Nikola Obreschkoff's contribution to the problem of isolating real roots of polynomials with continued fractions

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Abstract: In this talk we first mention some key facts of Obreschkoff's life and work and then delve into the influence of Obreschkoff's book *Verteilung und Berechnung der Nullstellen reeller Polynome*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1963, on the real root isolation problem.

Obreschkoff is one of only two authors in the literature (Uspensky being the other one) who mention Vincent's theorem of 1836. This theorem is the core of the continued fractions (CF) real root isolation method, which turns out to be the fastest method known today. A little known "secret" about this method is that it can be implemented only if we are able to compute "efficient" bounds on the values of the *positive* roots of polynomials — as opposed to bounds on the *absolute* values of the roots. Surprisingly, very little is still known about such positive root bounds. Therefore, it is remarkable that Obreschkoff is the *only* author we know of to have included Cauchy's theorem — for computing positive root bounds — in the above mentioned book of his that came out in 1963, the year he died.

Without Cauchy's theorem, the implementation of the CF algorithm would have been impossible and without Obreschkoff's 1963 book my Ph.D. thesis would not have been completed.

1 Nikola Obreschkoff — A short biographical note

First of all I would like to express my thanks to co-organizer of ACA 2006 Margarita Spiridonova, and Andrey Andreev, of the Bulgarian Academy of Sciences, for providing material difficult to obtain .

It is a humbling experience to hold this talk in Varna — Obreschkoff's birthplace — on the occasion of the 110th anniversary of his birthday. Additionally, for the first time in my life, I am in a position to publicly express my gratitude to Nikola Obreschkoff for his book *Verteilung und Berechnung der Nullstellen reeller Polynome*; that book alone provided, at a crucial junction, the help that was needed for the completion of my Ph.D. thesis.

During the '80s I tried very hard to find someone who knew Obreschkoff so that I can learn more about him. My efforts were successful only in July 1988, when during a conference in Piza, Italy, I met Professor Blagoj S. Popov, from the Mathematics Department, of the University of Skopje. During World War II Professor Popov was a student at the University of Sofia, and had Obreschkoff as a professor. He had only good words to say about him, both as a professor and as a mathematician. Here is what we can additionally learn from B. Penkov, who wrote about Nikola Obreschkoff on his centennial anniverssary.

Nikola Dimitrov Obreschkoff was born in this town of Varna — so that he can be rightfully called a "Varnalis" — on April 18, 1896 as one of the last children in a family of 10. His mother Kitza Obreschkova, a music lover and fluent in French, was the moral and intellectual force of the family. His father was a military officer.

At the turn of the century the family moved to Sofia, where Nikola graduated in 1915 from the Second Boys' High School. In the fall of 1915 Nikola was admitted as a student in the Physico-Mathematical Department of Sofia University. After a short interruption during World War I, Nikola graduated in 1920 and was immediately appointed assistant at the Chair of Calculus. Reflecting on the works of Obreschkoff one comes to the conclusion that he was simultaneously algebraist and analyst and probabilist.

In 1922, Obreschkoff received his "Habilitation" with his papers on the extension of the so-called Budan-Fourier theorem and the distribution of zeros of polynomials — a topic that remained his "true love" to the end.

In 1925 Obreschkoff was promoted to Associate Professor and in 1928 to Full Professor and Head of the Chair of Algebra. He remained at that position for 35 years. He received two Ph.D. degrees — from Palermo (1932) and Paris (1933) — and produced about 250 publications, with an average of 6 or 7 papers per year.

In 1945 Obreschkoff was elected directly as an Ordinary (instead of as a Corresponding) member of the Bulgarian Academy of Sciences and Arts — as it was then called. In 1950 he became director of the then recently established Mathematical Institute and held that position until his death 13 years later.

