Teaching Great Ideas of Mathematics with *Mathematica*

The authors use Mathematica to present some of the great ideas of mathematics to liberal arts students.

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In the spring semester of 1997 we introduced a new course at the University of Kansas. The course was titled *Journey through Genius*, after the book with the same title by William Dunham [Dunham 1990]. We used additional sources [Arnol'd 1990, Dunham 1994, and Polya 1954] and, at times, we presented additional perspectives.

Since the majority of the students majored neither in mathematics nor in physics, it was obvious that the class would have difficulties comprehending the subtle points involved, carrying out some of the intricate and complicated computations requisite in certain sections of the textbook, and completing the homework assignments. Therefore, we decided not to emphasize hand calculations but, instead to focus on mastering the general concepts involved and to rely on computer-aided symbolic manipulation with *Mathematica*.

The results were excellent and the students acknowledged that, after having taken this course, they had a better understanding of how mathematics was done. Below we describe some of the topics covered and the ways we made use of *Mathematica*'s functions and packages to enhance the students' understanding of these topics.

ERATOSTHENES, ACCURACY OF COMPUTATIONS AND INTERVAL ARITHMETIC

Before the time of Galileo and Copernicus, there existed a man whose accomplishments formed the framework of modern geography, astronomy, and number theory. This man, Eratosthenes (ca. 284-192 BC), was one of the great men of ancient Greece. He was born in Cyrene, the modern day Shahhat, in Libya. In his youth, Eratosthenes studied at Alexandria and Athens. When he was forty, he began tutoring the son of King Ptolemy III of Egypt. After that he became the director of the Alexandria Library. It was then that Eratosthenes' greatest works were created.

Eratosthenes was a man of many facets—he knew at least a little of everything. To his students he was known as "pentathlus", a champion in five athletic sports, but "beta" (β) is the nickname he is best known for. Historians have speculated that beta stood for being a second Plato, or second to his contemporary Archimedes, or just second best at everything.

By being knowledgeable in many areas, Eratosthenes made many contributions. He included the fundamentals of astronomy in his long poem, *Hermes*. He also created a chronology and a calendar that included leap years. Most of the works written by Eratosthenes have been lost. The only information about him and his work comes from commentators. Even some of his most famous writings, such as *On the Measurement of the Earth*, have never been found.

Eratosthenes computed the circumference of the earth at 250,000 stadia. His calculation was based on the formula

```
EarthCircumference = (360/7.2) * 5000 stadia
250000. stadia
```

where 7.2 is the angle by which the sun missed being directly overhead in Alexandria on the first day of summer, and 5000 is the distance, in stadia, between Alexandria and Syene, where (on the same day) the sun was directly overhead (Figure 1). (Alexandria and Syene are close to the meridian and their distance was known.) According to the best estimate scholars have been able to determine, 1 stadium = 0.1575 km, with accuracy assumed to 0.5 meters [Zachary 1997]. Therefore, the circumference of the earth comes out to about

```
EarthCircumference * 0.1575 km/stadia
39375. km
```

which equals

```
Needs[Miscellaneous`Units`]
Convert[39375 10^3Meter, Mile]
24466.5 Mile
```

or, about 24,466 miles. We pointed out to the students that Eratosthenes' answer (given his measurements) was as accurate as was possible to obtain—a statement the students later verified.

In the process, the students learned that any result that involves measurements, *cannot* be exact. We called a measurement *accurate* if its true value lay (or, was believed to lie) within a specified range; we also called a measurement *precise* if that range was "small".

We used the standard practice in science to describe the range in which a measurement lies by giving only the significant digits of this measurement (starting with the first nonzero integer). Thus, rounding to the nearest foot, when we say that a certain distance is 216.0 meters, we imply that its true value is somewhere between 215.5 and 216.5 meters. Saying that this distance is 216.732 meters is more precise (it has 6 significant digits) because now the range is much narrower—the true value lies between 216.7315 and 216.7325.

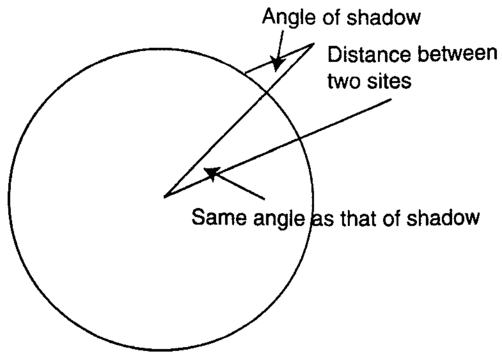

FIGURE 1

There is insufficient information on the accuracy of the figure of 5000 stadia as the distance between Alexandria and Syene. Since Eratosthenes used surveyors to obtain the figure, it is unlikely that the true value lies between 4999.5 and 5000.5. The means of measurement of the times lead us to believe that we can do no better than to place the actual value between 4950 and 5050 stadia. We then asked the students to verify our earlier statement. Using *Mathematica*'s `Interval` function (for doing interval arithmetic), the following expression yields the circumference of the earth as an interval

```
EarthCircumference =
           360
  ----------------------- *
  Interval[{7.15, 7.25}]
    (Interval[{4950, 5050}] *
       Interval[{0.15745, 0.15755}])
Interval [{38700.1, 40059.6}]
```

```
Map[Convert[# 10^3Meter, Mile] &,
  Last[EarthCircumference]]
{24047.1 Mile, 24891.9 Mile}
```

