CLASSICAL MATHEMATICS WITH MATHEMATICA

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Many topics in the history of mathematics can be used both as a motivation and as an introduction to concepts that seem formidable to the uninitiated. The task of conveying these ideas becomes a lot easier (and much more successful) using Mathematica. In this paper we present the way we taught two classical topics of mathematics using Mathematica's functions and packages to enhance the students' understanding.

Introduction

In 1997 we introduced a new course at the University of Kansas. The course was titled *Journey through Genius*, after the book with the same title by William Dunham [5]. We used additional sources [2, 6, and 7] and, at times, we presented additional perspectives.

Since the majority of the students majored neither in mathematics nor in physics, it was obvious that the class would have difficulties not only comprehending the subtle points involved, but also carrying out some of the intricate and complicated computations requisite in certain sections of the textbook, let alone the homework assignments. Therefore, we decided not to put any emphasis on fastidious hand calculations but, instead, to focus on mastering the general concepts involved and to rely on computer-aided symbolic manipulation with *Mathematica*.

The results were excellent and all the students acknowledged that, after having taken this course, they had a better understanding of how mathematics was done.

Below we describe how we covered two classical topics of mathematics: the approximation of π and the discovery of integral calculus by Archimedes—the greatest mathematician of the ancient world.

The Approximation of π

Circles and the approximation of π have intrigued people since ancient times. Below we examine various ways for approximating the value of π to an ever greater degree of accuracy.

Archimedes and his Approximation of π

Archimedes is considered one of the greatest mathematicians of all time. In his own time, he was known as "alpha" (α, the first—compared with Eratosthenes' β), "the wise one," "the master" and "the great geometer" and his works and inventions brought him fame that lasts to this very day. He was one of the last great Greek mathematicians.

Born in 287 B.C. in Syracuse, a Greek seaport colony in Sicily, Archimedes was the son of Phidias, an astronomer. Except for his studies at Euclid's school in Alexandria, he spent his entire life in his birthplace. Archimedes proved to be a master at mathematics and spent most of his time contemplating new problems to solve, becoming at times so involved in his work that he forgot to eat. Lacking the blackboards and paper of modern times, he used any available surface, from the dust on the ground to ashes from an extinguished fire, to draw his geometric figures. Never giving up an opportunity to ponder his work, after bathing and anointing himself with olive oil, he would trace figures in the oil on his own skin.

Much of Archimedes' fame comes from his relationship with Hiero, the king of Syracuse, and Gelon, Hiero's son. The great geometer had a close friendship with, and may have been related to, the monarch. In any case, he seemed to make a hobby out of solving the king's most complicated problems to the utter amazement of the sovereign. At one time, the king ordered a gold crown and gave the goldsmith the exact amount of metal to make it. When Hiero received it, the crown had the correct weight but the monarch suspected that some silver had been used instead of the gold. Since he could not prove it, he brought the problem to Archimedes. One day while considering the question, "the wise one" entered his bathtub and recognized that the amount of water that overflowed the tub was proportional to the volume of his body that was submerged. This observation is now known as *Archimedes' Principle* and gave him the means to solve the problem. He was so excited that he ran naked through the streets of Syracuse shouting "Eureka! eureka!" (I have found it!). The fraudulent goldsmith was brought to justice. Another time, Archimedes stated "Give me a place to stand on and I will move the earth." King Hiero, who was absolutely astonished by the statement, asked him to prove it. In the harbor was a ship that had proved impossible to launch even by the combined efforts of all the men of Syracuse. Archimedes, who had been examining the properties of levers and pulleys, built a machine that allowed him the single-handedly move the ship from a distance. He also had many other inventions including the Archimedes' watering screw and a miniature planetarium.

Though he had many great practical inventions, Archimedes considered his purely theoretical work to be his true calling. His accomplishments are numerous. His approximation of π between $3\frac{10}{71}$ and $3\frac{1}{7}$ was the most accurate of his time and he devised a new way to approximate square roots. Unhappy with the unwieldy Greek number system, he devised his own that could accommodate larger numbers more easily. He invented the field of hydrostatics with the discovery of Archimedes' Principle. However, as we will see below, his greatest invention was that of the integral calculus. He also anticipated the invention of differential calculus as he devised ways to approximate the slope of the tangent lines to his figures. In addition, he made many other discoveries in geometry, mechanics and other fields.

