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A Comparison of Various Methods for Computing Bounds for Positive Roots of Polynomials

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Abstract: The recent interest in isolating real roots of polynomials has revived interest in computing sharp upper bounds on the values of the positive roots of polynomials. Until now Cauchy's method was the only one widely used in this process. Ştefănescu's recently published theorem offers an alternative, but unfortunately is of limited applicability as it works *only* when there is an *even* number of sign variations (or changes) in the sequence of coefficients of the polynomial under consideration. In this paper we present a more general theorem that works for any number of sign variations *provided* a certain condition is met. We compare the method derived from our theorem with the corresponding methods by Cauchy and by Lagrange for computing bounds on the positive roots of polynomials. From the experimental results we conclude that it would be advantageous to extend our theorem so that it works without any restrictive conditions[1].

Key Words: upper bounds, positive roots, real root isolation
Category: F.2.1, G.1.5

1 Introduction

Computing an upper bound, ub, on the values of the (real) positive roots of a polynomial $p(x)$ is a very important operation because it can be used to isolate these roots—that is, to find intervals on the positive axis each containing exactly one positive root.

As an example, suppose that the positive roots of $p(x)$ lie in the open interval $]0, ub[$ and that we have a test for determining the number of roots in any interval $]a, b[$. Then, we can isolate these roots by repeatedly subdividing the interval $]0, ub[$ until each resulting interval contains exactly one root and every real root is contained in some interval. This bound, ub, is of practical use because we now work with a definite interval $]0, ub[$, instead of $]0, +\infty[$.

[1] **Note added in proof**: In the mean time the above mentioned theorem was extended by Akritas, Strzebonski, and Vigklas, (in their paper: *"Implementations of a New Theorem for Computing Bounds for Positive Roots of Polynomials"; Computing, 78, (2006), 355–367*) so that it now works in all cases; this extension was achieved by introducing the concept of *breaking up* a positive coefficient into several parts to be paired with "unmatched" negative coefficients of lower order terms.

Obviously, the sharper the upper bound, ub, the more efficient the real root isolation method becomes, since fewer bisections will be performed. Please note that the bisection method uses the upper bound only once and imagine the savings in time that would occur if an isolation method depends heavily on repeated computations of such bounds!

Such is the case with the continued fractions method for isolating the positive roots of polynomial equations. This method is based on Vincent's theorem of 1836, [Vincent 1836], which states:

Theorem 1. *If in a polynomial equation, $p(x)$, with rational coefficients and without multiple roots we perform sequentially replacements of the form*

$$x \leftarrow \alpha_1 + \frac{1}{x}, x \leftarrow \alpha_2 + \frac{1}{x}, x \leftarrow \alpha_3 + \frac{1}{x}, \ldots$$

where $\alpha_1 \geq 0$ is an arbitrary non negative integer and $\alpha_2, \alpha_3, \ldots$ are arbitrary positive integers, $\alpha_i > 0$, $i > 1$, then the resulting polynomial either has no sign variations or it has one sign variation. In the last case the equation has exactly one positive root, which is represented by the continued fraction

$$\alpha_1 + \frac{1}{\alpha_2 + \frac{1}{\alpha_3 + \frac{1}{\ddots}}}$$

whereas in the first case there are no positive roots.

Note that if we represent by $\frac{ax+b}{cx+d}$ the continued fraction that leads to a transformed polynomial $f(x) = (cx+d)^n p(\frac{ax+b}{cx+d})$, with one sign variation, then the single positive root of $f(x)$—in the interval $(0, \infty)$—corresponds to *that* positive root of $p(x)$ which is located in the open interval with endpoints $\frac{b}{d}$ and $\frac{a}{c}$. These endpoints are *not* ordered and are obtained from $\frac{ax+b}{cx+d}$ by replacing x with 0 and ∞, respectively. See the papers by Alesina & Galuzzi, [Alesina and Galuzzi 1998] and Chapter 7 in [Akritas 1989] for a complete historical survey of the subject and implementation details respectively.

Therefore, with Vincent's theorem we can isolate the (positive) roots of a given polynomial $p(x)$. The negative roots are isolated—as suggested by Sturm—after we transform them to positive with the replacement $x \leftarrow -x$ performed on $p(x)$. The requirement that $p(x)$ have no multiple roots does not restrict the generality of the theorem because in the opposite case we first apply square-free factorization and then isolate the roots of each one of the square-free factors.