within which lies Eratosthenes' answer. (The currently accepted value for the circumference of the earth is 24,860 miles!) Eratosthenes' estimation of the circumference of the earth implied a surface area much greater than the area of the earth known at the time. Thus the value of 250,000 stadia was not widely accepted. About two centuries later an "improved" circumference of 180,000 stadia was computed which, combined with an overestimate of the size of Asia, led people to believe that it was only 70,000 stadia from the Atlantic ocean to India. When Columbus was planning to sail westwards in 1492, hoping to reach India, the problem he was facing centered on the *size*, not the shape, of the earth. It was the "improved" circumference of the earth (180,000 stadia) that ultimately convinced Columbus to attempt his westward voyage to India.

THE BERNOULLI BROTHERS, AND EXACT VERSUS APPROXIMATE COMPUTATIONS

Jacob (1654-1705) and Johann (1667-1748) Bernoulli constitute mathematics' most important fraternal success story—despite the fact that their relationship was far from harmonious. Born in Switzerland, they were enthusiastic followers of Leibniz and were the major force in disseminating and promoting the calculus throughout Europe. Their efforts, perhaps as much as those of Leibniz himself, gave the subject the flavor and appearance that it retains today. (It should be kept in mind that, de l'Hospital's book *Analyse des infiniment petits*, written in the vernacular rather than in Latin, was almost exclusively Johann Bernoulli's in all but the name on the title page.)

Both Bernoulli brothers are associated with the harmonic series; Johann proved that it is infinite, and Jacob published the result in his *Treatise on Infinite Series* (acknowledging his brother's priority).

The harmonic series is an example of a series that defies intuition. One must add its first 12,367 terms to get 10, and for the sum to climb to 20 it takes about a quarter of a *billion* terms! To think that the harmonic series could eventually surpass one hundred, or one thousand seems completely out of the question. But, of course it does.

Regarding the harmonic series, $1 + \frac{1}{2} + \frac{1}{3} + \cdots$, one is tempted to try out several additions of ever increasing number of successive terms in order to get a feeling for the overall sum. These additions can be done in (exact) rational arithmetic, in which case the result is a fraction, or in (approximate) floating-point arithmetic. For example, the first few terms of the harmonic series can be computed as

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$$

$$\frac{137}{60}$$

or as

```
1. + 0.5 + 0.3333 + 0.25 + 0.2
2.2833
```

Given that $\frac{137}{60}$ is approximated by the same number 2.2833, it is tempting to conclude that there is no difference between the above two ways of adding terms. We demonstrated the opposite to the class as follows.

Floating-point arithmetic involves a finite set of numbers. Therefore, an actual calculation may result in a number outside the floating-point set. In this case, the result has to be approximated by its closest floating-point number—an act that generates round-off errors. (In case of ties, the conflict can be resolved by truncation, by rounding to the one even endpoint, or by rounding away from zero.) Therefore, when we add sufficiently many floating-point numbers, each one of which is subjected to a round-off error, these errors accumulate and may distort the final result.

To enhance the students' understanding of floating-point numbers and to make clear that floating point arithmetic is not always accurate and results depend on the order of the arithmetic operations, we used *Mathematica*'s standard add-on package.

```
Needs["NumericalMath`ComputerArithmetic`"]
```

(To our knowledge, this package is the only tool available in the scientific community for simulating floating point arithmetic.) With its help, we simulated a floating-point environment base 10, with 2 digits precision, and were able to show the main ideas using only 20 terms of the harmonic series. Otherwise, using the full precision of the computer, we would have had to employ several thousand terms of the harmonic series in order to demonstrate the result below. The expression

```
SetArithmetic[2, 10];
```

provides us with a base 10 environment with 2 digits precision (with our machine precision at 16). Then, we generate in this environment the first 20 terms of the harmonic series

```
table1 = Table[ComputerNumber[1/i], {i, 1, 20}]
```

```
{1.0000000000000000, 0.5000000000000000,
  0.3300000000000000, 0.2500000000000000,
  0.2000000000000000, 0.1700000000000000,
  0.1400000000000000, 0.1200000000000000,
  0.1100000000000000, 0.1000000000000000,
  0.0910000000000000, 0.0830000000000000,
  0.0770000000000000, 0.0710000000000000,
  0.0670000000000000, 0.0620000000000000,
  0.0590000000000000, 0.0560000000000000,
  0.0530000000000000, 0.0500000000000000}
```