The end of Archimedes life was anything but uneventful. King Hiero had been so impressed with his friend's inventions that he persuaded him to develop weapons to defend the city. These inventions would prove quite useful. In 212 B.C., Marcellus, a Roman general, decided to conquer Syracuse with a full frontal assault on both land and sea. The Roman legions were routed. Huge catapults hurled 500 pound boulders at the soldiers; large cranes with claws on the end lowered down on the enemy ships, lifted them in the air, and then threw them against the rocks; and systems of mirrors focused the sun rays to scorch enemy ships on fire. The Roman soldiers refused to continue the attack and fled at the mere sight of anything projecting from the walls of the city. Marcellus was forced to lay siege to the city, which fell after eight months. Archimedes was killed by a Roman soldier when the city was taken. The traditional story is that the mathematician was unaware of the taking of the city. While he was drawing figures in the dust, a Roman soldier stepped on them and demanded he come with him. Archimedes responded, "Don't disturb my circles!" The soldier was so enraged that he pulled out his sword and slew the great geometer. When Archimedes was buried, they placed on his tombstone the figure of a sphere inscribed inside a cylinder and the 2:3 ratio of the volumes between them, the solution to the problem he considered his greatest achievement.

To compute the ratio of the circumference of any circle to its diameter, Archimedes inscribed and circumscribed regular polygons, and concentrated on their perimeters. He started the process by inscribing a regular hexagon in a circle whose radius equaled the side of the hexagon; then he computed the perimeter of the hexagon. He then doubled the number of sides of his inscribed polygon, to get a regular dodecagon whose perimeter he then calculated (Figure 1).

```
<< Graphics`MultipleListPlot`;

circle = Graphics[Circle[{0, 0}, 1]];
Show[circle, Graphics[RegularPolygon[6]]
  Graphics[RegularPolygon[12]], AspectRatio -> Automatic];
```

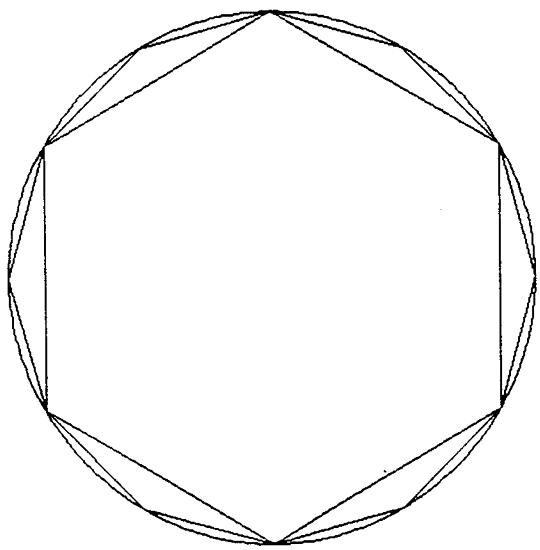

Figure 1

Archimedes' polygons inscribed in a circle to approximate π

He continued doubling the number of sides of the inscribed regular polygon again to get a regular 24-gon, then a regular 48-gon, and finally a regular 96-gon, and calculated their respective perimeters. He repeated the comparable calculations for the circumscribed polygons.

Archimedes found that π was less than $3\frac{1}{7}$ but greater than $3\frac{10}{71}$. Converting these rational numbers to floating-point arithmetic, we have $3.140845 < \pi < 3.142857$. What leaves modern mathematicians shaking their heads in wonder is the fact that at each stage of his computations he had to approximate sophisticated square roots, yet he never faltered. (For example, determining the perimeter of the dodecagon required a numerical value for the square root of 3.) With our calculators and computers, this strikes us as no real obstacle, but in Archimedes' time, not only were these devices unthinkable, but there was not even a good number system to facilitate such computations.

To demonstrate Archimedes' formidable task, and to show how slowly the accuracy of the above calculations increases, we asked the students to repeat Archimedes' calculations for the *inscribed* regular polygons using *Mathematica*. They used the formula $t = \sqrt{2 - \sqrt{4 - s^2}}$ (from an earlier exercise) to get the length t of the side of a regular polygon with twice as many sides as the polygon with sides of length s. They also used the functions `Nest`, and `NestList`, where, for example, `Nest[f, x, 2]` results in `f[f[x]]`, whereas `NestList[f, x, 2]` results in the list `{f[x], f[f[x]]}`.