Obreschkoff authored many and influencial books. Only few months before his unexpected death in 1963, two monographs were published: *Zeros of Polynomials* (in Bulgarian, and now also available in English) and *Verteilung und Berechnung der Nullstellen reeller Polynome*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1963 (in German). These books are the result of 40 years of active research in this field and are a "goldmine" of information. In particular, the book in German can be found in all major University Libraries in the United States and played a tremendous role in the development of the continued fractions method for the isolation of the real roots of polynomials.

In this talk I will not analyze Obreschjoff's work; others more capable than me have done it in the past and will do it again in the future. Instead, I will focus on the influence his 1963 book, in German, had on the development of the continued fractions method.

It is interesting that — back in the late '70s and '80s — I attempted to buy a copy of Obreschkoff's 1963 book in the United States, but this effort was not successful. I then asked my father in Greece to try and buy a copy of it through the Bulgarian Embassy in Athens, but that effort also failed. So, I resorted to making a copy of it from the library — since copying was not illegal back then. Unfortunately, the letters in that copy have faided away and so I am again back to square one.

2 An overview of the polynomial real root isolation problem

It is well known that in the 17th and 18th century many attempts were made to find the solution of the general *quintic* equation. Even though all these attempts failed, the works of Cardano and Desvartes led to a deeper understanding of the "nature" of the roots of polynomial equations.

In 1804 Paolo Ruffini proved that it is impossible to solve by radicals the general quintic equation. Later, in 1826, Abel proved the general case, namely that it is impossible to solve by radicals algebraic equations of degree greater than 4.

In the beginning of the 19th century the attention of the mathematicians had already shifted to *numerical methods* for the solution of polynomial equations. At this time Fourier suggested to proceed in two steps: to wit, first *isolate* the roots and then *approximate* them to any desired degree of accuracy.

Isolation of the real roots of polynomial equations is the process of finding real non-overlapping intervals such that each one of them contains one real root and each real root is contained in some interval. On the other hand, *approximation* is the process of narrowing the intervals such that the roots are computed to the required degree of accuracy.

Isolation is by far the most important problem and as such it attracted the attention of the great matematicians of the time. In the beginning of the 19th century F.D. Budan and J.B.J. Fourier presented two different — but equivalent — theorems that allow us to compute an *upper bound* on the *number* of the real roots that an equation with real coefficients has in a given open interval.

Budan's theorem was published in 1807 in the paper "Nouvelle méthode pour la résolution des équations numériques", whereas Fourier's theorem was first published in 1820 in "Le bulletin des sciences par la société philomatique de Paris." Due to the importance of these two theorems a great dispute erupted concerning priority rights. As F. Arago describes in his book *Biographies of distinguished scientific men* (p. 383) Fourier "deemed it necessary to receive statements from ex students of the Polytechnic School or professors of the University" to prove that he had taught his theorem in 1796, 1797 and 1803.

No wonder then that the "authorship" of these two equivalent theorems is attributed, in the literature, sometimes to Budan, sometimes to Fourier, and most of the times to both — as

Budan-Fourier or Fourier-Budan. The interesting thing to note is that no matter the name of the theorem, its statement is the one due to Fourier, whereas the statement of Budan's theorem was totally missing from the literature up until 1978, the year I completed my Ph.D. thesis. The latter found Budan's theorem *only* in Vincent's paper of 1836, and after that it has appeared in my publications.

Based on Fourier's statement of the theorem, C. Sturm presented in 1829 an improved theorem that allow us to compute the *exact number* of the real roots that an equation with real coefficients has in a given open interval. With the help of his theorem — and using bisection — Sturm became the first person in the history of mathematics to solve the real root isolation problem.

With this achievement Sturm became really famous and from 1830 till 1978 his bisection method was the only one widely known and used. Consequently, Budan's (version of the equivalent) theorem was totally forgotten and along with it Vincent's theorem that is based on it.

