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A Converse Rule of Signs for Polynomials

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Abstract — Zusammenfassung

A Converse Rule of Signs for Polynomials. Given a polynomial $P(x)$, it is well-known that the Cardano-Descartes rule of signs gives only an upper bound on the number of its positive roots, except in the case in which there is one or no sign variation, where it indicates that $P(x)$ has one or no positive root(s) respectively. In certain new root isolation methods, of great interest to symbolic mathematical computation, it is important to know under what conditions the existence of one positive root implies that $P(x)$ presents only one sign variation. These conditions are discussed and presented in this paper.

AMS Subject Classifications: 26C10, 65H05, 68C05, 68C20.

Key words: Cardano, Descartes, rule of signs, polynomial real root isolation.

Eine inverse Zeichenregel für Polynome. Für Polynome $P(x)$ ist lange bekannt, daß die Zeichenregel von Cardano-Descartes nur obere Schranken für die Anzahl der positiven Nullstellen liefert, ausgenommen die Fälle, in denen ein oder gar kein Vorzeichenwechsel stattfindet, woraus folgt, daß $P(x)$ genau eine oder keine positive Wurzel besitzt. In neuen Methoden zur Trennung der Nullstellen, die bei symbolischer Rechnung von großem Interesse sind, ist die umgekehrte Fragestellung wichtig: Unter welchen Bedingungen hat die Existenz genau einer positiven Wurzel zur Folge, daß bei den Koeffizienten von $P(x)$ nur ein Zeichenwechsel auftritt? Diese Bedingungen werden in der vorliegenden Arbeit vorgestellt und diskutiert.

1. Introduction

Recently, a new and very efficient method for the isolation of the real roots of a polynomial was developed [1], [2], [3]. The method is based on a theorem by A. J. H. Vincent [9], which, although formulated in 1836, was consigned to oblivion for almost 140 years [4], [5]. According to this theorem, if a polynomial equation, $P(x)=0$, with rational coefficients and without multiple roots, is successively transformed by transformations of the form

$$x=a_i+\frac{1}{y}, \quad i=1,2,\ldots \tag{1}$$

for arbitrary integral positive a_i's, one eventually obtains an equation with at most one sign variation. This means, according to the rule of signs of Cardano-Descartes [7], that the (transformed) polynomial has at most one positive root. This celebrated rule states that the number p of positive roots of a polynomial $P(x)$ cannot exceed the number v of sign variations in the sequence $c_0, c_1, \ldots, c_n$ of coefficients of the polynomial. If $v-p>0$, then it is an even number.

A closer examination of the Cardano-Descartes rule reveals that it is a rather weak proposition; it gives the exact number of positive roots only in the following two special cases:

(i) if there is no sign variation, there is no positive root, and

(ii) if there is one sign variation, there is one positive root.

But these are precisely the cases which are of great importance in Vincent's theorem. If it is known that a polynomial has only one positive root, then this root can be quickly isolated and approximated.

From a practical point of view, however, it is necessary to know whether the number of transformations required by Vincent's theorem is finite or not [1], [2]. The ability to design and analyze computer algorithms for the isolation of real roots, based on this theorem, hinges on the ability to prove this fact. It will be shown in the next section that this fact is true, if the converse of proposition (ii) is true.

2. The Possibility of a Converse Proposition

In general, it is not true that, if a polynomial has exactly one positive real root, then its sequence of coefficients presents exactly one sign variation. This is a consequence of the following lemma by Segner ([6], pp. 54 – 56, [7], p. 100), which we quote without proof.

Lemma: *If a polynomial $P(x)$ is multiplied by a linear factor $x-c$, $(c>0)$, then the sequence of coefficients of the resulting polynomial presents $v+2m+1$ sign variations, where m is an arbitrary non-negative integer.*

Nevertheless, under certain more restrictive conditions, the converse of proposition (ii) is true. We present it in the form of the following theorem.

Theorem: *Let $P(x)$ be a polynomial of degree $n>1$, without multiple roots, which has one positive real root x_0 and $n-1$ roots $x_1, x_2, \ldots, x_{n-1}$ with negative real parts — the complex roots appearing in conjugate pairs — and which can be expressed in the form*

$$x_j = -(1+\alpha_j),\ j=1,2,\ldots,n-1 \tag{2}$$

with $|\alpha_j|<\varepsilon_n$, where

$$\varepsilon_n = \left(1+\frac{1}{n}\right)^{\frac{1}{n-1}} - 1. \tag{3}$$

Then $P(x)$ has exactly one sign variation.

