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ON THE FORGOTTEN THEOREM OF MR. VINCENT

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SUMMARIES

A little known theorem concerning the isolation of roots of polynomial equations, published in 1836 by a mathematician known only as Mr. Vincent, is discussed. Mr. Vincent's method is of historical and practical interest because it requires fewer computations than Sturm's method. The advantages afforded by this theorem are particularly relevant to software systems for computerized algebra. Certain computational results which offer an empirical comparison of the two methods are also presented.

Ci'examiné est un théorème peu connu relatif à l'isolement des racines des équations polynômes, publié en 1836 par un mathématicien connu simplement sous le nom de Monsieur Vincent. La méthode de Vincent est d'un intérêt historique aussi bien que pratique puisqu'elle exige moins de calculs que celle de Sturm. Les avantages admis par ce théorème sont particulièrment applicables à des programmes de manipulations algèbriques. Certains résultats qui offrent une comparaison empirique des deux méthodes sont également présentés.

Diskutiert wird ein wenig bekannter Lehrsatz, der die Isolation von Wurzeln polynomischer Gleichungen betrifft und 1836 von einem nur als Herr Vincent bekannten Mathematiker veröffentlicht wurde. Herrn Vincents Methode hat historisches und praktisches Interesse, weil sie weniger Berechnungen als Sturms Methode verlangt. Die Vorteile, die dieser Lehrsatz hat, sind für Software Systeme für Computeralgebra besonders wichtig. Eingeschlossen sind auch gewisse Berechnungsergebnisse, die einen empirischen Vergleich der beiden Methoden darbieten.

1. INTRODUCTION

In 1834 a certain Mr. Vincent published a "note" (of thirty pages) in the *Mémoires de la Société royale de Lille* concerning the solution of polynomial equations with numerical coefficients. The same memorandum appeared two years later, with a few additions, under the title "Note sur la résolution des équations numérique" in the October issue, 1836, of the *Journal de Mathématiques Pures et Appliquées* [Vincent 1836]. According to a footnote, the article was reprinted "for the benefit of the professors". Nevertheless, the article and the remarkable method described therein were consigned to oblivion for more than a century [1], although it seems that several people had dealt with variations of this method; Mr. Vincent even mentions that a similar "note" was included in the sixth edition of Bourdon's Algebra [Bourdon 1831].

We may attempt to explain the fact that Vincent's theorem was forgotten by noting the careful manner in which he pays tribute to Sturm and notes the "beauty" and usefulness of Sturm's celebrated theorem on the location of the roots of equations. In 1834, the same year in which Vincent first published his paper, Sturm published his work on second order differential equations, known today as the Sturm-Liouville theory, for which he received the "Grand Prix des Sciences Mathématiques" from the Académie des Sciences. Two years later, when Vincent's paper was reprinted, Sturm was elected to the Académie des Sciences. It is not surprising that Sturm's method outshone all the others. Vincent uses in his approach a theorem on the isolation of roots, given by the French physician F. D. Budan in 1807. This theorem was enunciated in a somewhat different form by J. B. J. Fourier and included in his *Analyse des Équations* published posthumously by C. L. M. H. Navier. Vincent indicates his surprise that Fourier did not try to go further and prove the proposition that was the main subject of Vincent's paper. He states, however, the belief that such a proof may exist in other manuscripts which were not published because of the untimely death of Navier.

The Budan-Fourier theorem on the isolation of roots uses a sequence of derivatives, $f(x)$, $f^{(1)}(x)$, ..., $f^{(n)}(x)$. The procedure is similar to that used later (1829), by Sturm, to calculate the number of variations of sign (as defined by Descartes' rule) in the two sequences of real numbers obtained when x is replaced by the real numbers p and q ($p < q$). This method, however, involves less computation than Sturm's method.

Another possible reason that Vincent's original method may not have proved popular is his somewhat inefficient procedure for obtaining the transformed equation for the substitution $x = y + \alpha$. He obtains the coefficients of this equation as coefficients of the Taylor expansion of the polynomial. As it

will be seen in the next section, Uspensky simplified the method considerably by using synthetic division (Horner's rule in essence) to obtain the coefficients of the transformed equation [Uspensky 1948, 127-137]. Moreover, for $\alpha = 1$ the synthetic division does not include any multiplication. With this modification Vincent's theorem becomes a powerful tool for the isolation of real roots of equations, a tool that represents an essential improvement over methods based on Sturm's theorem.

So far as we have been able to determine, Vincent's theorem and the implied method for isolating roots are not mentioned by any author with the exception of Uspensky [1948, 128] and Obreschkoff [1963, 248-249]. Uspensky notes that even such a capital work as the *Enzyclopädie der mathematischen Wissenschaften* ignores it. We will show that Vincent's contribution has practical significance as well as historical.

A statement of the theorem and of the propositions on which it is based will be given in the following sections, together with a description of its use and an assessment of the related method for root isolation.