Many of these numbers are not in the simulated floating-point set and have to be approximated by the nearest floating-point equivalent, resulting in an abundance of round-off errors. Therefore, the following expression computes their sum as

```
sum1 = Fold[Plus, 0, table1]
```

```
3.800000000000000
```

while the true value is 3.59774 (to 5 decimals)

$$\sum_{i=1}^{20} \frac{1.}{i}$$

```
3.59774
```

We can actually observe the running sum being formed in Table 1 that follows.

```
sum1 = FoldList[Plus, 0, table1];
TableForm[ SetAccuracy [
    Transpose[{table1, Drop[sum1, 1]}], 4],
  TableHeadings → {Automatic, {Harmonic
Series, Running Sum}}, TableSpacing → {1, 2}]
```

	Harmonic Series	Running Sum
1	1.	1.
2	0.5	1.5
3	0.33	1.8
4	0.25	2.
5	0.2	2.2
6	0.17	2.4
7	0.14	2.5
8	0.12	2.6
9	0.11	2.7
10	0.1	2.8
11	0.091	2.9
12	0.083	3.
13	0.077	3.1
14	0.071	3.2
15	0.067	3.3
16	0.062	3.4
17	0.059	3.5
18	0.056	3.6
19	0.053	3.7
20	0.05	3.8

TABLE 1

Taking the sum of the first 40 terms, we again obtain the same value 3.80000, instead of the correct result (to five decimals) 4.27854.

```
table2 = Table[ComputerNumber[1/i], {i, 1, 40}];
Fold[Plus, 0, table2]
```

```
3.800000000000000
```

Reversing the order of the addition, that is, starting the addition from the last of the twenty terms, yields quite different results:

```
sum2 = Fold[Plus, 0, Reverse[table1]]
```

```
3.600000000000000
```

In the case of the 40 terms, adding them in reverse order, results in 4.20000, instead of 4.27854.

```
Fold[Plus, 0, Reverse[table2]]
```

```
4.200000000000000
```

The students found these results impressive and learned to appreciate the dangers of floating-point arithmetic and of accumulating errors that do not cancel each other.

EUCLID AND THE PRIME NUMBERS

Euclid (330?-275? BC) is one of the most influential and best read mathematicians of all time. His prize work, *Elements*, was the textbook of elementary geometry and logic up to the early twentieth century. For his work in the field, he is known as the *father of geometry* and is considered one of the great Greek mathematicians.

Very little is known about the life of Euclid. Both the dates and place of his birth and death are unknown. It is believed that he was educated at Plato's academy in Athens and stayed there until he was invited by Ptolemy I to teach at his newly-founded university in Alexandria. There, Euclid founded the school of mathematics and remained for the rest of his life. As a teacher, he was probably one of the mentors to Archimedes. Euclid's *Elements* is a thirteen-volume compilation of Greek mathematics and geometry. It is unknown how much, if any, of the work included in *Elements* is Euclid's original work; many of the theorems found can be traced to previous thinkers including Eudoxus, Thales, Hippocrates and Pythagoras. However, the *format* of *Elements*—definitions and postulates followed by theorems whose proofs were based on those definitions and postulates—belongs to Euclid alone.. Every statement, no matter how obvious, was proved. Euclid chose his postulates carefully, picking only the most basic and self-evident propositions as the basis of his work. Before Euclid, rival schools of geometry each had a different set of postulates, some of which were questionable. Euclid's format helped standardize Greek mathematics. As for the subject matter, it ran the gamut of ancient thought. Especially noteworthy subjects include the method of *exhaustion*, which would be used by Archimedes in the invention of integral calculus, and the proof that the set of all prime numbers is infinite.

In preparation for Euclid's proof of the infinitude of primes, we introduced the students to the historical worry of the Greeks that the number of primes may dwindle to nothing as their sizes increase. This was demonstrated by such *Mathematica* computations as

```
PrimePi[100000]
9592
```

(the number of primes less than or equal to 100,000) whereas

```
PrimePi[900000] - PrimePi[800000]
7323
```

ARCHIMEDES

Archimedes is considered one of the greatest mathematicians of all time. In his own time, he was known as "alpha" (α, the first—compared with Eratosthenes' β), "the wise one," "the master" and "the great geometer" and his works and inventions brought him fame that lasts to this very day. He was one of the last great Greek mathematicians.

Born in 287 B.C. in Syracuse, a Greek seaport colony in Sicily, Archimedes was the son of Phidias, an astronomer. Except for his studies at Euclid's school in Alexandria, he spent his entire life in his birthplace. Archimedes proved to be a master at mathematics and spent most of his time contemplating new problems to solve, becoming at times so involved in his work that he forgot to eat. Lacking the blackboards and paper of modern times, he used any available surface, from the dust on the ground to ashes from an extinguished fire, to draw his geometric figures. Never giving up an opportunity to ponder his work, after bathing and anointing himself with olive oil, he would trace figures in the oil on his own skin. Much of Archimedes' fame comes from his relationship with Hiero, the king of Syracuse, and Gelon, Hiero's son. The great geometer had a close friendship with, and may have been related to, the monarch. In any case, he seemed to make a hobby out of solving the king's most complicated problems to the utter amazement of the sovereign. The story of the discovery of *Archimedes' Principle* ("Eureka!"—I have found it!), which led to the field of hydrostatics, is well known. Another time, Archimedes stated "Give me a place to stand on and I will move the earth." King Hiero, who was absolutely astonished by the statement, asked him to prove it. In the harbor was a ship that had proved impossible to launch even by the combined efforts of all the men of Syracuse. Archimedes, who had been examining the properties of levers and pulleys, built a machine that allowed him the single-handedly move the ship from a distance. He also had many other inventions including the Archimedes' watering screw and a miniature planetarium.

