipulation, ACM, 1971.

[15] J. V. Uspensky, Theory of Equations, McGraw-Hill, New York, 1948.

[16] A. J. H. Vincent, Sur la résolution des équations numériques, J. Math. Pures Appl., 1 (1836) 341–372.

[17] L. Weisner, Introduction to the Theory of Equations, MacMillan, New York, 1938.

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Hector's paces

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—RICHARD MOORE