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THE ROLE OF THE FIBONACCI SEQUENCE IN THE ISOLATION OF THE REAL ROOTS OF POLYNOMIAL EQUATIONS

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1. INTRODUCTION

Isolation of the real roots of polynomials in $\mathbb{Z}[x]$ is the process of finding real, disjoint intervals each of which contains exactly one real root and every real root is contained in some interval. This process is of interest because, according to Fourier, it constitutes the first step involved in the computation of real roots, the second step being the approximation of these roots to any desired degree of accuracy.

Various propositions have been used to isolate the real roots of polynomial equations with integer coefficients; due to their relation to Fibonacci numbers in this paper we will only examine Vincent's theorem [10] and Wang's generalization of it as presented by Chen in her dissertation [8].

In its *original* statement Vincent's theorem of 1836 states the following [7]:

<u>Theorem 0:</u> If in a polynomial equation with rational coefficients and without multiple roots one makes successive substitutions of the form

$$x:=a_1 + 1/x', \; x':=a_2 + 1/x'', \; x'':=a_3 + 1/x''', \ldots,$$

where a_1 is an arbitrary nonnegative integer and $a_2, a_3, \ldots$ are any positive integers, then the resulting, transformed equation has either zero or one sign variation. In the latter, the equation has a single positive root represented by the continued fraction

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}$$

whereas in the former case there is no root.

Obviously, this theorem only treats positive roots: the negative roots are investigated by replacing x by $-x$ in the original polynomial equation. The generality of the thoerem is not restricted by the fact that there should be no multiple roots, because we can first apply square-free factorization [6]. Vincent himself states that his theorem was hinted in 1827 by Fourier, who never did give any proof of it (or if he did, it was never found): moreover, Lagrange had used the basic principle of this theorem much earlier.

Notice that Vincent's theorem does not give us a bound on the number of substitutions of the form $x{:=}a_i + 1/x$ that have to be performed; this bound was computed with the help of the Fibonacci sequence by Uspensky (with a correction by Akritas) and is described below.

In 1960, and without being aware of Vincent's theorem, Wang generalized it so that it can be applied to polynomial equations with multiple roots; more precisely, using Wang's theorem we obtain not only the isolating intervals of the roots but also their multiplicites. Like Vincent, Wang did not give us a bound on the number of substitutions of the form $x{:=}a_i + 1/x$ that have to be performed; and again, this bound was computed with the help of the Fibonacci sequence by Chen (in her Ph.D. thesis) and is also described below.

2. VINCENT'S THEOREM OF 1836 AND WANG'S THEOREM OF 1960

We begin with a formal definition of sign variations in a number sequence.

Definition: We say that a sign variation exists between two nonzero numbers c_p and c_q $(p < q)$ of a finite or infinite sequence of real numbers $c_1, c_2, \ldots,$ if the following holds:

for $q = p + 1$, c_p and c_q have opposite signs;

for $q \geq p + 2$, the numbers $c_{p+1}, \ldots, c_{q-1}$ are all zero and c_p and c_q have opposite signs.

We next present the extended version of Vincent's theorem of 1836 which, by the way, is based on Budan's theorem of 1807 [5]. Notice how the Fibonacci numbers are used to bound the number of partial quotients that need to be computed.

Theorem 1: Let $p(x) = 0$ be a polynomial equation of degree $n > 1$, with rational coefficients and without multiple roots, and let $\Delta > 0$ be the smallest distance between any two of its roots. Let m be the smallest index such that

$$F_{m-1}\Delta/2 > 1 \text{ and } F_{m-1}F_m\Delta > 1 + 1/\varepsilon_n, \qquad (V)$$

where F_k is the k-th member of the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, ... and

$$\varepsilon_n = (1 + 1/n)^{1/(n-1)} - 1.$$

Let a_1 be an arbitrary nonnegative integer and let $a_2, \ldots, a_m$ be arbitrary positive integers. Then the substitution

$$x{:=}a_1 + \cfrac{1}{a_2 + \cfrac{\ddots}{\ddots + \cfrac{1}{a_m + \cfrac{1}{y}}}} \qquad (CF)$$

(which is equivalent to the series of successive substitutions of the form $x{:=}a_i + 1/y$, $i = 1, 2, \ldots, m$) transforms the equation $p(x) = 0$ into the equation $p_{ti}(y) = 0$, which has no more than one sign variation in the sequence of its coefficients.