To show the square roots Archimedes had to approximate, we started with a regular hexagon, s = 1, and obtained the length of the sides of a regular 12-gon, 24-gon, 48-gon, and 96-gon using exact arithmetic. Based on these lengths, the approximate value of π is as shown in Table 1.

lengthsOfSides = NestList[$\sqrt{2-\sqrt{4-\#^2}}$ &, 1, 4];

TableForm[Prepend[Table[
{6 * 2^i, lengthsOfSides[[i + 1]], lengthsOfSides[[i + 1]] * 6. * 2^i,
lengthsOfSides[[i + 1]] * 6. * 2^i / 2}, {i, 4}], {6, 1, 6, 3}],
TableHeadings -> {None, {"No. of\nSides",
"Length of each Side", "Peri-\n meter", " π\nApprox\n"}},
TableSpacing → {1, 2}]

No. of Sides	Length of each Side	Peri- meter	π Approx
6	1	6	3
12	$\sqrt{2-\sqrt{3}}$	6.21166	3.10583
24	$\sqrt{2-\sqrt{2+\sqrt{3}}}$	6.26526	3.13263
48	$\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{3}}}}$	6.2787	3.13935
96	$\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}$	6.28206	3.14103

Table 1

The last square root is the expression for the exact value of the length of the side of the 96-gon. Based on this length we obtain 3.14103 as a lower bound on the value of π , as compared to Archimedes' 3.140845. Archimedes' interval is a tad wider.

Table 1 demonstrates how slowly the accuracy increases. As another example, we started again with a regular hexagon, s = 1, and doubled the number of sides 7 times. The following expression approximates (with 20-digit precision) the length of the side of the resulting 768-gon (768 = 6*2^7).

Nest[$\sqrt{2-\sqrt{4-\#^2}}$ &, N[1, 20], 7]

0.008181208052469579189

When multiplied by 768 (the number of the sides), the perimeter of the 768-gon is 6.283167784296636817. Half of this last result is 3.141583892148318409, an approximation of π accurate to four decimals 3.1415.

After the establishment of the decimal notation, Francois Vieta (1540-1603) computed π to 9 places of accuracy. After Vieta, early in the 17th century, Ludolph van Ceulen computed π to 35 places of accuracy—starting with a square and going up to a 2^{62}-gon. He was the last to compute digits of π using Archimedes' way.

ArcTan[x] and the Approximation of π

Archimedes' method has one liability: the need to calculate square roots of square roots. With each additional doubling of the number of sides, we embed another square root in the design and further complicate the process. With the development of calculus and infinite series in the seventeenth century mathematicians found truly efficient approximations of π.

In trigonometry we encounter an important function called the inverse tangent and denoted by ArcTan[x] or $\tan^{-1}(x)$. What is important about this function is that it can be expressed as an infinite series

```
TraditionalForm[ArcTan[x]] == Series[ArcTan[x], {x, 0, 19}]
```

$$\tan^{-1}(x) == x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \frac{x^{13}}{13} - \frac{x^{15}}{15} + \frac{x^{17}}{17} - \frac{x^{19}}{19} + O[x]^{20}$$

where the summation process continues indefinitely in the obvious pattern. The further we carry the arithmetic, the closer we get to the true value of $\tan^{-1}(x)$. But what does this have to do with π?

Gregory Series (1671)

Gregory's series relating π with $\tan^{-1}(x)$ is probably among the best known. It states that

$$\pi == 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

```
True
```

and it is obtained from the identity

$$\pi == 4\,\text{ArcTan}[1.]$$

```
True
```

where we have replaced x by 1 in the infinite series expansion of ArcTan[x] above.

However, for computational purposes Gregory's formula is practically useless. It takes 2000 terms to compute π with an accuracy of 2 digits!

$$4 \sum_{n=0}^{2000} \frac{(-1)^n}{2n+1} \; // \; N$$

```
3.14209
```

Machin's Formula (1706)

Machin's formula brings some improvement to the process.

```
π == 4 (4 ArcTan[1/5] - ArcTan[1./239])
```

```
True
```

Indeed, using only 20 terms of the infinite series expansion of the ArcTan[x] we have π with 14 digits of accuracy.

```
N[4 ((4 Normal[Series[ArcTan[x], {x, 0, 20}]] /. x → 1/5) -
      Normal[Series[ArcTan[x], {x, 0, 20}]] /. x → 1/239),
  16]
```

```
3.141592653589792
```

```
N[π, 14]
```

```
3.1415926535898
```

Quite an improvement. Virtually all calculations for π from the beginning of the 18th century until the early 1970s have relied on variations of Machin's formula.

