There is no "Uspensky's method"

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In honor of the 150th anniversary
of Vincent's theorem

Abstract. In this paper an attempt is made to correct the misconception of several authors[1]that there exists a method by Upensky(based on Vincent's theorem) for the isolation of the real roots of a polynomial equation with rational coefficients. Despite Uspensky's claim, in the preface of his book [2], that he invented this method, we show that what Uspenshy actually did was to take Vincent's method and double its computing time. Uspensky must not have understood Vincent's method probably because he was not aware of Budan's theorem [3]. In view of the above, it is historically incorrect to attribute Vincent's method to Uspensky.

1. Introduction

Isolation of the real roots of a polynomial equation is the process of finding real, disjoint intervals such that each contains exactly one real root and every real root is contained in some interval.

Basically, isolation can be achieved in one of the following two "classical" ways:

(a) using Sturm's theorem (1829), which is based on a theorem by Fourier(1820), or,

(b) using Vincent's theorem (1836), which is based on a theorem by Budan (1807).

For the relation between the theorems by Budan and Fourier see [3]. In this note we focus our attention on Vincent's theorem which, without loss of generality, deals only with positive roots. We have [4]:

Theorem 1 (Vincent 1836). If in a polynomial equation with rational coefficients and without multiple roots, one makes successive transformations of the form

$$x = a + \frac{1}{x'},\ x' = b + \frac{1}{x''},\ x'' = c + \frac{1}{x'''},\ \ldots,$$

where a,b and c are any positive numbers greater than or equal to one, then the resulting transformed equation either has zero variations or it has a single variation. In the second case the equation has a single positive real root represented by the continued fraction

$$a + \cfrac{1}{b + \cfrac{1}{c + \ldots}};$$

in the first case there is no root.

Two root-isolation methods actually result from Vincent's theorem, corresponding to the two different ways in which the quantities a,b,c,... can be computed; the difference between these two methods can be thought of as being analogous to the difference between the integrals of Riemann and Lebesque.

The first method is due to Vincent and basically consists of computing each of the quantities a,b,c... by a series of unit incrementations, i.e. $a \leftarrow a+1$. This "brute force" approach results in a method with an exponential behavior and, hence, of very little practical value.

The second method has polynomial computing time and is due to the author of this article; details can be found elsewhere [5].

Uspensky [2] claims to have invented another method, based also on Vincent's theorem. As we see below, Uspensky, just like Vincent, computes each of the quantities a,b,c... by a series of unit incrementations, i.e. $a \leftarrow a+1$ which corresponds to the substitution $x \leftarrow x+1$; moreover, Uspensky performs with each substitution $x \leftarrow x+1$ the redundant substitution $x \leftarrow \frac{1}{x+1}$, thus doubling the computing time of Vincent's method. This fact cannot be taken as an excuse to name Vincent's method after Uspensky. If there is ever any justification to place one person's name on another person's method, it is when the former improved on the latter, not the opposite as happened here. What then is Uspensky's method?

2. What Uspensky missed

As we mentioned above, Uspensky apparently was not aware of Budan's theorem, the importance of which is underestimated in the existing literature. It appeared in 1807 in the memoir "Nouvelle méthode pour la résolution des équations numériques" ([6] p. 219), but to the best of our knowledge its statement can be found only in [3], [4], and [7]. The following is an excerpt from Vincent's paper ([4], p. 342).

Theorem 2 (Budan 1807). *If in an equation in x, $P(x) = 0$, we make two transformations, $x = p + x'$ and $x = q + x''$, where p and q are real numbers such that $p < q$, then*

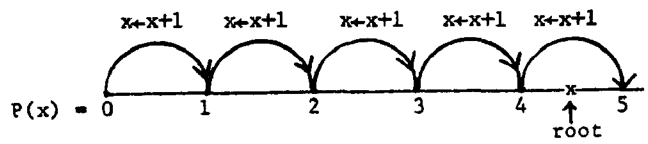
(i) *the transformed equation in $x' = x - p$ cannot have fewer sign variations than the equation in $x'' = x - q$;*

(ii) *the number of real roots of the equation $P(x) = 0$, located between p and q, can never be more than the number of sign variations lost in passing from the transformed equation in $x' = x - p$ to the transformed equation in $x'' = x - q$;*

(iii) *when the first number is less than the second, the difference is always an even number.*

