contributions

A SHORT NOTE ON A NEW METHOD FOR POLYNOMIAL REAL ROOT ISOLATION

Alkiviadis G. Akritas
North Carolina State University
Raleigh, North Carolina 27650

The purpose of this short note is to make known to our community some of the results of my Ph.D. Dissertation [1]. More details can also be found elsewhere [2], [3].

Isolation of the real roots of a polynomial is the process of finding real intervals such that each contains exactly one real root and every real root is contained in some interval. According to Fourier, this is the first step in solving an equation numerically, the second being the approximation of the roots to any desired degree of accuracy.

Since 1830 the only real root isolation method widely known and used was that of Sturm, which actually is a bisection method. Heindel [6] implemented it in the computer algebra system SAC-1, and proved that it is

$$0(n^{13}L(|P|_\infty)^3),$$

where n is the degree of the integral, univariate, square-free polynomial P, and $L(|P|_\infty)$, the length, in bits, of its maximum coefficient in absolute value.

Quite recently the author discovered Vincent's remarkable theorem of 1836, [8], [5], according to which if an equation with rational coefficients and without multiple roots, is successively transformed by substitutions of the form $x \leftarrow a_i + \frac{1}{x}$, for arbitrary, integral, positive a_i's, one eventually obtains an equation with at most one sign variation. This theorem is carefully studied, some new concepts are introduced, and an extension of it is presented (known as the Vincent-Uspensky-Akritas theorem [4]), along with two important theoretical bounds, one on the number of executions of the substitutions of the form $x \leftarrow a_i + \frac{1}{x}$, and another one on the values of the positive integral quantities a_i.

The calculation of the quantities a_1, a_2, $\cdots$ constitutes the real root isolation procedure. Two methods actually result-Vincent's and Akritas'-corresponding to the two different ways of computing these a_i's. These methods, which have been implemented in REDUCE-2 and SAC-1, are fully described and their computing times are carefully analyzed. It turns out that the difference between these two methods can be thought of as being analogous to the difference between the integrals of Riemann and Lebesgue.

Vincent's method, [8] basically consists of computing a particular a_i by a series of unit incrementations, $a_i \leftarrow a_i + 1$. This "brute force" approach results in a method with an exponential behavior, which however, will work extremely fast when the a_i's are all very small.

The author's method, on the contrary, consists of immediately computing a particular a_i as the "positive lower root bound" of a polynomial. This aesthetically pleasing interpretation of the Vincent-Uspensky-Akritas theorem results in a method, which by far surpasses Sturm's (and all the other existing [1]) in beauty and simplicity; moreover, our method is the only one with polynomial computing time bound, which

(i) isolates the real roots of a polynomial by continued fraction approximation, instead of by bisection, and

(ii) can yield isolating intervals of "minimum" length.

We have been able to show that our method is

$$0(n^5 L(|P|_\infty)^3);$$

this is the best theoretical computing time bound achieved thus far, and empirical results verify the superiority of Akritas' method over all others existing.

As an illustration we present the following table, ([1] p. 145), obtained by using SAC-1:

Random Polynomials

Degree	Sturm	Akritas
5	2.05	.26
10	33.28	.46
15	156.40	.94
20	524.42	2.36

The integral polynomials have coefficients, which are 10 decimal digits long; the times are in seconds.

Historical Note. Uspensky ([7], p. 128) describes a method similar to Vincent's, which he claims, in the preface of his book, that he himself invented!!! As the author observed, ([1], pp. 85-86) what can be considered a contribution on Uspensky's part, is only the fact that he used the Ruffini-Horner method in order to perform the substitutions $x \leftarrow 1 + x$ and $x \leftarrow \frac{1}{1+x}$; Vincent on the contrary used Taylor's expansion theorem [8].

References

1. Akritas, A. G., Vincent's Theorem in Algebraic Manipulation, Ph.D. Dissertation, North Carolina State University, 1978.

2. Akritas, A. G., "Vincent's Theorem and Akritas' Method for Polynomial Real Root Isolation," Proceedings of the North Carolina Academy of Sciences, to appear in the Journal of the Elisha Mitchell Society, 1978.

3. Akritas, A. G., "A New Method for Polynomial Real Root Isolation," Proceedings of the 16th Annual Southeast Regional ACM Conference, Atlanta, Georgia, April 1978, pp. 39-43 (this paper received the First Prize in the Student Paper Competition).

4. Akritas, A. G., "A Correction on a Theorem by Uspensky," Bulletin of the Greek Mathematical Society, submitted for publication.

5. Akritas, A. G., and S. D. Danielopoulos, "On the Forgotten Theorem of Mr. Vincent," Historia Mathematica, in press.

6. Heindel, L. E., "Integer Arithmetic Algorithms for Polynomial Real Zero Determination," Journal of the ACM, Vol. 18, No. 4, Oct. 1971, pp. 533-548.

7. Uspensky, J. V., Theory of Equations, McGraw-Hill, New York, 1948.

8. Vincent, M., "Sur la Résolution des Équations Numériques," Journal de Mathématiques, Pures et Appliquées, Vol. 1, 1836, pp. 341-372.