Vincent's theorem of 1836 is the basis of the *continued fractions* method for the isolation of the real roots of polynomial equations with integer coefficients, a method by far surpassing Sturm's in efficiency. The continued fractions method is the fastest method existing and has been implemented in *Mathematica*. By contrast, Sturm's method is not used in any computer algebra system. Below is a diagram of the development of Polynomial Real Root Isolation

Fourier's theorem (1820)	⟺	Budan's theorem (1807)
⇓ ⇓		⇓ ⇓
Sturm's theorem (1829)		Vincent's theorem (1836)
⇓ ⇓	↙ ↙	⇓ ⇓
Sturm's bisection method (1829)	↙ ↙	Vincent's Continued Fractions *exponential* method (1836)
	↙ ↙ ↙	⇓ ⇓ ⇓
Collins-Akritas bisection method. (1975-76), Maple		Akritas with Strzebonski CF polynomial method (1978, 1994), *Mathematica*
⇓ ⇓		
Rouillier-Zimmermann		

bisection method
(2004), gmp

Figure 1. The theorems by Budan and Fourier and the two classical methods for the isolation of the real roots of polynomial equations. The methods by Collins-Akritas and Rouillier-Zimmermann are based on a total modification of Vincent's theorem and are not considered classical.

Vincent's first names were forgotten to such an extent that I thought his first initial was “M”, since the 1836 paper is attributed to a certain M. Vincent. However, as Lloyd pointed out, that “M” stands for the French word Monsieur, and according to Poggendorff's *Biographisch-Literarisches Handwörterbuch der exakten Wissenschaften* Vincent's true first names are Alexandre Joseph Hidulf.

Vincent's theorem was so totally forgotten that even such a major work as *Enzyclopaedie der mathematischen Wissenschaften* ignores it. Before 1978 it is not mentioned by anyone, save for **Obreschkoff** (1963) and **Uspensky** (1948). What a coincidence that both authors are of slavic origin! I discovered Vincent's theorem in Uspensky's book in 1975-76 and did my Ph. D. thesis on it (1978).

3 Vincent's theorem and the continued fractions method for the isolation of the real roots

We present Vincent's theorem as found in his paper of 1836 and in **Obreschkoff**'s book of 1963.

Vincent theorem (1836):
If in a polynomial equation, $p(x)$, with rational coefficients and without multiple roots we perform sequentially replacements of the form

$$x \leftarrow a_1 + \frac{1}{x}, x \leftarrow a_2 + \frac{1}{x}, x \leftarrow a_3 + \frac{1}{x}, \ldots$$

where $a_1 \geq 0$ is a random *non negative* integer *and* a_2, a_3, … are random *positive* intagers, $a_i > 0$, $i > 1$, then the resulting polynomial either has *no* sign variations or it has *one* sign variation. In the last case the equation has exactly one positive root, which is represented by the continued fraction

$$a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\ddots}}}$$

whereas in the first case there are *no* positive roots.

Note that if we represent by $\frac{ax+b}{cx+d}$ the conntinued fraction that leads to a polynomial $f(x) = p(\frac{ax+b}{cx+d})$, with one sign variation, then the single positive root of $f(x)$ — in the interval $(0, \infty)$ — corresponds to that positive root of $p(x)$ which is located in the open interval with endpoints $\frac{b}{d}$ and $\frac{a}{c}$. These endpoints are *not* ordered and are obtained from $\frac{ax+b}{cx+d}$ by replacing x with 0 and ∞, respectively.

Therefore, with Vincent's theorem we can isolate the (positive) roots of a given polynomial $p(x)$. The negative roots are isolated — as suggested by Sturm — after we transform them to positive with the replacement $x \leftarrow -x$ performed on $p(x)$. The requirement that $p(x)$ have no multiple roots does not restrict the generality of the theorem because in the opposite case we first apply square-free factorization and then isolate the roots of each one of the square-free factors.