There are two ways for computing the partial quotients α_i —and, therefore, two ways for isolating the positive roots of $p(x)$ using continued fractions. The first method was developed by Vincent in 1836, whereas the second was developed by Akritas in his Ph.D. thesis (1978).

In his 1836 paper Vincent demonstrates his method for computing the partial quotients (exponential behavior) with several examples. In all of these examples he computes a partial quotient α_i with unit increases of the form $\alpha_i \leftarrow \alpha_i + 1$. Each one of these increases corresponds to the replacement $x \leftarrow x + 1$ which is performed on some polynomial $f(x)$.

During this process Vincent uses Budan's theorem, [Akritas 1982], in order to determine when the computation of one partial quotient α_i (whose *initial value* is set to 0) has been *completed*, so that he can move on to the computation of the next one. To wit, Vincent keeps performing replacements of the form $x \leftarrow x + 1$ (and unit increases of the form $\alpha_i \leftarrow \alpha_i + 1$) until he detects sign variation losses in the polynomials $f(x + \alpha_i)$ and $f(x + \alpha_i + 1)$. Then, and only then, does Vincent perform the replacement of the form $x \leftarrow \frac{1}{x+1}$, in order to start the computation of the next partial quotient α_{i+1}.

Vincent's method is exponential—something that was first observed by Sturm and then by Uspensky. The exponential behavior appears only in the cases of very large partial quotients α_i. However, for small α_i Vincent's method is astonishingly fast; see Tables 1 and 3 in [Akritas and Strzebonski 2005] for experimental results, which were independently verified in the SYNAPS implementation of the CF method by Emiris and Tsigaridas, [Tsigaridas and Emiris 2006].

In 1978, [Akritas 1978], [Akritas and Strzebonski 2005], it was realized that each partial quotient α_i is the integer part of a real number—i.e. $\alpha_i = \lfloor \alpha_s \rfloor$, where α_s is the smallest positive root of some polynomial $f(x)$—and, hence, that it can be computed as the lower bound, ℓb, on the values of the positive roots of a polynomial. So assuming that $\ell b = \lfloor \alpha_s \rfloor$ the exponential behavior of the continued fractions method can be *eliminated* by setting $\alpha_i \leftarrow \ell b$, $\ell b \geq 1$, and performing the replacement $x \leftarrow x + \ell b$, $\ell b \geq 1$— which takes about the same time as the replacement $x \leftarrow x + 1$.

A *lower* bound, ℓb, on the values of the positive roots of a polynomial $f(x)$, of degree n, is found by first computing an *upper* bound, ub, on the values of the positive roots of $x^n f(\frac{1}{x})$ and then setting $\ell b = \frac{1}{ub}$. So, clearly, what is needed is an efficient method for computing upper bounds on the values of (just) the positive roots of polynomial equations[2].

The rest of this paper is structured as follows:

In Section 2 we present the well known classical theorems by Cauchy and Lagrange as well as our main result, Theorem 5, which works for any number of sign variations—provided inequality (2) holds.

In Section 3 we present experimental results comparing the methods by Cauchy, Langrange and the one obtained from Theorem 5. The experiments were

[2] with suitable transformations $p(x) \equiv p(-x) = 0$ and $p(x) \equiv x^n p(-\frac{1}{x}) = 0$ one can find the lower $-ub$ and upper $-\frac{1}{ub}$ bounds of the negative roots x_- of $p(x)$ respectively, $-ub \leq x_- \leq -\frac{1}{ub}$

performed using the computer algebra system *Mathematica*, which provides "infinite precision" arithmetic. Note that we work with polynomials having *only* integer coefficients of any (arbitrary) bit length.

2 Methods for computing bounds on the values of the positive roots of polynomials

Several methods for computing bounds on the values of the positive roots of polynomials exist in the literature [Demidovich and Maron 1987], [Kioustelidis 1986], [Obreschkoff 1963]. Recently, Ştefănescu, [Ştefănescu 2005], presented a new theorem giving yet another alternative for such a computation, but unfortunately, it is of limited applicability as it works *only* when there is an *even* number of sign variations (or changes) in the sequence of coefficients of the polynomial under consideration.

In this section we present the two classical theorems by Cauchy and Lagrange-MacLaurin along with Theorem 5, which works for any number of sign variations. The reason for our choice is to evaluate (in the following section) the performance of a new rule based on Theorem 5 against the classical methods, which are widely used.