Proof: Except for a constant factor, the polynomial $P(x)$ can be written in the form

$$\begin{aligned} P(x) &= (x-x_0)(x-x_1)\ldots(x-x_{n-1}) \\ &= (x-x_0)(x+1+\alpha_1)\ldots(x+1+\alpha_{n-1}) \\ &= (x-x_0)(x^{n-1}+R_1 x^{n-2}+\ldots+R_{n-1}), \end{aligned} \tag{4}$$

where $$R_k = \sum (1+\alpha_1)(1+\alpha_2)\dots(1+\alpha_k),$$

is a sum consisting of $\binom{n-1}{k}$ terms ($k \le n-1$). The polynomial (4) can be further written in the form

$$P(x) = x^n + (R_1 - x_0)x^{n-1} + (R_2 - R_1 x_0)x^{n-2} + \dots$$
$$+ (R_{n-1} - R_{n-2} x_0)x - R_{n-1} x_0 .$$

If $R_k > 0$, $k = 1, 2, \dots, n-1$ and the ratio R_k/R_{k-1}, with $R_0 \equiv 1$, diminishes with increasing k, then obviously it can be said that the coefficients of the polynomial $P(x)$, in its expanded form, have exactly one sign variation. In order to show that $R_k > 0$, $k = 1, 2, \dots, n-1$, we notice that for each of the $\binom{k-1}{k}$ terms we have

$$|(1+\alpha_1)(1+\alpha_2)\dots(1+\alpha_k) - 1| \le (1+|\alpha_1|)(1+|\alpha_2|)\dots(1+|\alpha_k|) - 1 .$$

Since by hypothesis $|\alpha_j| < \varepsilon_n$ for $j = 1, 2, \dots, n-1$, we obtain

$$(1+|\alpha_1|)(1+|\alpha_2|)\dots(1+|\alpha_k|) - 1 \le (1+\varepsilon_n)^k - 1 \le (1+\varepsilon_n)^{n-1} - 1 = \frac{1}{n}$$

and, therefore, it is possible to write

$$R_k = \binom{n-1}{k}(1+\delta_k) \tag{5}$$

where $|\delta_k| \le \frac{1}{n}$. Hence $R_k > 0$, $k = 1, 2, \dots, n-1$. Next we need to show that the ratio R_k/R_{k-1} diminishes with increasing k, i.e.,

$$\frac{R_k}{R_{k-1}} > \frac{R_{k+1}}{R_k}, \quad k = 1, 2, \dots, n-1 . \tag{6}$$

Using (5) we obtain

$$\frac{R_k}{R_{k-1}} = \frac{n-k}{k}\,\frac{1+\delta_k}{1+\delta_{k-1}} \quad \text{and} \quad \frac{R_{k+1}}{R_k} = \frac{n-k-1}{k+1}\,\frac{1+\delta_{k+1}}{1+\delta_k}$$

and thus the inequality (6) requires that

$$\frac{k(n-k-1)}{(k+1)(n-k)} < \frac{(1+\delta_k)^2}{(1+\delta_{k-1})(1+\delta_{k+1})} .$$

This, indeed, is true, since on one hand

$$\frac{k(n-k-1)}{(k+1)(n-k)} = 1 - \frac{n}{(k+1)(n-k)} \le 1 - \frac{4n}{(n+1)^2} = \frac{(n-1)^2}{(n+1)^2},$$

and, on the other hand,

$$\frac{(1+\delta_k)^2}{(1+\delta_{k-1})(1+\delta_{k+1})} > \frac{\left(1-\frac{1}{n}\right)^2}{\left(1+\frac{1}{n}\right)^2} = \frac{(n-1)^2}{(n+1)^2} .$$

This completes the proof of the theorem.

One may wonder whether the relatively small size of the interval ε_n, defined by (3), is too restrictive and thus reduces the usefulness of the theorem. This is not true. It is proved [1] that after m applications of the transformation (1), where m is the smallest index such that

$$F_{m-1} \cdot \frac{\Delta}{2} > 1 \quad \text{and} \quad F_{m-1} \cdot F_m \cdot \Delta > 1 + \frac{1}{\varepsilon_n}, \tag{7}$$

all the real roots, except for at most one, and the real parts of all complex roots are forced to cluster around -1 within an interval of size ε_n. These are precisely the requirements of our theorem.

In the above relations (7), Δ is the smallest distance of any two of its roots and F_k is the k-th element of the Fibonacci sequence $1, 1, 2, 3, 5, 8, \ldots$.

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