Ramanujan's Series (1914)

Ramanujan's formula [4] is quite extraordinary. It states that

$$\frac{1}{\pi} == \frac{\sqrt{8}}{9801} \sum_{n=0}^{\infty} \frac{(4\,n)!}{(n!)^4} \frac{(1103 + 26390\,n)}{396^{4\,n}}$$

and converges to the true value of $\frac{1}{\pi}$ much faster: each successive term in the sequence adds roughly eight more correct digits. The first five terms of the series give π with an accuracy of almost 50 digits.

$$N\left[\frac{\sqrt{8}}{9801} \sum_{n=0}^{5} \frac{(4\,n)!}{(n!)^4} \frac{(1103 + 26390\,n)}{396^{4\,n}},\ 50\right]$$

```
0.31830988618379067153776752674502872406891929148804
```

$$N\left[\frac{1}{\pi},\ 50\right]$$

```
0.31830988618379067153776752674502872406891929148809
```

Borwein and Borwein's Series (1987)

Here [3] we have quite an improvement in the number of digits that each new term adds to the true value of $\frac{1}{\pi}$. The formula used is

$$\frac{1}{\pi} == 12 \sum_{n=0}^{\infty} (-1)^n (6n)! \left(212175710912\sqrt{61} + 1657145277365 + n\left(13773980892672\sqrt{61} + 107578229802750\right)\right) \Big/ \left((n!)^3 (3n)! \left(5280\left(236674 + 30303\sqrt{61}\right)\right)^{3n+\frac{3}{2}}\right)$$

Using again only five terms of the series we obtain π with an accuracy of almost 150 digits.

$$N\left[12 \sum_{n=0}^{5} (-1)^n (6n)! \left(212175710912\sqrt{61} + 1657145277365 + n\left(13773980892672\sqrt{61} + 107578229802750\right)\right) \Big/ \left((n!)^3 (3n)! \left(5280\left(236674 + 30303\sqrt{61}\right)\right)^{3n+\frac{3}{2}}\right), 200\right]$$

```
0.318309886183790671537767526745028724068919291480912897495334688117\
 793595268453070180227605532506171912145685453515916073785823692229\
 573057559348214620630367477626159772666044587506491182347470558173
```

$$N\left[\frac{1}{\pi}, 200\right]$$

```
0.318309886183790671537767526745028724068919291480912897495334688117\
 793595268453070180227605532506171912145685453515916073785823692229\
 573057559348214633996784584799338748181551461554927938506153774348
```

A Change in Direction: Bailey, Borwein and Plouffe's Base-16 Digit Extraction (1997)

As stated by Adamchik and Wagon [1], one of the charms of mathematics is that it is possible to make elementary discoveries about objects that have been studied for millenia. This was the case with the discovery of the following new formula for π [3]. (We call it the BBP series, after the authors' names in [3].)

$$\pi == \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right)$$

Although the formula is not hard to prove, finding it in the first place took some effort. (We do not present the proof; it can be found in [1]. Below we follow their presentation.) The authors of [3] report that their discovery was "a combination of inspired guessing and extensive searching" using a special algorithm for recognizing whether a constant is a combination of other, more fundamental, constants.

Notice that we can use the BBP series to extract some hexadecimal digit of π far down the line. Consider, for example, the real

$$s == \sum_{k=0}^{\infty} \frac{1}{16^k} \frac{4}{8k+1}$$

The far-away digits (say, five digits starting from the dth) arise as the fractional part of $16^d \times s$. Multiplying 16^d into the sum and then splitting it into a sum from 0 to d and a sum from d+1 to infinity does the job. The second sum is

$$s == \sum_{k=d+1}^{\infty} \text{FractionalPart}\left[\frac{1}{16^{k-d}} \frac{4}{8\,k+1}\right]$$

and it is easy to see that 15 terms of this are more than enough for the required accuracy. The first sum is

$$s == \sum_{k=0}^{d} \text{FractionalPart}\left[\frac{1}{16^{k-d}} \frac{4}{8\,k+1}\right]$$

and this is computed by reducing the numerator modulo 8k+1. As an example, notice that the 21th through 25th digits in the base-16 expansion of π are 8a2e0 as can be seen from

```
BaseForm[N[π, 40], 16]
```