In the sequel we will refer to polynomials of the type

$$p(x) = \alpha_n x^n + \alpha_{n-1} x^{n-1} + \ldots + \alpha_0, \quad (\alpha_n > 0) \tag{1}$$

with real coefficients $\alpha_n, \alpha_{n-1}, \ldots, \alpha_0$ and having at least one sign variation.

2.1 Cauchy's Method

Theorem 2. *Let $p(x)$ be a polynomial as in Eq. (1), of degree $n > 0$, with $\alpha_{n-k} < 0$ for at least one k, $1 \leq k \leq n$. If λ is the number of negative coefficients, then an upper bound on the values of the positive roots of $p(x)$ is given by*

$$ub = \max_{\{1 \leq k \leq n : \alpha_{n-k} < 0\}} \sqrt[k]{-\frac{\lambda \alpha_{n-k}}{\alpha_n}}$$

Note that if $\lambda = 0$ there are no *positive roots.*

Proof. : From the definition above we have

$$ub^k \geq \left(-\frac{\lambda \alpha_{n-k}}{\alpha_n}\right)$$

for every k such that $\alpha_{n-k} < 0$. For these k, the inequality above could be written

$$ub^n \geq \left(-\frac{\lambda \alpha_{n-k}}{\alpha_n}\right) ub^{n-k}$$

Summing for all the k's we have

$$\lambda ub^n \geq \lambda \sum_{1 \leq k \leq n: \alpha_{n-k} < 0} \left(-\frac{\lambda \alpha_{n-k}}{\alpha_n} \right) ub^{n-k}$$

or

$$ub^n \geq \sum_{1 \leq k \leq n: \alpha_{n-k} < 0} \left(-\frac{\alpha_{n-k}}{\alpha_n} \right) ub^{n-k}$$

i.e., dividing $p(x) = 0$ by α_n, making unitary the leading coefficient, and replacing x with ub, $x \leftarrow ub$, the first term, i.e. ub^n, would be *greater than*, or *equal to*, the sum of the absolute values of the terms with negative coefficient. Hence, for all $x > ub$, $p(x) > 0$. □

2.2 Lagrange's and MacLaurin's Method

Theorem 3. *Suppose α_{n-k}, $k \geq 1$, is the first of the negative coefficients of a polynomial $p(x)$, as in Eq. (1), then an upper bound on the values of the positive roots of $p(x)$ is given by*

$$ub = 1 + \sqrt[k]{\frac{B}{\alpha_n}},$$

where B is the largest absolute value of the negative coefficients of the polynomial $p(x)$.

Proof. Set $x > 1$. If in $p(x)$ each of the nonnegative coefficients $\alpha_{n-1}, \alpha_{n-2}, \ldots,$ α_{k-1} is replaced by zero, and each of the remaining coefficients $\alpha_k, \alpha_{k+1}, \ldots, \alpha_0$ is replaced by the negative number $-B$, we obtain

$$p(x) \geq \alpha_n x^n - B(x^{n-k} + x^{n-k-1} + \ldots + 1) = \alpha_n x^n - B\frac{x^{n-k+1} - 1}{x - 1}$$

Hence for $x > 1$ we have

$$\begin{aligned} p(x) > \alpha_n x^n - \frac{B}{x-1} x^{n-k+1} &= \frac{x^{n-k+1}}{x-1}(\alpha_n x^{k-1}(x-1) - B) \\ &> \frac{x^{n-k+1}}{x-1}(\alpha_n (x-1)^k - B) \end{aligned}$$

Consequently for

$$x \geq 1 + \sqrt[k]{\frac{B}{\alpha_n}} = ub$$

we have $p(x) > 0$ and all the positive roots x_+ of $p(x)$ satisfy the inequality $x_+ < ub$. □

2.3 A New Theorem

In 2005 Ştefănescu, [Ştefănescu 2005], proved the following theorem:

Theorem 4 Stefanescu, 2005. *Let $p(x) \in R[x]$ be such that the number of variations of signs of its coefficients is* ***even****. If*

$$p(x) = c_1 x^{d_1} - b_1 x^{m_1} + c_2 x^{d_2} - b_2 x^{m_2} + \ldots + c_k x^{d_k} - b_k x^{m_k} + g(x),$$

with $g(x) \in R_+[x]$,$c_i > 0, b_i > 0, d_i > m_i > d_{i+1}$ for all i, the number

$$B_3(p) = \max \left\{ \left(\frac{b_1}{c_1}\right)^{1/(d_1 - m_1)}, \ldots, \left(\frac{b_k}{c_k}\right)^{1/(d_k - m_k)} \right\}$$

is an upper bound for the positive roots of the polynomial p for any ***choice*** *of $c_1, \ldots, c_k$.*

We point out that Ştefănescu's theorem introduces the concept of *matching* or *pairing* a positive coefficient with the negative coefficient of a lower order term. However, as stated above, Theorem 4 works only for polynomials with an even number of sign variations.