Though he had many great practical inventions, Archimedes considered his purely theoretical work to be his true calling. His accomplishments are numerous. His approximation of π between $3\frac{10}{71}$ and $3\frac{1}{7}$ was the most accurate of his time and he devised a new way to approximate square roots. Unhappy with the unwieldy Greek number system, he devised his own that could accommodate larger numbers more easily. However, as we will see below, his greatest invention was that of the integral calculus. He also anticipated the invention of differential calculus as he devised ways to approximate the slope of the tangent lines to his figures. In addition, he made many discoveries in geometry, mechanics and other fields.

Archimedes was killed by a Roman soldier during the siege of Syracuse. The traditional story is that the mathematician was unaware of the taking of the city. While he was drawing figures in the dust, a Roman soldier stepped on them and demanded he come with him. Archimedes responded, "Don't disturb my circles!" The soldier was so enraged that he pulled out his sword and slew the great geometer. When Archimedes was buried, they placed on his tombstone the figure of a sphere inscribed inside a cylinder and the 2:3 ratio of the volumes between them, the solution to the problem he considered his greatest achievement.

Archimedes' Approximation of π

To compute the ratio of the circumference of any circle to its diameter, Archimedes inscribed and circumscribed regular polygons, and concentrated on their perimeters. He started the process by inscribing a regular hexagon in a circle whose radius equaled the side of the hexagon; then he computed the perimeter of the hexagon. He then doubled the number of sides of his inscribed polygon, to get a regular dodecagon whose perimeter he then calculated (Figure 2).

```
Needs[Graphics`MultipleListPlot`];
circle = Graphics[Circle[{0, 0}, 1]];

Show[circle, Graphics[RegularPolygon[6]],
  Graphics[RegularPolygon[12]],
  AspectRatio → Automatic];
```

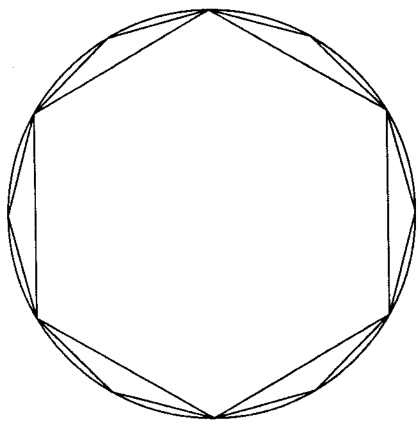

FIGURE 2

Archimedes continued doubling the number of sides of the inscribed regular polygon to get a regular 24-gon, then a regular 48-gon, and finally a regular 96-gon, and calculated their respective perimeters. He repeated the comparable calculations for the *circumscribed* polygons.

Archimedes found that π was less than $3\frac{1}{7}$ but greater than $3\frac{10}{71}$. Converting these rational numbers to floating-point arithmetic, we have $3.140845 < \pi < 3.142857$. What leaves modern mathematicians shaking their heads in wonder is the fact that at each stage of his computations he had to approximate sophisticated square roots, yet he never faltered. (For example, determining the perimeter of the dodecagon required a numerical value for the square root of three.) With our calculators and computers, this strikes us as no real obstacle, but in Archimedes' time, not only were these devices unknown, but there was not even a good number system to facilitate such computations.

To demonstrate Archimedes' formidable task, and to show how slowly the accuracy of the above calculations increases, we asked the students to repeat Archimedes' calculations for the *inscribed* regular polygons using *Mathematica*. They used the formula $t = \sqrt{2 - \sqrt{4 - s^2}}$ (from an earlier exercise) to get the length t of the side of a regular polygon with twice as many sides as the polygon with sides of length s. They also used the functions `Nest`, and `NestList`, where, for example, `Nest[f, x, 2]` results in `f[f[x]]`, whereas `NestList[f, x, 2]` results in the list `{f[x], f[f[x]]}`.

To show the square roots Archimedes had to approximate, we started with a regular hexagon, $s = 1$, and obtained the length of the sides of a regular 12-gon, 24-gon, 48-gon, and 96-gon using exact arithmetic. Based on these lengths, the approximate value of π is as shown in Table 2.

```
lengthsOfSides = NestList[√(2 - √(4 - #²)) &, 1, 4];
TableForm[Prepend[
   Table[{6 * 2^i, lengthsOfSides[[i + 1]],
       lengthsOfSides[[i + 1]] * 6. * 2^i,
       lengthsOfSides[[i + 1]] * 6. * 2^i/2},
     {i, 4}], {6, 1, 6, 3}],
  TableHeadings → {None, {No.of
Sides, LengthofeachSide, Peri-
meter, π
Approx
}}, TableSpacing → {1, 2}]
```

No. of Sides	Length of each Side	Perimeter	π Approx
6	1	6	3
12	$\sqrt{2-\sqrt{3}}$	6.21166	3.10583
24	$\sqrt{2-\sqrt{2+\sqrt{3}}}$	6.26526	3.13263
48	$\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{3}}}}$	6.2787	3.13935
96	$\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}$	6.28206	3.14103

TABLE 2

The last square root is the expression for the *exact* value of the length of the side of the 96-gon. Based on this length we obtain 3.14103 as a lower bound on the value of π , as compared to Archimedes' 3.140845. Archimedes' interval is slightly wider.