The proof can be found in the literature [4], [6]. Since the transformed equation $p_{ti}(y) = 0$ has either 0 or 1 sign variation, the above theorem is closely related to the Cardano-Descartes rule of signs which states that the number p of positive roots of a polynomial equation $p(x) = 0$ cannot exceed the number v of sign variations in the sequence of coefficients of $p(x)$, and if $n = v - p > 0$, then n is an even number. Notice that the Cardano-Descartes rule of signs gives the exact number of positive roots only in the following two special cases:

(i) if there is no sign variation, there is no positive root, and

(ii) if there is one sign variation, there is one positive root.

(Observe how these two special cases are used in Theorem 1 above.)

Theorem 1 can be used in the isolation of the real roots of a polynomial equation. To see how it is applied, observe the following:

i. The continued fraction substitution (CF) can also be written as

$$x := \frac{p_m y + p_{m-1}}{q_m y + q_{m-1}}, \qquad \text{(CF1)}$$

where p_k/q_k is the k-th convergent to the continued fraction

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}},$$

and, as we know, for $k \geq 0$, $p_0 = 1$, $p_{-1} = 0$, $q_0 = 0$, and $q_{-1} = 1$ we have:

$$p_{k+1} := a_{k+1}p_k + p_{k-1},$$

$$q_{k+1} := a_{k+1}q_k + q_{k-1}.$$

ii. The distance between two consecutive convergents is

$$|p_{m-1}/q_{m-1} - p_m/q_m| = 1/q_{m-1}q_m.$$

Clearly, the smallest values of the q_i occur when $a_i = 1$ for all i. Then, $q_m = F_m$, the m-th Fibonacci number. This explains why there is a relation between the Fibonacci numbers and the distance Δ in Theorem 1.

iii. Let $p_{ti}(y) = 0$ be the equation obtained from $p(x) = 0$ after a substitution of the form (CF1), corresponding to a series of translations and inversions. Observe that (CF1) maps the interval $0 < y < \infty$ onto the x-interval whose unordered endpoints are the consecutive convergents p_{m-1}/q_{m-1} and p_m/q_m. If this x-interval has length less than Δ, then it contains at most one root of $p(x) = 0$, and the corresponding equation $p_{ti}(y) = 0$ has at most one root in $(0, \infty)$.

iv. If y' were this positive root of $p_{ti}(y) = 0$, then the corresponding root x' of $p(x) = 0$ could be easily obtained from (CF1). We only know though, that y' lies in the interval $(0, \infty)$; therefore, substituting y in (CF1) once by 0 and once by ∞ we obtain for the positive root x its isolating interval whose unordered endpoints are p_{m-1}/q_{m-1} and p_m/q_m. To each positive root there corresponds a different continued fraction; at most m partial quotients have to be computed for the isolation of any positive root. (As we mentioned before, negative roots can be isolated if we replace x by $-x$ in the original equation.)

The calculation of the partial quotients (for each positive root) constitutes the real root isolation procedure. There are two methods, Vincent's and the continued fractions method of 1978 (developed by Akritas), corresponding to the two different ways in which the computation of the a_i's may be performed. The difference between these two methods can be thought of as being analogous to the difference between the integrals of Riemann and Lebesgue. That is, the sum 1+1+1+1+1 can be computed in the following two ways:

(a) $1+1 = 2,\ 2+1 = 3,\ 3+1 = 4,\ 4+1 = 5$ (Riemann) and
(b) $5 \cdot 1 = 5$ (Lebesgue).