2. VINCENT'S THEOREM

Most methods for the isolation of the roots of polynomials with numerical coefficients rely on Descartes' rule of signs. According to this rule, the number of positive real roots of a polynomial equation with real coefficients is never greater than the number of variations in the sequence of its coefficients, $a_0, a_1, \ldots, a_n$, and if less, the difference is an even number. A variation is defined as a change of sign in two consecutive terms of the sequence of coefficients. Zero coefficients are disregarded in counting the number of variations. It must be noted that Descartes' rule gives the exact number of roots only if there is either one or no variation. In the first case there is one positive real root; in the second there is no root.

As mentioned previously, Vincent states in his paper that he based his proposition on a result which was formulated somewhat differently by both Budan and Fourier. Vincent states Budan's theorem as follows:

> If in an equation in x, $f(x) = 0$, we make two transformations $x = p + x'$ and $x = q + x''$, where p and q are real numbers such that $p < q$, then
>
> (i) the transformed equation in $x' = x - p$ cannot have fewer variations than the transformed equation in $x'' = x - q$;
>
> (ii) the number of real roots of the equation $f(x) = 0$ located between p and q can never be more than the number of variations lost in passing from the transformed equation in

$x' = x - p$ to the transformed equation in $x'' = x - q$;

(iii) when the first number is less than the second, the difference is always an even number.

Fourier's version of this result is stated as follows:

If in the sequence of (m+1) functions

$$f(x),\ f^{(1)}(x),\ \ldots,\ f^{(m)}(x),$$

we replace x by any two real numbers p, q ($p < q$), and if we represent by P and Q the two resulting sequences of numbers, then

(i) the sequence P cannot present fewer variations than the sequence Q;

(ii) the number of real roots of the equation $f(x) = 0$, located between p and q, can never be more than the number of variations lost in passing from the substitution $x = p$ to the substitution $x = q$;

(iii) when the first number is less than the second, the difference is an even number.

Using this result, Vincent carries out several consecutive transformations in order that the transformed equation will have only one or zero variations; in this case the number of roots can be determined umambiguously. He states his proposition as follows:

If in a polynomial equation with rational coefficients and without multiple roots, one makes successive transformations of the form

$$x = a + \frac{1}{x'},\ x' = b + \frac{1}{x''},\ x'' = c + \frac{1}{x'''},\ \ldots, \tag{1}$$

where a, b and c are any positive numbers greater than one, then the resulting transformed equation either has zero variations or it has a single variation. In the second case the equation has a single positive real root represented by a continued fraction

$$a + \cfrac{1}{b + \cfrac{1}{c + \ldots}}\ ; \tag{2}$$

in the first case there is no root.

It is obvious that Vincent's method relies heavily on transformations which consist of a translation and an inversion. While the inversion can be easily obtained by reversing the order of the polynomial coefficients, the translation operation requires a computation which Vincent does not perform in the easiest

possible way. He uses Taylor's expansion theorem to obtain the coefficients of the transformed polynomial. If the substitution $x = y + a$ is made in the polynomial equation $f(x) = c_0 + c_1x + \ldots + c_nx^n = 0$, then the coefficients

$$c_k' = \frac{f^{(k)}(\alpha)}{k!}, \quad k = 0, 1, \ldots, n,$$

of the transformed polynomial in y may be expressed by

$$c_k' = \sum_{j=k}^{n} \binom{j}{k}\alpha^{j-k}c_j, \quad k = 0, 1, \ldots, n.$$

The computation is somewhat simplified when $\alpha = 1$. Since Horner's method had been known from 1819, it is rather surprising that Vincent did not employ it for polynomial evaluation [Horner 1819, 308-335] in order to perform translations. The applicability and superiority of this method can be seen from the following: consider a polynomial

$$f(x) = c_0x^n + c_1x^{n-1} \ldots + c_n \tag{3}$$

and the transformation $x = y + \alpha$. This substitution produces a polynomial in y having coefficients b_i which satisfy the relations

$$\sum_{i=0}^{n} c_ix^{n-i} = \sum_{i=0}^{n} c_i(y+\alpha)^{n-i} = \sum_{i=0}^{n} b_iy^{n-i} = \sum_{i=0}^{n} b_i(x-\alpha)^{n-i}. \tag{4}$$

From the first and last expressions in (4), we deduce that the coefficients b_i of the transformed polynomial, can be obtained with a sequence of applications of the synthetic division algorithm (i.e., Horner's rule for the evaluation of polynomials). Indeed, (4) suggests that we can write

$$f(x) = (x-\alpha)g(x) + R_n, \tag{5}$$

where $g(x)$ is a polynomial of degree n-1, and R_n is the coefficient b_n in (4). If we express $f(x)$ as in (3), and $g(x)$ as a polynomial of degree n-1 with coefficients $a_0, a_1, \ldots, a_{n-1}$, and then equate the coefficients of equal powers of x in (5), we obtain

$$a_0 = c_0 \quad a_j = c_j + \alpha a_{j-1} \quad j = 1, 2, \ldots, n, \tag{6}$$