Table 2 verifies how slowly the accuracy increases. As another example, we started again with a regular hexagon, $s = 1$, and doubled the number of sides 7 times. The following expression approximates (with 20-digit precision) the length of the side of the resulting 768-gon (768 = 6*2^7).

```
Nest[√(2 - √(4 - #²)) &, N[1, 20], 7]
0.00818120805246957919
```

When multiplied by 768 (the number of the sides), the perimeter of the 768-gon is 6.2831677842966 36817. Half of this last result is 3.1415838921483 18409, an approximation of π accurate to four decimals 3.1415.

Archimedes' Discovery of the Integral Calculus

Archimedes computed the volume of a sphere with his now famous *variable cross-sections* method that is the forerunner to Riemann integration. The presentation of this method below, is that of Polya [Polya 1954] interspersed with *Mathematica* diagrams for better understanding.

Archimedes regards the sphere as generated by a revolving circle, and he regards the circle as a locus, characterized by a relation between the distances of a variable point from two fixed rectangular axes of reference. Written in modern notation, this relation is

$$x^2 + y^2 = 2\,ax, \qquad (1)$$

a circle with radius a that touches the y-axis at the origin, as shown in Figure 3 below for a circle of radius 1. (Note that $a = 1$ throughout our discussion.)

```
a = 1;
sphere2D = ParametricPlot[
    {{x, √(2 * a * x - x²)}, {x, - √(2 * a * x - x²)}},
    {x, 0, 2a}, AspectRatio → Automatic,
    Axes → True, AxesLabel → {x, y},
    Ticks → {{0, 1, 2}, Automatic}];
```

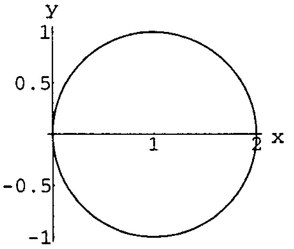

FIGURE 3

The circle, revolving about the x-axis, generates a sphere. As Polya indicates, using modern notation does not distort Archimedes' idea. On the contrary, it seems to be suggestive. It suggests motives which may lead us to Archimedes' idea today and which are, perhaps, not too different from the motives that led Archimedes himself to his discovery.

Consider the term y^2 *in* the equation of the circle. Observe that πy^2 is the area of a *variable cross-section of the sphere.* Yet Democritus found the volume of the cone by examining the variation of its cross-section. This leads us to rewrite the equation of the circle in the form

$$\pi x^2 + \pi y^2 = \pi\, 2ax \qquad (2)$$

Now we can interpret πx^2 as the variable cross-section of a cone, generated by the rotation of the line $y = x$ about the x-axis (Figure 4).

```
a = 1;
cone2D = ParametricPlot[{{x, x}, {x, - x}},
    {x, 0, 2a}, AspectRatio → Automatic,
    Axes → True, AxesLabel → {x, y},
    Ticks → {{0, 1, 2}, Automatic},
    DisplayFunction → Identity];

Show[%,
  Graphics[Line[{{2a, 2a}, {2a, - 2a}}]],
  DisplayFunction → $DisplayFunction];
```

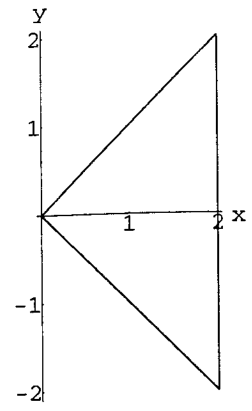

FIGURE 4

This suggests that we seek an analogous interpretation of the remaining term $\pi 2ax$. To see such an interpretation, we experiment with rewriting the equation, and so we hit upon the form

$$2a(\pi y^2 + \pi x^2) = x\pi(2a)^2 \qquad (3)$$

Now the term on the right-hand side of equation (3) is the cross-section of a cylinder generated by the rotation of the rectangle shown in Figure 5 about the x-axis.

```
a = 1;
cylinder2D = Show[
    Graphics[Line[{{2a, 0}, {2a, 2a}, {0, 2a},
          {0, - 2a}, {2a, - 2a}, {2a, 0}}]],
    AspectRatio → Automatic,
    Axes → True, AxesLabel → {x, y},
    Ticks → {{0, 1, 2}, Automatic}] ;
```