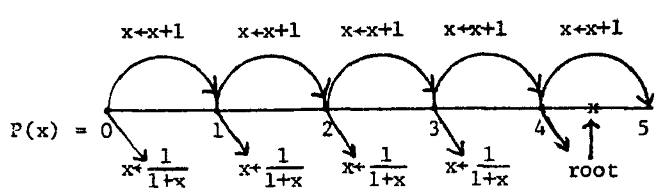
Consider now the following important application of Budan's theorem, which is precisely the point missed by Uspensky. In a given polynomial equation, $P(x) = 0$, replace x by $x + 1$ ($x \leftarrow x + 1$, for short); this results in the positive roots of $P(x) = 0$ being shifted one unit to the left. Now, if $P(x + 1) = 0$ has (in the sequence of its coefficients) the same number of sign variations as $P(x) = 0$, then it follows from Budan's theorem that $P(x) = 0$ has no real roots in the interval $(0,1)$. If, however, $P(x + 1) = 0$ has fewer sign variations than $P(x) = 0$, then the number of real roots of $P(x) = 0$ located in the interval $(0,1)$ can never be more than the number of sign variations lost; generally, in this case, it must be determined whether these roots are real or complex and how many they are.

Vincent makes full use of the above observation and while searching for a root he proceeds as shown below



That is, he successively replaces x by $x + 1$ in the original equation $P(x) = 0$ (whose positive root - for this example - lies inside $(4,5)$) until the number of sign variations is decreased. Since in general the location of the root is not known in advance, the root can be found by this decrease in the number of sign variations. Vincent then easily obtains a first continued fraction approximation to this root as $x = 4 + \frac{1}{x'}$, as stated in his theorem. Please note that Vincent performs those, and only those, transformations that are described in his theorem. This is not done by Uspinsky as is shown below.

Uspensky, on the contrary, can not take advantage of the above mentioned observation because he is apparently not aware of Budan's theorem. So, for him it does not suffice that $P(x + 1) = 0$ has the same number of sign variations as $P(x)=0$ in order to conclude that $P(x) = 0$ has no roots inside $(0,1)$; to make sure, he also performs the redundant substitution $x \leftarrow \frac{1}{1 + x}$ in $P(x) = 0$, which unfailingly results in an equation with no sign variations and hence no positive roots. Uspensky uses the information obtained from both the needed transformation $x \leftarrow x + 1$ and the not needed one $x \leftarrow \frac{1}{1 + x}$, to realize that $P(x) = 0$ has no roots in the interval $(0,1)$. In other words, searching for a root, Uspensky advances as illustrated below ([2],p.128)



Uspensky's transformations are not the ones described in Vincent's theorem, and consequently, his transformations take twice as much computation time as the ones needed for Vincent's method.

In [8], Vincent's method is illustrated by his own example([4],p. 358); this example can also be found in Uspensky's book ([2], pp. 129-137). Copies are shown of the original pages in French, followed by commentaries in English.

3. Conclusions

From the above, it is clear that Uspensky had nothing to do with inventing Vincent's method. Vincent may not have presented things in the clearest way, [8], but he did have clear priority.

One may understand an occasional error [9], but to continue attributing Vincent's method to Uspensky (and to talk about a "modified Uspensky's method" [1]) despite all historical evidence (and despite our previous attempts to set the record straight [5], [10]) is to willfully ignore scientific realities. It is our hope that scientists will give Vincent the credit he so justly deserves.

References

[1] Collins, G.E. and R. Loos: Real Zeros of Polynomials. In: Computer Algebra, Symbolic and Algebraic Computations. Edited by B. Buchberger, G.E. Collins and R. Loos, Springer Verlag, Wien, New York, 83-95, 1982.

[2] Uspensky, J.V.: Theory of Equations. McGraw-Hill Co., New York 1948.

[3] Akritas, A.G.: Reflections on a pair of theorems by Budan and Fourier. Mathematics Magazine, Vol. 55, No. 5, 292-298, 1982.

[4] Vincent, A.J.H.: Sur la résolution des équations numériques. Journal de Mathématiques Pures et Appliquées, Vol. 1, 341-372, 1836

[5] Akritas, A.G.: An implementation of Vincent's theorem. Numerische Mathematik, Vol. 36, 53-62, 1980.

[6] Cajori, F.: A history of the arithmetical methods of approximation to the roots of numerical equations of one unknown quantity. Colorado College Publications, General Series No. 51, Science Series Vo. XII, no. 7, Colorado Springs, CO., 171-215, 1910.

[7] Akritas, A.G. and S.D. Danielopoulos: On the forgotten theorem of Mr. Vincent. Historia Mathematica, Vol. 5, 427-435, 1978.

[8] Akritas, A.G.: There is no "Uspensky's Method". TR-86-10, University of Kansas, Department of Computer Science, Lawrence Ks 66045, 1986.

[9] Collins, G.E. and A.G. Akritas: Polynomial real root isolation using Descartes' rule of signs. Proceedings of the 1976 ACM Symposium on Symbolic and Algebraic Computations. Yorktown Heights, N.Y., 272-275, 1976.

[10] Akritas, A.G.: A remark on the proposed Syllabus for an AMS short course on Computer Algebra. ACM - SIGSAM Bulletin, Vol. 14, No. 2, 24-25, 1980.