$3.243f6a8885a308d313198a2e03707345_{16}$

We can get those sigits using the BBP series by forming the following sum (with the factor 16^{20} incorporated in the formula):

$$s = \sum_{k=0}^{35} \frac{1}{16^{k-20}} \left(\frac{4}{8\,k+1} - \frac{2}{8\,k+4} - \frac{1}{8\,k+5} - \frac{1}{8\,k+6}\right);$$

and taking its fractional part base 16:

```
N[FractionalPart[s], 7]
```

0.5397646

```
BaseForm[%, 16]
```

$0.8a2e03_{16}$

Notice that the FractionalPart[$\frac{20}{3}$] == $\frac{\text{Mod}[20,3]}{3}$. Because we want the fractional part, only the residue counts, and that can be divided by *8k+1* using floating-point arithmetic. Of course, in the case of π we have four such series to consider. Below is the code for such a `DigitExtractor`.

```
DigitExtractor[d_, b_, coeffs_List] := Module[{
    n = d - 1, mainSum = 0, s = Sign[b], base = Abs[b], nc = N[coeffs],
    rd, m = Length[coeffs]},
   Do[mainSum = Mod[mainSum + s^k nc . Table[If[coeffs[[i]] == 0, 0,
            N[PowerMod[base, n - k, (8 k + i)]] / (8 k + i)], {i, m}], 1],
     {k, 0, n}];
    rd = RealDigits[Mod[mainSum + Sum[s^k nc . (base^(n-k) / (8 k + Range[m])),
       {k, n + 1, n + 15}], 1], base];
   Take[Join[Array[0 &, -rd[[2]]], rd[[1]]], 5]]
```

```
DigitExtractor[0, 16, {4, 0, 0, -2, -1, -1}] // Timing
{0.17 Second, {3, 2, 4, 3, 15}}

BaseForm[N[π], 16]
3.243f₁₆
```

Archimedes' Discovery of the Integral Calculus

Archimedes computed the volume of a sphere with his now famous variable cross-sections method that is the forerunner to Riemann integration. The presentation of this method below is that of Polya [7], interspersed with Mathematica diagrams for better understanding.

Archimedes regards the sphere as generated by a revolving circle, and he regards the circle as a locus characterized by a relation between the distances of a variable point from two fixed rectangular axes of reference. Written in modern notation, this relation is

$$x^2 + y^2 = 2ax, \quad (1)$$

a circle with radius a that touches the y-axis at the origin, as shown in Figure 2 below for a circle of radius 1. (Note that a = 1 throughout our discussion.)

```
a = 1;
sphere2D = ParametricPlot[{{x, √(2*a*x - x²)}, {x, -√(2*a*x - x²)}},
  {x, 0, 2 a}, AspectRatio → Automatic, Axes -> True,
  AxesLabel -> {"x", "y"}, Ticks -> {{0, 1, 2}, Automatic}];
```

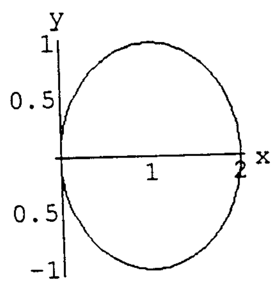

Figure 2

The circle of radius 1 generating the sphere.

The circle, revolving about the x-axis, generates a sphere. As Polya indicates [7], using modern notation does not distort Archimedes' idea. On the contrary, it seems to be suggestive. It suggests motives which may lead us to Archimedes' idea today and which are, perhaps, not too different from the motives that led Archimedes himself to his discovery.