In the sequel we present a generalization of Ştefănescu's theorem in the sense that Theorem 5 works for any number of sign variations provided inequality (2) holds.

Theorem 5. *Let $p(x)$ be a polynomial as in Eq. (1) and denote by $d(p)$ and $t(p)$ the degree of $p(x)$ and the number of its terms, respectively.*

Moreover, let

$$p(x) = q_1(x) - q_2(x) + q_3(x) - q_4(x) + \ldots + q_{2m-1}(x) - q_{2m}(x) + g(x),$$

where all the polynomials $q_i(x)$, $i = 1, 2, \ldots, 2m$ and $g(x)$ have only positive coefficients, and the exponent of each term in $q_i(x)$ is greater than the exponent of each term in $q_{i+1}(x)$, $i = 1, 2, \ldots, 2m - 1$. In addition, for $i = 1, 2, \ldots, m$, assume that

$$t(q_{2i-1}) \geq t(q_{2i}), \tag{2}$$

and that

$$q_{2i-1}(x) = c_{2i-1,1} x^{e_{2i-1,1}} + \ldots + c_{2i-1,t(q_{2i-1})} x^{e_{2i-1,t(q_{2i-1})}}$$

$$q_{2i}(x) = b_{2i,1} x^{e_{2i,1}} + \ldots + b_{2i,t(q_{2i})} x^{e_{2i,t(q_{2i})}}$$

where

$$e_{2i-1,1} = d(q_{2i-1}) \quad and \quad e_{2i,1} = d(q_{2i})$$

Then an upper bound of the values of the positive roots of $p(x)$ is given by

$$ub = \max_{\{i=1,2,\ldots,m\}} \left\{ \left(\frac{b_{2i,1}}{c_{2i-1,1}}\right)^{\frac{1}{e_{2i-1,1}-e_{2i,1}}}, \ldots, \left(\frac{b_{2i,t(q_{2i})}}{c_{2i-1,t(q_{2i})}}\right)^{\frac{1}{e_{2i-1,t(q_{2i})}-e_{2i,t(q_{2i})}}} \right\}$$

Proof. Suppose $x > 0$. We have

$$\begin{aligned}
|p(x)| &\geq c_{1,1}x^{e_{1,1}} + \ldots + c_{1,t(q_1)}x^{e_{1,t(q_1)}} - b_{2,1}x^{e_{2,1}} - \ldots - b_{2,t(q_2)}x^{e_{2,t(q_2)}} \\
&+ \\
&\vdots \\
&+ c_{2m-1,1}x^{e_{2m-1,1}} + \ldots + c_{2m-1,t(q_{2m-1})}x^{e_{2m-1,t(q_{2m-1})}} \\
&- b_{2m,1}x^{e_{2m,1}} - \ldots - b_{2m,t(q_{2m})}x^{e_{2m,t(q_{2m})}} + g(x) \\
&= x^{e_{2,1}}(c_{1,1}x^{e_{1,1}-e_{2,1}} - b_{2,1}) + \ldots \\
&+ x^{e_{2m,t(q_{2m})}}(c_{2m-1,t(q_{2m})}x^{e_{2m-1,t(q_{2m})}-e_{2m,t(q_{2m})}} - b_{2m,t(q_{2m})}) + g(x)
\end{aligned}$$

which is strictly positive for

$$x > \max_{\{i=1,2,\ldots,m\}} \left\{ \left(\frac{b_{2i,1}}{c_{2i-1,1}}\right)^{\frac{1}{e_{2i-1,1}-e_{2i,1}}}, \ldots, \left(\frac{b_{2i,t(q_{2i})}}{c_{2i-1,t(q_{2i})}}\right)^{\frac{1}{e_{2i-1,t(q_{2i})}-e_{2i,t(q_{2i})}}} \right\}$$

□

3 Empirical Results

We compare the performance of the three methods described above. As we mentioned earlier, the method obtained from Theorem 5 works only when inequality (2) holds. In that case, if λ is the number of negative coefficients of the polynomial under consideration, we match them with the first λ positive coefficients of higher order terms.