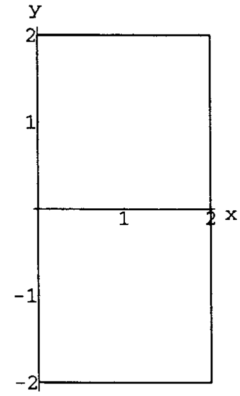

FIGURE 5

We combine Figures 3, 4 and 5 to obtain Figure 6.

```
Show[sphere2D, cone2D,
  cylinder2D, AspectRatio → Automatic,
  Axes → True, AxesLabel → {x, y},
  Ticks → {{0, 1, 2}, Automatic}] ;
```

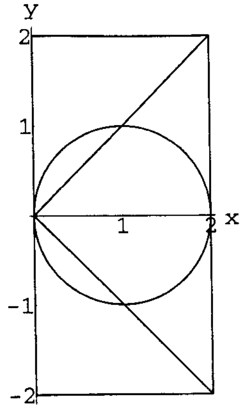

FIGURE 6

Revolving Figure 6 about the x-axis results in a sphere, a cone and a cylinder. Notice that the cone and the cylinder have the same base and height (the vertex of the cone is at the origin (0,0,0)). The *radius* of the common *base* and the *height* have the same length $2a$. In Figure 7 below we generate these solids using *Mathematica*'s add-on package `Graphics'Shapes'`. (The function `RotateShape` below rotates a three-dimensional object by the specified Euler angles.)

```
Needs[Graphics'Shapes']

a = 1;
g1 = RotateShape[
     Graphics3D[Cylinder[2a, 1, 20]], 0,
   π/2, π/2] //WireFrame;
g2 = RotateShape[Graphics3D[Cone[2a, 1, 20]], 0,
   π/2, π/2] //WireFrame;
 \ = RotateShape[
     Graphics3D[Sphere[a, 20, 15]], 0,
   π/2, π/2] //WireFrame;
Show[RotateShape[
   {g1, g2, g3}, π, 0, 0], Boxed → False,
  ViewPoint → {-1.01, -5.546, -0.484}];
```

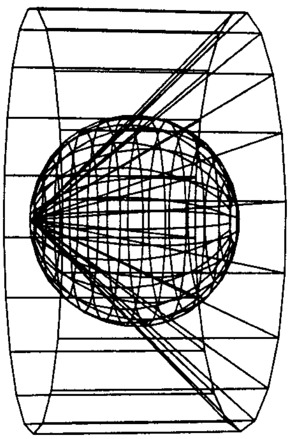

FIGURE 7

Several ideas are contained in equation (3) above. We notice the areas of three circular disks, πy^2, πx^2, and $\pi(2a)^2$. The three circles are the intersections of the same plane with three solids of revolution. The plane is perpendicular to the x-axis, and at the distance x from the origin ($0 \le x \le 2a$). With *Mathematica,* we view this construction in Figure 8.

```
a = 1;
Show[Graphics3D[Cylinder[2a, 1, 20]],
  Graphics3D[Cone[2a, 1, 20]],
  Graphics3D[Sphere[a, 20, 15]],
  PlotRange → {-1, 0.6}, Boxed → False];
```

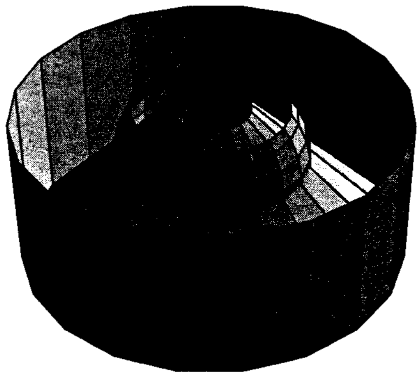

FIGURE 8

Archimedes treats differently the disks, the areas of which appear on different sides of equation (3). He leaves the disk with radius $2a$, the cross-section of the cylinder, in its original position. Yet he removes the disks with radii y and x, cross-sections of the sphere and the cone, respectively, from their original position and transports them to the point (-2a, 0) of the x-axis. (We show the suspended disk of the sphere with radius y between two empty circles, and the disk of the cone with radius x between two filled squares. The disk of the cylinder, at abscissa x, is shown between two filled circles.) We let these disks with radii y and x hang with their center vertically under (-2a, 0), suspended by a string of negligible weight (Figure 9). (This string is an addition to Archimedes' original figure.)

```
a = 1;
Show[sphere2D, cone2D, cylinder2D,
    Graphics[{{Dashing[{0.02, 0.02}],
        Circle[{-2, -1}, 1]},
      Circle[{-2.91652, -0.6}, 0.1],
      Circle[{-1.08348, -0.6}, 0.1],
      Circle[{0.59, 0.9}, 0.1],
      Circle[{0.59, -0.9}, 0.1]}],
  Graphics[
    {Disk[{0.6, 2}, 0.1], Disk[{0.6, -2}, 0.1]}],
  Graphics[{Line[{{0.6, 2}, {0.6, -2}}],
      Line[{{-1.08348, -0.6}, {-2.91652, -0.6}}],
  {Dashing[{0.02, 0.02}], Line[{{-2a, -2a},
          {-4a, -4a}, {0, -4a}, {-2a, -2a}}]}],
```

```
  Line[{{-2.6, -2.6}, {-1.4, -2.6}}],
  Line[{{-2a, 0}, {-2a, -2.6}}]}],
  Graphics[{Point[{-2a, 0}],
     Point[{-2a, -0.6}], Point[{-2a, -2.6}]}],
  Graphics[{Text[<---2--->, {-a, 0.2}],
  Text[<---2--->, {a, 2a + 0.2}],
  Text[-x-, {0.3, -2a - 0.1}]}],
  Graphics[{Rectangle[{-2a - 0.7, -2a - 0.5},
       {-2a - 0.5, -2a - 0.7}], Rectangle[
       {-2a + 0.7, -2a - 0.5}, {-2a + 0.5, -2a - 0.7}],
     Rectangle[{0.7, 0.5}, {0.5, 0.7}],
     Rectangle[{0.7, -0.5}, {0.5, -0.7}]}],
  Prolog → AbsolutePointSize[3],
AspectRatio → Automatic,
  Axes → True,   AxesLabel → {x, y},
  PlotRange → {{-4, 4}, Automatic}] ;
```