Consider the term y^2 in equation (1). Observe that πy^2 is the area of a variable cross-section of the sphere. Yet Democritus found the volume of the cone by examining the variation of its cross-section. This leads us to rewrite the equation of the circle in the form

$$\pi x^2 + \pi y^2 = \pi 2ax \quad (2)$$

We can now interpret πx^2 as the variable cross-section of a cone, generated by the rotation of the line y = x about the x-axis (Figure 3).

```
a = 1;
cone2D = ParametricPlot[{{x, x}, {x, -x}},
  {x, 0, 2 a}, AspectRatio → Automatic, Axes -> True,
  AxesLabel -> {"x", "y"}, Ticks -> {{0, 1, 2}, Automatic},
  DisplayFunction -> Identity];

Show[%, Graphics[Line[{{2 a, 2 a}, {2 a, -2 a}}]],
 DisplayFunction -> $DisplayFunction];
```

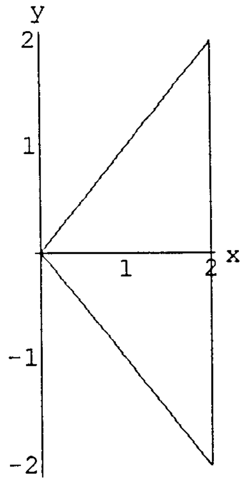

Figure 3

The line y=x generating the cone.

This suggests that we seek an analogous interpretation of the remaining term $\pi 2ax$. If we do not see such an interpretation, we may try to rewrite the equation in still other ways, and so we hit upon the form

$$2\,a\,(\pi y^2 + \pi x^2) = x\pi\,(2\,a)^2 \tag{3}$$

Now the term on the right-hand side of equation (3) is the cross-section of a cylinder generated by the rotation of the rectangle shown in Figure 4 about the x-axis.

```
a = 1;
cylinder2D = Show[Graphics[Line[
    {{2 a, 0}, {2 a, 2 a}, {0, 2 a}, {0, -2 a}, {2 a, -2 a}, {2 a, 0}}]],
  AspectRatio → Automatic, Axes -> True, AxesLabel -> {"x", "y"},
  Ticks -> {{0, 1, 2}, Automatic}] ;
```

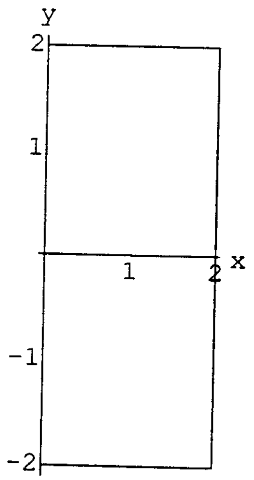

Figure 4

The rectangle generating the cylinder.

We combine Figures 2, 3 and 4 to obtain Figure 5.

```
Show[sphere2D, cone2D, cylinder2D,
 AspectRatio → Automatic, Axes -> True, AxesLabel -> {"x", "y"},
 Ticks -> {{0, 1, 2}, Automatic}] ;
```

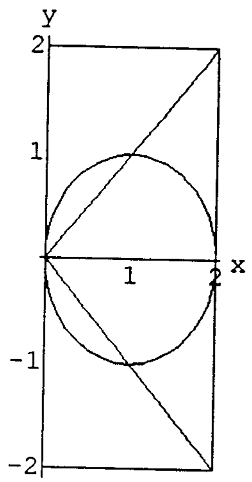

Figure 5

The figure generating a sphere, a cone and a cylinder.

When Figure 5 is revolved about the x-axis, there result a sphere, a cone and a cylinder. Notice that the cone and the cylinder have the same base and height (the vertex of the cone is at the origin (0,0,0)). The radius of the common base and the height have the same length 2a. In Figure 6 below we generate these solids using Mathematica's add-on package Graphics`Shapes`. (The function RotateShape below rotates a three-dimensional object by the specified Euler angles.)

```
<< Graphics`Shapes`;
```

```
a = 1;
g1 = RotateShape[Graphics3D[Cylinder[2 a, 1, 20]], 0,
   π/2, π/2] // WireFrame;
g2 = RotateShape[Graphics3D[Cone[2 a, 1, 20]], 0,
   π/2, π/2] // WireFrame;
g3 = RotateShape[Graphics3D[Sphere[a, 20, 15]], 0,
   π/2, π/2] // WireFrame;
Show[RotateShape[{g1, g2, g3}, π, 0, 0], Boxed -> False,
 ViewPoint -> {-1.010, -5.546, -0.484}];
```

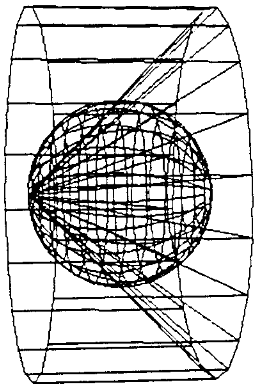

Figure 6

The 3 solids generated by *Mathematica*.