From the statement of Vincent's theorem it becomes clear that in order to isolate the positive roots of a polynomial $p(x)$ we have to compute the partial quotients $a_1, a_2, \ldots, a_m$ for the replacements that will lead to polynomials $f(x)$ with exactly one sign variation. As we will see immediately below, it is at this point that Obreschkoff's 1963 book comes into play.

There are two ways for computing the partial quotients a_i — and, therefore, two ways for isolating the positive roots of $p(x)$ using continued fractions. The first method was developed by Vincent in 1836, whereas the second was developed by me in my Ph.D. thesis (1978). The difference between these two methods for computing the partial quotients a_i is analogous to the difference that exists between the integrals of Riemann and Lebesgue. As we know, the sum 1 + 1 + 1 + 1 + 1 can be computed in the following two ways: according to Riemann as 1 + 1 = 2, 2 + 1 = 3, 3 + 1 = 4, 4 + 1 = 5, whereas according to Lebesgue as 5·1 = 5.

Besides these two ways we will see a “failed” attempt by Uspensky to compute the partial quotients a_i in yet another way. Uspensky's attempt failed because he was unaware of Budan's theorem.

Computing the partial quotients a_i according to Vincent (exponential behavior):
In his 1836 paper Vincent demonstrates his method with several examples. In all of these examples he computes a partial quotient a_i with unit increases of the form $a_i \leftarrow a_i + 1$. Each one of these increases corresponds to the replacement $x \leftarrow x + 1$ which is performed on some polynomial $f(x)$.

During this process Vincent uses Budan's theorem in order to determine when the computation of one partial quotient a_i (whose *initial value* is set to 0) has been *completed*, so that we can move on to the computation of the next. To wit, Vincent keeps performing replacements of the form $x \leftarrow x + 1$ (and unit increases of the form $a_i \leftarrow a_i + 1$) until he detects sign variations losses in the polynomials $f(x + a_i)$ and $f(x + a_i + 1)$. Then, and only then, does Vincent perform the replacement of the form $x \leftarrow \frac{1}{x+1}$, in order to start the computation of the next partial quotient a_{i+1}

Vincent's method is exponential — something that was first observed by Sturm and then by Uspensky. The exponential behavior appears only in the cases of very large partial quotients a_i. However, for small a_i Vincent's method is astonishingly fast.

Computing the partial quotients a_i according to Uspensky (doubly exponential behavior):
Note that Vincent made full use of Budan's theorem. To wit, if the polynomials $f(x)$ and $f(x+1)$ have the same number of sign variations Vincent proceeds to the next unit increase $a_i \leftarrow a_i + 1$ — which is accompanied by the next replacement of the form $x \leftarrow x + 1$. He knew from Budan's theorem that in this case $f(x)$ has no roots in the interval (0, 1).

Uspensky in his book (1949) also computes a partial quotient a_i with unit increases of the form $a_i \leftarrow a_i + 1$ — which are accompanied by unit replacements of the form $x \leftarrow x + 1$. However, if the polynomials $f(x)$ and $f(x+1)$ have the same number of sign variations, Uspensky — not being aware of Budan's theorem — *cannot* conclude that $f(x)$ has no roots in the interval (0, 1). To verify that, in addition to the replacement $x \leftarrow x + 1$, Uspensky has to perform at *each* step the *additional* replacement $x \leftarrow \frac{1}{x+1}$ — from which he expects a polynomial with *no* sign variations.

So, the only thing that Uspensky actually achieved was to double the computing time of Vincent's method. His claim that he discovered a new continued fractions method does not hold water.

Uspensky's contributions to Vincent's continued fractions method:

a. Uspensky performed the replacements $x \leftarrow x + 1$ using the Ruffini-Horner method, whereas Vincent used Taylor's expansion theorem.

b. To tackle the exponential behavior of the method Uspensky proposed replacements of the form $x \leftarrow x + k$, where k is randomly chosen and successively increased. Obviously, the nature of the partial quotients a_i had not been understood.