Vincent's method consists of computing a particular a_i by a series of unit incrementations $a_i := a_i + 1$, to each one of which corresponds the translation $p_{ti}(x) := p_{ti}(x+1)$ for some polynomial equation $p_{ti}(x)$. This brute force approach results in a method with exponential behavior and hence is of little practical importance.

The continued fractions method of 1978 on the contrary, consists of computing a particular a_i as the lower bound b on the values of the positive roots of a polynomial equation; actually, we can safely conclude that $b = \lfloor a_s \rfloor$ where a_s is the smallest positive root of some equation obtained during the transformations described in Theorem 1. Implementation details can be found in the literature [1], [2]. Here we simply mention that to compute this lower bound b on the values of the positive roots we use Cauchy's rule [3] (actually presented for upper bounds).

Cauchy's Rule: Let $p(x) = x^n + c_{n-1}c^{n-1} + \ldots + c_n x + c_0 = 0$ be a monic polynomial equation with integer coefficients of degree $n>0$, with $c_{n-k}<0$ for at least one k, $1 \leq k \leq n$, and let λ be the number of its negative coefficients. Then

$$b = \max_{\substack{1 \leq k \leq n \\ c_{n-k} < 0}} \{ |\lambda c_{n-k}|^{1/k} \}$$

is an upper bound on the values of the positive roots of $p(x) = 0$.

Notice that the lower bound is obtained by applying Cauchy's rule to the polynomial $p(1/x) = 0$.

Moreover, we used Mahler's [9] bound on Δ

$$\Delta \geq \sqrt{3} \cdot n^{-(n+2)/2} \cdot |p(x)|_1^{-(n-1)}, \tag{M}$$

(where n is the degree of $p(x)$ and $|p(x)|_1$ is the sum of the absolute values of the coefficients).

According to Chen [8], and without being aware of Vincent's theorem, Wang in 1960 independently stated a more general theorem which includes the one by Vincent as a special case.

Again a bound was needed on the number m of substitutions of the form $x:=a_i + 1/y$ that must be performed; this bound on m was computed, again with the help of Fibonacci numbers, by Chen [8] and is described in Theorem 2 below.

Theorem 2: Let $p(x) = 0$ be an integral polynomial equation of degree $n \geq 3$, and assume that it has at least 2 sign variations in the sequence of its coefficients; moreover, let $\Delta > 0$ be the smallest distance between any two of its roots. Let m' be the smallest positive index such that

$$(F_{m'-1})^2 > 1/\Delta, \qquad \text{(FIB)}$$

where F_k is the k-th member of the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, ..., and let m'' be the smallest positive integer such that

$$m'' > 1 + \lceil \log_\phi n \rceil / 2.$$

If we let

$$m = m' + m'',$$

then the arbitrary continued fraction substitution

$$x:=a_1 + \cfrac{1}{a_2 + \cfrac{\ddots}{\quad + \cfrac{1}{a_m + \frac{1}{y}}}}$$

with a_1 nonnegative integer and $a_2, \ldots, a_m$ positive integers, transforms $p(x) = 0$ into the equation $p_{\tilde{a}}(y) = 0$, which has r sign variations in the sequence of its coefficients. If $r = 0$, then there are no roots of $p(x)$ in the interval I_m with (unordered) endpoints p_m/q_m, p_{m-1}/q_{m-1} (obtained from (CF1)). If $r > 0$, then $p(x) = 0$ has a unique positive position real root of multiplicity r in I_m.

Notice how this theorem includes the one by Vincent as a special case; however, as was mentioned before, this proposition is of theoretical interest only. It has been demonstrated, both theoretically [1] and empirically [2], that, when classical arithmetic algorithms are used, Vincent's theorem together with square-free factorization is the best approach to the problem of isol g the real roots of a polynomial equation with integer coefficients.

CONCLUSION

We have illustrated the importance of the Fibonacci sequence in computing an upper bound on the number of substitutions of the form $x:=a_i+1/x$, which are required for polynomial real root isolation using Theorem 1 (Vincent) or Theorem 2 (Wang).

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