Much is concentrated in equation (3) above. We notice the areas of three circular disks, πy^2, πx^2, and $\pi(2\ a)^2$. The three circles are the intersections of the same plane with three solids of revolution. The plane is perpendicular to the x-axis, and at the distance x from the origin ($0 \le x \le 2a$). (With Mathematica, the only posible view of this intersection is that of Figure 7.)

```
a = 1;
Show[Graphics3D[Cylinder[2 a, 1, 20]], Graphics3D[Cone[2 a, 1, 20]],
 Graphics3D[Sphere[a, 20, 15]], PlotRange -> {-1, .6},
 Boxed -> False, ColorOutput -> GrayLevel];
```

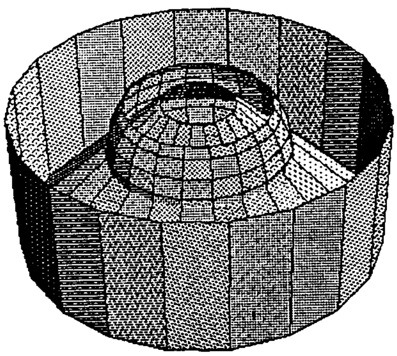

Figure 7

The intersection of the three solids of revolution with the same plane.

Archimedes treats differently the disks, the areas of which appear on different sides of the equation (3). He leaves the disk with radius 2a, cross-section of the cylinder, in its original position. Yet he removes the disks with radii y and x, cross-sections of the sphere and the cone, respectively, from their original position and transports them to the point (-2a, 0) of the x-axis. (We show the suspended disk of the sphere with radius y between two empty circles, and the disk of the cone with radius x between two filled squares. The disk of the cylinder, at abscissa x, is shown between two filled circles.) We let these disks with radii y and x hang by their center vertically under (-2a, 0), suspended by a string of negligible weight (Figure 8). (This string is an addition to Archimedes' original figure.)

```
a = 1;
Show[sphere2D, cone2D, cylinder2D,
  Graphics[{{Dashing[{0.02, 0.02}], Circle[{-2, -1}, 1]},
   Circle[{-2.91652, -.6}, .1], Circle[{-1.08348, -.6}, .1],
   Circle[{.59, .9}, .1], Circle[{.59, -.9}, .1]}],
 Graphics[{Disk[{.6, 2}, .1], Disk[{.6, -2}, .1]}],
 Graphics[{Line[{{.6, 2}, {.6, -2}}],
   Line[{{-1.08348, -.6}, {-2.91652, -.6}}],
 {Dashing[{0.02, 0.02}],
    Line[{{-2 a, -2 a}, {-4 a, -4 a}, {0, -4 a}, {-2 a, -2 a}}]},
 Line[{{-2.6, -2.60}, {-1.4, -2.60}}],
 Line[{{-2 a, 0}, {-2 a, -2.60}}]}],
 Graphics[
  {Point[{-2 a, 0}], Point[{-2 a, -.6}], Point[{-2 a, -2.60}]}],
 Graphics[{Text["<---2--->", {-a, .2}],
 Text["<---2--->", {a, 2 a + .2}],
 Text["-x-", {.3, -2 a - .1}]}],
 Graphics[{Rectangle[{-2 a - .7, -2 a - .5}, {-2 a - .5, -2 a - .7}],
   Rectangle[{-2 a + .7, -2 a - .5}, {-2 a + .5, -2 a - .7}],
   Rectangle[{.7, .5}, {.5, .7}], Rectangle[{.7, -.5}, {.5, -.7}]}],
 Prolog -> AbsolutePointSize[3],
AspectRatio → Automatic, Axes -> True,  AxesLabel -> {"x", "y"},
 PlotRange -> {{-4, 4}, Automatic}] ;
```