We followed the standard practice and used as benchmark the Laguerre[3], Chebyshev (first[4] and second[5] kind), Wilkinson[6] and Mignotte[7] polynomials, as well as several types of randomly generated polynomials of degrees $\{5, 15, 35, 45,$

[3] recursively defined as: $L_0(x) = 1$, $L_1(x) = 1 - x$, and $L_{n+1}(x) = \frac{1}{n+1}((2n + 1 - x)L_n(x) - nL_{n-1}(x))$

[4] recursively defined as: $T_0(x) = 1$, $T_1(x) = x$, and $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$

[5] recursively defined as: $U_0(x) = 1$, $U_1(x) = 2x$, and $U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$

[6] defined as: $W(x) = \prod_{i=1}^{n}(x - i)$

[7] defined as: $M_n(x) = x^n - 2(5x - 1)^2$

$55, 65, 100, 200\}$. For the random polynomials the size of the coefficients ranges from -2^{20} to 2^{20}. Along with the bounds we also compute numerically (using *Mathematica*'s function *NSolve*) the *Maximum Positive Root*, MPR, of each polynomial. $\nexists$ means that there are no positive roots for this polynomial and N/A that the method here is Non Applicable.

More precisely, N/A means that the polynomial has negative coefficients, which cannot be matched with positive coefficients of a higher order terms.

In Table 1 we used some polynomials presented in [Ştefănescu 2005]. As we see, the method of Theorem 5 gives slightly better results in most cases.

Polynomials	Cauchy	Lagrange	Theorem 5	MPR
Q_1	1.25992	2.00000	0.42857	0.42152
Q_2	2.02000	2.10000	1.04881	1.00347
Q_3	1.14870	2.00000	0.75395	0.72543
Q_4	1.35791	2.14186	1.14186	1.12041
P_1	7.81025	8.81025	7.81025	4.27293
P_2	1.31607	2.07722	1.58740	1.16541
P_3	2.08008	2.58740	1.44225	1.12612
P_4	1.64375	2.31951	1.31951	1.06815

Table 1: Bounds for positive roots of the polynomials used in [Ştefănescu 2005].

In Table 2 we used the Laguerre polynomials. For these polynomials the method of Theorem 5 is by far better than the others two.

Degree	Cauchy	Lagrange	Theorem 5	MPR
5	75	601	25	12.6408
15	1800	7.44×10^{13}	225	48.0261
35	22050	2.33×10^{43}	1225	123.173
45	46575	1.35×10^{60}	2025	161.459
55	84700	5.11×10^{77}	3025	199.987
65	139425	1.03×10^{96}	4225	238.686
100	500000	4.89×10^{164}	10000	374.984
200	4.00×10^{6}	1.19×10^{385}	40000	767.815

Table 2: Bounds for positive roots of Laguerre polynomials.

The bounds for Chebyshev polynomials of the first kind are given in Table 3 whereas the bounds for Chebyshev polynomials of the second kind are given in Table 4. In both cases the method of Theorem 5 is much better than Lagrange's method and slightly better than Cauchy's.

Degree	Cauchy	Lagrange	Theorem 5	MPR
5	1.11803	2.11803	1.11803	0.95106
15	3.87298	3.07289	1.93649	0.99452
35	8.87412	13.7421	2.95804	0.99899
45	11.1243	31.2022	3.35410	1.00064
55	13.8744	72.7588	3.70810	1.41983
65	16.1245	184.152	4.03113	1.27649
100	25.0000	4440.05	5.00000	1.67201
200	50.0000	4.60×10^7	7.07107	1.68006

Table 3: Bounds for positive roots of Chebyshev polynomials of the first kind.

Degree	Cauchy	Lagrange	Theorem 5	MPR
5	1.00000	2.00000	1.00000	0.866025
15	3.74166	2.87083	1.87083	0.980785
35	8.74643	12.7969	2.91548	0.996195
45	11.0000	28.7539	3.31662	0.997669
55	13.7477	68.0370	3.67423	0.998427
65	16.0000	171.000	4.00000	0.998867
100	24.8747	4093.60	4.97494	0.999516
200	49.8748	4.25×10^7	7.05337	0.999878

Table 4: Bounds for positive roots of Chebyshev polynomials of the second kind

The bounds for the Wilkinson polynomials are given in Table 5. Here the superiority of the method of Theorem 5 over the other two is remarkable.

The bounds for the Mignotte polynomials are given in Table 6. Here all three methods are about the same.