which is the synthetic division algorithm. We notice that the last coefficient a_n is precisely the remainder R_n in (5), or equivalently, the coefficient b_n in (4). Further, application

of the same process to $g(x)$, gives

$$g(x) = (x-\alpha)\, h(x) + R_{n-1}, \tag{7}$$

where $h(x)$ is of degree $n-2$. Combining (5) and (7), we obtain

$$f(x) = (x-\alpha)^2\, h(x) + R_{n-1}(x-\alpha) + R_n. \tag{8}$$

If the process is repeated n times we obtain

$$f(x) = R_0(x-\alpha)^n + R_1(x-\alpha)^{n-1} + \ldots + R_n,$$

in which the coefficients R_i are equal to the coefficients b_i, and appear as remainders in each application of algorithm (6). In particular, for $\alpha = 1$ this algorithm does not require any multiplication, so that the transformation $x = y + 1$ can be performed in a very efficient manner. To find the exact number of positive roots of a polynomial, we consider separately the positive roots of a polynomial which are > 1 or < 1, excluding the case when 1 is a root. The positive roots that are greater than 1 may be written in the form $x = 1 + y$, while those less than 1 may be written in the form $x = 1/(1 + y)$ where $y > 0$. We can therefore transform a polynomial equation by the substitutions $x = 1 + y$ and $x = 1/(1 + y)$ and examine the number of variations of the transformed equations [Uspensky 1948, 128]. If the equation obtained by the transformation $x = 1 + y$ has no variations, it means that the original equation has no roots > 1, whereas the presence of one variation indicates precisely one root > 1 of the given equation. Similar conclusions hold for the equation resulting from the transformation $x = 1/(1 + y)$. If one or both of the transformed equations have more than one variation, we transform them again by the substitutions $y = 1 + z$, $y = 1/(1 + z)$, and if necessary continue to make similar substitutions until the transformed equations have no more than one variation. This necessarily must happen after a finite number of steps. The negative roots are investigated by replacing x by $-x$ in the original equation and by investigating the positive roots of the transformed equation.

Uspensky has given a proof of Vincent's theorem which includes also the concept of root separation. For the sake of completeness we state this theorem in the words of Uspensky:

> *Let N_k be the k-th term of the series*
>
> $$1, 1, 2, 3, 5, 8, 13, 21, \ldots$$
>
> *in which each term is the sum of the preceding two and where $\Delta > 0$ is the smallest distance between any two roots of the equation $f(x) = 0$ of degree n and without multiple roots. Let the number m be so chosen that*

$$N_{m-1}\Delta > \frac{1}{2},\ \Delta N_m N_{m-1} > 1 + \frac{1}{\varepsilon}$$

where

$$\varepsilon = (1 + \frac{1}{n})^{\frac{1}{n-1}} - 1.$$

Then the substitution

$$x = a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_m + \cfrac{1}{\xi}}}$$

presented in the form of a continued fraction with arbitrary positive integral elements $a_1, a_2, \ldots, a_m$ *transforms the equation* $f(x) = 0$ *into the equation* $F(\xi) = 0$, *which has not more than one variation.*

[Uspensky 1948, 298-299]

3. SOME APPLICATIONS AND ASSESSMENT OF VINCENT'S METHOD

It is obvious that this method for isolating the roots of polynomials is of great significance for numerical mathematics. But its most significant application is in the area of computerized algebra otherwise known as *symbolic and algebraic manipulation*, in which the exact computation of the roots of polynomials is required. Among the existing major algebra systems, SAC-1 is well known for its polynomial algebra capabilities [Collins 1971]. Most algorithms for root isolation have been based on Sturm's theorem [2]. Further, the decision methods for elementary algebra developed by Tarski [1951] and by Seidenberg [1954] involve root isolation techniques which use Sturm's theorem [Akritas 1973]. Such techniques are likewise important in quantifier elimination algorithms [Collins 1975].

Heindel [1971] has developed an algorithm for root isolation, based on Sturm's theorem, for which he shows that the computing time is bounded by $n^{13}L(d)^3$, where n is the degree of the polynomial and L(d) the length (number of bits in binary representation) of the sum of the absolute values of the polynomial coefficients. It is obvious that algorithms which use more efficient methods, such as Vincent's, are very desirable.

The superiority of Vincent's method is confirmed by empirical comparisions of the computation time for root isolation performed with algorithms based on the two theorems [Collins and Akritas 1976]. The table below shows the computation time in seconds for randomly generated polynomials of degrees 5-25.

DEGREE	STURM	VINCENT
5	0.58	0.07
10	6.83	0.21
15	28.8	0.32
20	89.6	0.68
25	208.2	0.74

A theoretical analysis of Vincent's method is currently being prepared by the authors.

NOTES

1. One of the authors of this article (AGA) came across Vincent's theorem while reviewing methods for the isolation of real roots of equations as presented by Uspensky [Uspensky 1948].

2. Recently, another algorithm based on Rolle's theorem has been developed [Collins and Loos 1976].

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