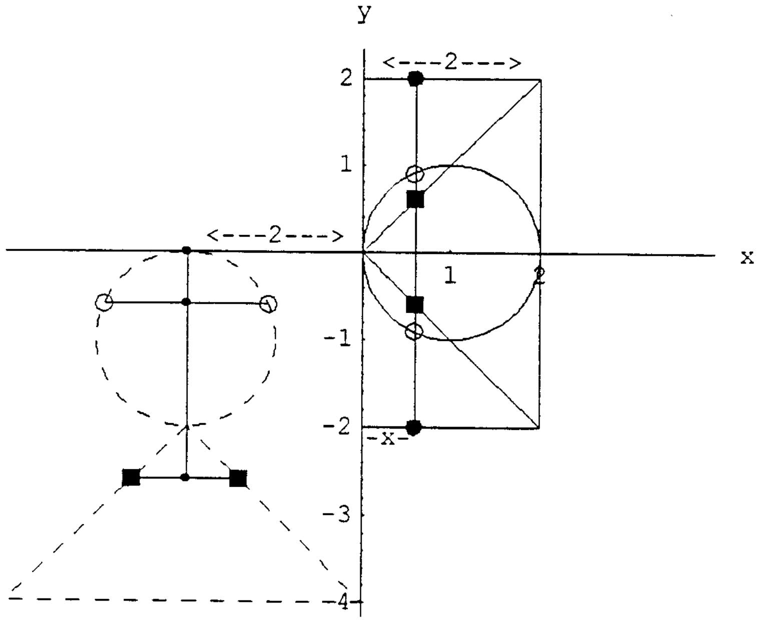

Figure 8

Archimedes' original figure depicting the suspended disks.

Let us regard the x-axis as a lever, a rigid bar of negligible weight, and the origin o as its point of suspension. Equation (3) deals with moments. (A moment is the product of the weight and the arm of the lever). Equation (3) says that the moment of the two disks on the left-hand side equals the moment of the one disk on the right-hand side and so, by the mechanical law discovered by Archimedes, the lever is at equilibrium.

As x varies from 0 to 2a, we obtain all cross-sections of the cylinder; these cross-sections fill the cylinder. To each cross-section of the cylinder there correspond two cross-sections suspended from the point (-2a, 0) and these cross-sections fill the sphere and the cone respectively. As their corresponding cross-sections, the sphere and the cone, hanging from (-2a, 0), are in equilibrium with the cylinder. Therefore, By Archimedes' mechanical law, the moments must be equal. Let V denote the volume of the sphere; let us recall the expression for the volume of the cone (due to Democritus) and also the volume of the cylinder and the obvious location of its center of gravity. Passing from the moments of the cross-sections to the moments of the corresponding solids, equation (3) leads to

$$2a\left(V+\frac{\pi\,(2a)^2\,2a}{3}\right) = a\pi\,(2a)^2\,2a, \tag{4}$$

which yields

$$V = \frac{4\pi a^3}{3}. \tag{5}$$

Looking back at the foregoing, we see that the decisive step is that from equation (3) to equation (4), from the filling cross-sections to the full solids. Yet this step is only heuristically assumed, not logically justified. It is plausible but not demonstrative. It is a guess, not a proof. And Archimedes, representing the great tradition of Greek mathematical rigor, knows this full well: "The fact at which we arrived is not actually demonstrated by the argument used; but the argument has given a sort of indication that the conclusion is true." This guess, however, is a guess with prospect. The idea goes much beyond the requirements of the problem at hand, and has an immensely greater scope.

The passage from (3) to (4), from the cross-section to the whole solid, in more modern language is the transition from the infinitesimal part to the whole quantity, from the differential to the integral. This transition is a great beginning and Archimedes, who was able to see himself in historical perspective, knew it full well: "I am persuaded that this method will be of no little service to mathematics. For I foresee that this method, once understood, will be used to discover other theorems that have not yet occurred to me, by other mathematicians, now living or yet unborn."

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SOME ELEMENTARY MATHEMATICS REVISITED AND REVISED

David J. Jeffrey

The developers of computer algebra systems have had to reconsider, and sometimes rewrite, results in elementary mathematics in order to use them reliably. This talk describes several topics that most mathematicians would consider elementary and «solved», but which have had to be modified for use in a computer algebra system. For example, quadratic, cubic and quartic equations should not be solved following the standard books. Inverse functions could be implemented more easily in computer algebra systems if they were formulated differently. The changes, and the background to the changes, will be described. Demonstrations in Derive and Maple will be given if possible. Teaching Error-Correcting Codes, Discrete Mathematics and Modern Algebra with Computer Algebra.