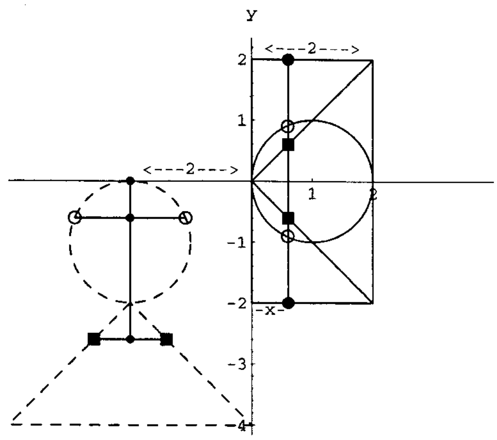

FIGURE 9

Let us regard the x-axis as a *lever*, a rigid bar of negligible weight, and the origin (0,0) as its point of suspension. A moment is the product of the weight and the arm of the lever. Our equation (3) says that the moment of the two disks on the left-hand side equals the moment of the one disk on the right-hand side and so, by the mechanical law discovered by Archimedes, the lever is in equilibrium.

As x varies from 0 to $2a$, we obtain all cross-sections of the cylinder; these cross-sections *fill* the cylinder. To each cross-section of the cylinder there correspond two cross-sections suspended from the point ($-2a$, 0) and these cross-sections *fill* the sphere and the cone respectively. *As their corresponding cross-sections, the sphere and the cone, hanging from (-2a, 0), are in equilibrium with the cylinder.* Therefore, by Archimedes' mechanical law, the moments must be equal. Let V denote the volume of the sphere, let us recall the expression for the volume of the cone (due to Democritus) and also the volume of the cylinder and the *obvious location of its center of gravity.* Passing from the moments of the cross-sections to the moments of the corresponding solids, equation (3) leads to

$$2a\left(V+\frac{\pi(2a)^2 2a}{3}\right)=a\pi(2a)^2 2a, \qquad (4)$$

which yields $V = \frac{4\pi a^3}{3.}$

Looking back at the foregoing, we see that the decisive step is that from equation (3) to (4), from the filling cross-sections to the full solids. Yet this step is only heuristically assumed, not logically justified. It is plausible but not demonstrative. It is a guess, not a proof. And Archimedes, representing the great tradition of Greek mathematical rigor, knows this full well: "The fact at which we arrived is not actually demonstrated by the argument used; but the argument has given a sort of indication that the conclusion is true." This guess, however, is a guess with prospect. The idea goes much beyond the requirements of the problem at hand, and has an immensely greater scope. The passage from the cross-section to the whole solid, is in more modern language the transition from the infinitesimal part to the whole quantity, from the differential to the integral. This transition is a great beginning, and Archimedes, who was a great enough man to see himself in historical perspective, knew it full well: "I am persuaded that this method will be of no little service to mathematics. For I foresee that this method, once understood, will be used to discover other theorems that have not yet occurred to me, by other mathematicians, now living or yet unborn."

Introduction to Modern Integral Calculus

We used the concepts and methods of the preceding section and the tools of *Mathematica* to make the students understand the "approach to the limit", the concept of area under the curve as an integral and the concept of successive approximation; without the formality of limits and integration.

Most of the students were unfamiliar with integration, either the concept or the practice. Archimedes' variable cross-sections method, described above, served both as a motivation and an introduction to the subject. We imitated Archimedes' method of the moving cross-section in the case of the (two-dimensional) ellipse defined by the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where we took a = 1, and b = 0.6, and a variable cross-section is a rectangle.