Computing the partial quotients a_i so as to eliminate the exponential behavior:
In my Ph.D. dissertation, in 1978, I realized that each partial quotient a_i is the integer part of a real number — i.e. $a_i = \lfloor \alpha_s \rfloor$, where α_s is the smallest positive root of some polynomial $f(x)$ — and, hence, that it can be computed as the lower bound, ℓb, on the values of the positive roots of a polynomial. So assuming that $\ell b = \lfloor \alpha_s \rfloor$ the exponential behavior of the continued fractions method can be *eliminated* by setting $a_i \leftarrow \ell b$, $\ell b \geq 1$, and performing the replacement $x \leftarrow x + \ell b$, $\ell b \geq 1$ — which takes about the same time as the replacement $x \leftarrow x + 1$.

A *lower* bound, ℓb, on the values of the positive roots of a polynomial $f(x)$, of degree n, is found by first computing an *upper* bound, ub, on the values of the positive roots of $x^n f(\frac{1}{x})$ and then setting $\ell b = \frac{1}{ub}$. So what was needed was an efficient method for computing upper bounds on the values of (just) the positive roots of polynomial equations.

It is at this point that Obreschkoff's 1963 book played a tremendous role. Whereas the English mathematical literature contained numerous methods for computing upper bounds on the *absolute* values of roots of polynomials, not a single author could be found including in his book an upper bound on *just* the *positive* roots of polynomial equations. You can imagine the agony, sleepless nights and sweat lost during that crucial period of writing the dissertation.

You can also imagine the relief felt when I discovered that the University Library in Raleigh, North Carolina, had a copy of Nikola Obreschkoff's 1963 book: *Verteilung und Berechnung der Nullstellen reeller Polynome*. As I mentioned, this book contained the results of 40 years of research on the “distribution and computation of zeros of polynomials”, a topic that was Obreschkoff's “true love.” As such it did not fail to include the all important Cauchy's rule for computing upper bounds on *just* the positive roots of polynomial equations!

We present Cauchy's rule in our own notation.

Cauchy's rule:
Let $p(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 = 0$ be a polynomial equation of degree $n > 0$, with *integer* coefficients and $c_{n-k} < 0$ for at least one k, $1 \leq k \leq n$. (Please note that $c_n > 0$!) If λ is the number of the negative coefficients then

$$b = \max_{\{1 \leq k \leq n \,:\, c_{n-k} < 0\}} \sqrt[k]{-\frac{\lambda\, c_{n-k}}{c_n}}$$

is an upper bound on the values of the positive roots of $p(x) = 0$.

This efficient rule was immediately programmed and the exponential behavior of the continued fractions method was eliminated once and for all. The results were presented in the ACM conference in Atlanta, Georgia (1978) and won the first prize in the student paper competition.

Please note that in general $\ell b \neq \lfloor \alpha_s \rfloor$ and, therefore, several applications of Cauchy's rule will be needed in order to exactly compute $\lfloor \alpha_s \rfloor$. Even though these computations are relatively “cheap”, nonetheless the need is highlighted for discovering new and better bounds on the values of the positive roots of polynomial equations.

In this respect we point out a new paper (Akritas and Vigklas, 2006) presented here in Varna and in which we compare Cauchy's rule with a recently discovered new rule by Doru Stefanescu (2005). This new rule is a special case of the *alternating sums* method found in the (Russian) literature. Unfortunately, Stefanescu's rule does not always work, but when it does it gives bounds equal to or better — and in certain classes of polynomials *much* better — than those obtained by Cauchy's rule. Who knows what the future holds for us in this direction.

■ Appendix to this section

4 Comparison of two methods for the isolation of the real roots

As we have seen Obreschkoff made significant contributions, through his 1963 book, to the development of the continued fractions method for the isolation of the real roots of polynomials.