The bounds for random polynomials with unitary leading coefficient are given in Table 7, whereas the bounds for random polynomials with randomly generated leading coefficient are given in Table 8. Here, when applicable, the method of

Degree	Cauchy	Lagrange	Theorem 5	MPR
5	45	226	15	5
15	960	6.17×10^{12}	120	15
35	11340	8.05×10^{40}	630	35
45	23805	1.14×10^{57}	1035	45
55	43120	1.51×10^{74}	1540	55
65	70785	1.16×10^{92}	2145	65
100	252500	1.81×10^{159}	5050	100
200	2.01×10^{6}	2.99×10^{376}	20100	200

Table 5: Bounds for positive roots of Wilkinson polynomials

Degree	Cauchy	Lagrange	Theorem 5	MPR
5	4.64159	4.68403	3.68403	3.54410
15	1.42510	2.35111	1.35111	1.31731
35	1.14976	2.12586	1.12586	1.11242
45	1.11304	2.09524	1.09524	1.08491
55	1.09078	2.07660	1.07660	1.06821
65	1.07584	2.06406	1.06406	1.05700
100	1.04811	2.04073	1.04073	1.03618
200	1.02353	2.01995	1.01995	1.01770

Table 6: Bounds for positive roots of Mignotte polynomials

Degree	Cauchy	Lagrange	Theorem 5	MPR
5	17.4356	32.6386	N/A	2.1557
15	47.0319	27.0384	17.7764	$\nexists$
35	9240.00	1007.00	N/A	616.94
45	13920.0	1018.00	696.000	695.16
55	20332.0	958.000	N/A	883.39
65	1.12×10^{7}	1.01×10^{6}	N/A	339158
100	83545.0	32589.6	N/A	1.6884
200	4.52×10^{6}	1.05×10^{6}	N/A	1.31275

Table 7: Bounds for positive roots of random polynomials with unitary leading coefficient

Theorem 5 is slightly or much better than the other two.

Degree	Cauchy	Lagrange	Theorem 5	MPR
5	2.78167	3.36560	2.20782	0.45627
15	7.41029	3.16699	N/A	1.32124
35	7.00776	2.07539	N/A	1.19835
45	31.5639	3.07302	1.66126	0.99823
55	2.39960	2.26034	1.28591	0.93821
65	1.49472	2.01141	1.39012	$\nexists$
100	12.5311	1.97929	N/A	1.28714
200	338.386	4.72440	N/A	2.65901

Table 8: Bounds for positive roots of random polynomials

Degree	Cauchy	Lagrange	Theorem 5	MPR
5	1714	858	857	856.342
15	2616	908	N/A	327.983
35	8853	908	681	679.695
45	10.0199	6.65547	N/A	$\nexists$
55	23.2463	11.0761	8.29619	$\nexists$
65	164.302	102.105	52.3026	$\nexists$
100	2217.16	1007.53	N/A	1.12060
200	4.69×10^{19}	1.11×10^{18}	N/A	4.99×10^{17}

Table 9: Bounds for positive roots of random polynomials with seed "1001" and unitary leading coefficient

Degree	Cauchy	Lagrange	Theorem 5	MPR
5	1.72435	1.86217	1.09090	$\nexists$
15	3.05607	2.05958	N/A	$\nexists$
35	33.0336	4.38433	2.54104	$\nexists$
45	21.6805	2.68709	N/A	1.14158
55	54.1144	3.15890	N/A	1.68826
65	6.02084	2.45091	N/A	1.24624
100	6.11942	3.77805	N/A	1.11428
200	136.564	4.22554	N/A	0.48657

Table 10: Bounds for positive roots of random polynomials with randomly generated leading coefficient and seed "1001"

The bounds for random polynomials with seed "1001" and unitary leading coefficient are given in Table 9, whereas the bounds for random polynomials with seed "1001" and randomly generated leading coefficient are given in Table 10. Same conclusions here as in Tables 7 and 8.

4 Conclusion

From the tables presented here we conclude that the method derived from Theorem 5, when applicable, gives in most cases a better, or much better, upper bound on the values of the positive roots of polynomials than the well known and widely used methods by Cauchy and Lagrange.

So in order to compute sharper upper bounds on the positive roots of polynomials we are tempted to pursue the matter further and extend Theorem 5 so that it works for the cases when inequality (2) fails; if we succeed, then there would be no reason at all to continue using the "classical" methods in our real root isolation methods.

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