First we established an approximation of the area of the ellipse as the goal. We took the area of the ellipse as the area between the functions defined by f(x) and g(x) below.

a = 1; b = 0.6;

$$\texttt{f[x\_]} := \texttt{b} * \sqrt{1-\frac{x^2}{a^2}};\ \texttt{g[x\_]} := -\ \texttt{b} * \sqrt{1-\frac{x^2}{a^2}};$$

Using *Mathematica* we obtained the value 1.88496.

$$\int_{-1}^{1} 2 * \texttt{f[x]}\, dx$$

```
1.88496
```

We then used the Mathematica package `Area-BetweenCurves.m` [Finch and Lehmann 1992], modified to give upper and lower bounds, and computed the approximate area for 6 approximating rectangles (see Figure 10). We then did the same for 40 and 80 rectangles (see

Figures 11, and 12) at which point we abandoned the diagrammatic illustration as too dense, but continued computing successive approximations, as shown in Table 3 below.

```
<< AreaBetweenCurves.m
```

```
AreaBet[f, g, -1, 1, 6]
```

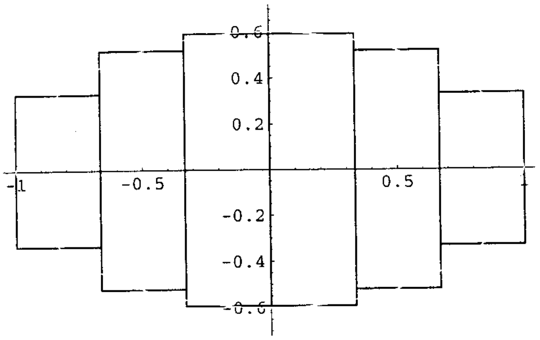

FIGURE 10

```
AreaBet[f,g, -1,1,40]
```

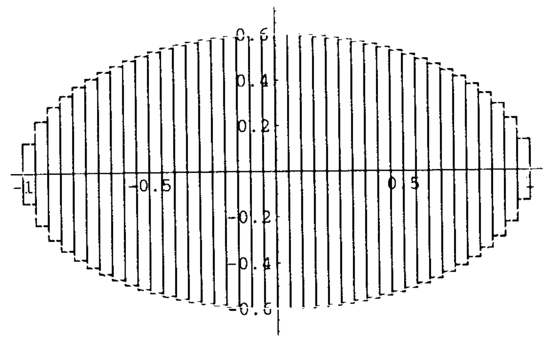

FIGURE 11

```
AreaBet[f,g, -1,1,80]
```

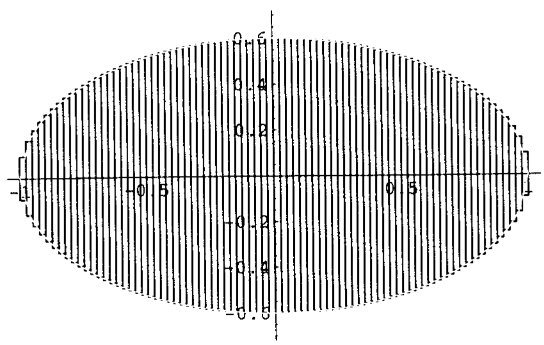

FIGURE 12

The approximation of the area by rectangles resulted from `NAreaBetLower[f,g,-1,1,n]`, `NAreaBet[f,g,-1,1,n]`, and `NAreaBetUpper[f,g,-1,1,n]` for n = 6, 40, 80, 250, 500, and 1000.

```
$RecursionLimit = 20000;
TableForm[
  Map[{#, NAreaBetLower[f, g, -1, 1, #],
        NAreaBet[f, g, -1, 1, #],
        NAreaBetUpper[f, g, -1, 1, #]} &,
     {6, 40, 80, 250, 500, 1000}]],
  TableHeadings → {None, {"Recta\nngles",
  "Lower\nBound", "Mid\npoint", Upper\nBound"
}}, TableSpacing → {1, 2}]
```

Rectangles	Lower Bound	Midpoint	Upper Bound
6	1.35053	1.92385	2.15053
40	1.81708	1.88726	1.93708
80	1.85217	1.88577	1.91217
250	1.87485	1.8851	1.89405
500	1.87998	1.88501	1.88958
1000	1.88249	1.88497	1.88729

TABLE 3

CONCLUSION

It is not our intention to advocate that students do not study calculus or other means of calculating. But even undergraduate students in mathematics, let alone the multitude who have not studied mathematics, would not be able to acquire most of the concepts presented in the *Journey Through Genius* course without the facilities of a symbolic computation system such as *Mathematica*. We would be able to present only a small fraction of the present content of the course, and even then the students would have only a vague idea without the visual demonstrations that *Mathematica* makes possible. In fact we champion the judicious use of *Mathematica* in such mathematics courses as calculus, not to deprive the students of learning computational techniques, but to enhance their understanding and perception of the principles and concepts involved.

ACKNOWLEDGEMENTS

Historical material was obtained from the following web sites: http://www.shu.edu/~wachsmut/reals/history, (on Archimedes, and Euclid), and http://www.thisweb.com/ThisMath/text/eratos.htm, (on Eratosthenes). Helpful discussions were held with Dr. Stelios Kapranidis of the University of South Carolina-Aiken.

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Included in the distribution for each electronic subscription is the file `Teaching.nb` containing *Mathematica* code for the material described in this article.