In this section we present several tables, *in two different time frames*, that compare the continued fractions method with various bisection methods. In particular, the results — Akritas and Strzebonski (2005) — of the comparison with the bisection method of Rouillier and Zimmermann (2004) have been independently verified by Emiris and Tsigaridas (2006).

Spring of 1978:

The first three tables are from my Ph.D. thesis. They were created using the computer system **sac-1** on an IBM S/370 Model 175 computer and compare the continued fractions method — without Strzebonski's improvement, of course — with that of Sturm. Please note that the max degree of the polynomials is at most 20 — a degree considered "large" at the time!

Polynomials with randomly generated roots from the interval $(0, 10^3)$

Degree	Continued Fractions V-A	Bisection Sturm
5	0.71	0.73
10	23.22	22.50
15	95.35	151.42
20	288.49	> 600

Table 1. In this table each polynomial of degree *n*, (where *n* = 5, 10, 15 and 20) was formed by taking the product of a corresponding number of linear terms.

Polynomials with randomly generated 10-digit coefficients

Degree	Continued Fractions V-A	Bisection Sturm
5	0.26	2.05
10	0.46	33.28
15	0.94	156.40
20	2.36	524.42

Table 2. In this table the coefficients of each polynomial are all different from 0, 10-digits long and randomly generated.

From the first two tables we see that the continued fractions method is much faster than that of Sturm. How does one compare it though with the Collins-Akritas bisection method — a method developed in 1976 and which is also faster than that of Sturm?

The answer was given indirectly as follows: The polynomials of Table 2 are the same polynomials used to compare the bisection method of Sturm with the Collins-Akritas bisection method. So in Table 3 below we compare the ratios of the times of the continued fractions method and of the Collins-Akritas method to the corresponding times of Sturm's method.

Comparison of the continued fractions method with the Collins-Akritas bisection method for the polynomials of Table 2

Degree	Continued Fractions / Sturm	Collins-Akritas / Sturm
5	0.13	0.28
10	0.014	0.10
15	0.004	0.05
20	0.0045	0.03

Table 3. In this table we compare the ratios of the times of the continued fractions method and of the Collins-Akritas method to the corresponding times of Sturm's method.

Even though the comparison is not extensive, from Table 3 we see that the continued fractions method is better than the Collins-Akritas bisection method.

Spring of 2002:

In their recent work, Rouillier *και* Zimmermann (2004) present a new bisection method for the isolation of the real roots of polynomials, "… which is optimal in terms fo memory usage and as fast as the Collins-Akritas method …"

In the following tables we compare the continued fractions method (CF), as it was modified to include Strzebonski's improvement, with the method REL of Rouillier and Zimmermann. Both methods became part of *Mathematica*'s kernel and can be found in the site

http://members.wolfram.com/webMathematica/Users/adams/RootIsolation.jsp

These two methods were tried on the polynomials Chebyshev, Laguerre, Wilkinson and Mignotte, which were used by Rouillier and Zimmermann as well as on three types of random polynomials.

All computations were carried out on a 850 MHz Athlon PC with 256 MB RAM. The data on memory requirements was obtain using *Mathematica*'s MaxMemoryUsed function. At the beginning of the computations *Mathematica*'s kernel occupied 1.6 MB.

Special Polynomials

Polynomials	Degree	# of roots	CF T (s)/M (MB)	REL T (s)/M (MB)
Chebyshev	1000	1000	2172/9.2	7368/8.5
Chebyshev	1200	1200	4851/12.8	15660/11.8
Laguerre	900	900	3790/8.7	22169/14.1
Laguerre	1000	1000	6210/10.4	34024/17.1
Wilkinson	800	800	73.4/3.24	3244/10
Wilkinson	900	900	143/3.66	5402/12.5
Wilkinson	1000	1000	256/4.1	8284/15.1
Mignotte	300	4	0.12/1.75	803/7.7
Mignotte	400	4	0.22/1.77	3422/15.8
Mignotte	600	4	0.54/1.89	26245/49.1

Table 4. For the special polynomials CF, the continued fractions method, was from 3 up to 50000 times faster than REL — for the Chebyshev and Mignotte polynomials, respectively.

As we mentioned CF isolates first the positive roots and then the negative ones. Should it happen that a polynomial is symmetric CF isolates only its positive roots. The Chebyshev polynomials *are* symmetric and so CF takes advantage of this fact; on the contrary, REL does not!

Polynomials with randomly generated coefficients

Coeffs. (# of bits)	Degree	# of roots	CF T (s)/M (MB)	REL T (s)/M (MB)
10	500	3.6	0.78/2.2	1.66/2.81
10	1000	4.4	6.67/3.75	34.2/7.5
10	2000	5.6	215/11.4	562/22.8
1000	500	3.2	0.56/2.28	2.19/2.97
1000	1000	3.6	12.7/5.1	31.4/6.5
1000	2000	6	329/14.2	510/24.3

Table 5. For polynomials with randomly generated coefficients CF, the continued fractions method, was from 1.5 up to 5 times faster than REL.

In Table 6 below each result is the average over a set of 5 polynomials. The number of roots is also the average. The same sets of polynomials were used for both methods.

Polynomials with randomly generated coefficients and unitary leading coefficient

Coeffs. (# of bits)	**Degree**	**# of roots**	**CF** T (s)/M (MB)	**REL** T (s)/M (MB
10	500	5.2	1.43/2.48	8.48/3.84
10	1000	4.8	7.12/3.74	80.7/10.1
10	2000	6.8	263/11.4	1001/37.1
1000	100	4.4	0.01/1.75	56.8/5.5
1000	200	6	0.086/1.93	252/17
1000	500	5.6	0.57/2.28	1917/96.8
1000	1000	6	25.5/5.2	>5000/?

Table 6. The case of polynomials with randomly generated coefficients and unitary leading coefficient proved extremely "difficult" for REL, which was again thousand times slower than CF.

It is *not* a coinsidence that in Table 6 REL was again thousand times slower than CF. Polynomials with randomly generated coefficients and unitary leading coefficient have both very large *and* very small roots. This forces any bisection method to begin the process with a very big interval which must be bisected many times before the small roots get isolated.

Polynomials with randomly generated roots

Roots (# of bits)	**Degree**	**# of roots**	**CF** T (s)/M (MB)	**REL** T (s)/M (MB
10	100	100	0.8/1.82	0.61/1.92
10	200	200	2.45/2.07	10.1/2.64
10	500	500	33.9/2.07	878/8.4
1000	20	20	0.12/1.88	0.044/1.83
1000	50	50	16.7/3.18	4.27/2.86
1000	100	100	550/8.9	133/6.49

Table 7. The case of polynomials with randomly generated roots of order of magnitude 10^{300} was the only case in which CF, the continued fractions method was 4 times slower than REL — as expected from the previous discussion.

The result in Table 7 was as expected from the previous discussion, because the partial quotients a_i are extremely big to wit, of order of magnitude 10^{300}.

From these tables it becomes clear that the continued fractions method is almost always faster than any bisection method. Regarding memory usage, we see that it is the same as, and some times better than, REL.

5 Conclusions

We have seen, in sufficient detail, the continued fractions method for the isolation of the real roots of polynomial equations. From the description of the method it became clear that its implementation would have been impossible without an efficient rule for computing upper, and lower, bounds on the values of just the *positive* roots of polynomials. Such a rule could not be found in any book in the English mathematical literature, back in 1978.

Closing I would like to express my gratidude to Nikola Obreschkoff for his 1963 book *Verteilung und Berechnung der Nullstellen reeller Polynome*, which included Cauchy's rule. Without that rule the implementation of the continued fractions method would have been impossible and without Obreschkoff's 1963 book my Ph.D. thesis would not have been completed.

Nikola Obreschkoff, bolschoe cpaciba!

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