A new Look at one of the Bisection Methods Derived from Vincent's Theorem or There is no Descartes' Method

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Dedicated to the memory of James Victor Uspensky (1883-1947)[1]

Abstract In 1976, G.E. Collins and A.G. Akritas developed a bisection method for the isolation of the real roots of polynomials. Even though this method is based on Vincent's theorem of 1836, credit to Vincent has been denied; to wit, from 1976 up until 1986 it was called "modified Uspensky's method", whereas from 1986 to the present day it is being called either "Collins-Akritas method" or "Descartes' method". In this paper we track the development of this bisection method, show its relation to Vincent's theorem and justify the name "Vincent-Collins-Akritas" given to it recently in France.

Key Words: Vincent's theorem, isolation of the real roots, real root isolation methods, bisection method, continued fractions method, Descartes' method/solver, modified Uspensky's method.

1 Introduction

As one of the two authors who developed the bisection method under discussion [19], Akritas has followed closely, and with great interest, the "maturation process" of his spiritual "child". So far he has made only one corrective intervention, back in 1986 [5], when the name of the method was wrongly attributed to Uspensky. The closing sentence in that article was: "*It is our hope that scientists will give Vincent the credit he so justly deserves.*"

Thirty years later Vincent still does not get the proper credit. Due to various misunderstandings, the bisection method derived from his theorem is referred to either as the "Collins-Akritas method" [28], which is almost right, or as the "Descartes' method" ([21], [22], [24], [25], [31]), which is totally misleading; moreover, Vincent's paper [36] is not cited in certain articles ([21], [22],

[1] For his book *Theory of Equations*, which kept Vincent's theorem "alive". See http://www.apmath.spbu.ru/ru/misc/uspenskii.html for an interesting biography of Uspensky (in Russian) [35].

[23]). Therefore, in an effort to dispel the existing misconceptions we attempt yet another corrective intervention.

Let us begin with a review of Descartes' rule of signs [16].

Consider the polynomial $p(x) \in \mathbb{R}[x]$, $p(x) = a_n x^n + \cdots + a_1 x + a_0$ and let $var(p)$ represent the number of sign *variations* or *changes* (positive to negative and vice-versa) in the sequence of coefficients $a_n, a_{n-1}, \ldots, a_0$.

Descartes' rule of signs: The number $\varrho_+(p)$ of real roots — multiplicities counted — of the polynomial $p(x) \in \mathbb{R}[x]$ in the open interval $]0, +\infty[$ is bounded above by $var(p)$; that is, we have $var(p) \geq \varrho_+(p)$.

According to Descartes' rule of signs if $var(p) = 0$ it follows that $\varrho_+(p) = 0$.

Additionally, according to Descartes' rule of signs, the mean value theorem and the fact that the polynomial functions are continuous, if $var(p) = 1$ it follows that $\varrho_+(p) = 1$ [16].

Therefore, Descartes' rule of signs yields the *exact* number of positive roots *only* in the two special cases mentioned above[2].

These two special cases of Descartes' rule are used in Vincent's theorem of 1836, [36], which states:

Theorem 1. *(Vincent's original theorem — "continued fractions" version) If in a polynomial, $p(x)$, of degree n, with rational coefficients and without multiple roots we perform sequentially replacements of the form*

$$x \leftarrow \alpha_1 + \frac{1}{x}, x \leftarrow \alpha_2 + \frac{1}{x}, x \leftarrow \alpha_3 + \frac{1}{x}, \ldots$$

where $\alpha_1 \geq 0$ is an arbitrary non negative integer and $\alpha_2, \alpha_3, \ldots$ are arbitrary positive integers, $\alpha_i > 0$, $i > 1$, then the resulting polynomial either has no sign variations or it has one sign variation. In the last case the equation has exactly one positive root, which is represented by the continued fraction

$$\alpha_1 + \frac{1}{\alpha_2 + \frac{1}{\alpha_3 + \frac{1}{\ddots}}}$$

whereas in the first case there are no positive roots.

For a detailed discussion of this theorem, its extension, the geometrical interpretation of the transformations involved and three different proofs see [12], [13] and [14]; a fourth proof is presented by Ostrowski [27], who rediscovered a special case of a previously stated theorem by Obreschkoff, ([26], p. 81).

[2] These two special cases were known to Cardano; in other words, what Descartes did was to generalize "Cardano's *special* rule of signs". This detail is mentioned in [6].

The negative roots are treated in the same way — as suggested by Sturm — after we transform them to positive with the replacement $x \leftarrow -x$ performed on $p(x)$. The requirement that $p(x)$ have no multiple roots does not restrict the generality of the theorem because in the opposite case we first apply square-free factorization and then isolate the roots of each one of the square-free factors.

1.1 Isolating the Real Roots of a Polynomial with Vincent's Theorem

By cleverly utilizing the two special cases of Descartes' rule — the case of 0 or 1 sign variation — Vincent's theorem can be used to isolate the positive roots of a given polynomial $p(x)$. To see this, note that if we represent by the Möbius transformation $M(x) = \frac{ax+b}{cx+d}$ the continued fraction that leads to a transformed polynomial

$$f(x) = (cx+d)^n p(\frac{ax+b}{cx+d}), \tag{1}$$

with one sign variation, then the single positive root of $f(x)$ — in the interval $]0, +\infty[$ — corresponds to *that* positive root of $p(x)$ which is located in the open interval with endpoints $\frac{b}{d}$ and $\frac{a}{c}$. These endpoints are *not* ordered and correspond to $M(0)$ and $M(\infty)$, respectively[3].

Therefore, to isolate the positive roots of a polynomial, all we have to do is compute — for *each* root — the variables a, b, c, d of the corresponding Möbius transformation $M(x) = \frac{ax+b}{cx+d}$ that leads to a transformed polynomial $f(x) = (cx+d)^n p(\frac{ax+b}{cx+d})$, with one sign variation.

Crucial observation: The variables a, b, c, d of a Möbius transformation $M(x) = \frac{ax+b}{cx+d}$ (in Vincent's theorem) leading to a transformed polynomial with one sign variation can be computed:

- *either* by *continued fractions*, leading to the continued fractions method developed by Akritas and Strzeboński — which in the sequel will be called the `VAS` *continued fractions* method[4],
- *or*, as we will see in the sequel, by *bisection*, leading to (among others) the *bisection* method developed by Collins and Akritas — which in the sequel will be called the `VCA` *bisection* method[5].

[3] As we will see in the sequel, the endpoints may also be computed from $M(0)$ and $M(1)$, if we work in the interval $]0, 1[$; in that case, Descartes' rule of signs does not apply and we use Uspensky's test instead.

[4] To distinguish it from other continued fraction methods such as [7], [18], [32] cited in ([37], pp. 470–478). In [31] `VAS` is referred to as the "Akritas' continued fractions method". See also [12] and [15].

[5] To distinguish it from Sturm's bisection method [37].

It is *not* an accident that Vincent's theorem is exactly what is needed to prove termination of both real root isolation methods mentioned above; and that proof of termination was masterly presented by Alesina and Galuzzi [12], [14].

The "bisection part" of this all important observation is missing from major works such as ([37], pp. 470–478) and almost every paper on the subject; to our knowledge, it appears *only* in the papers by Alesina and Galuzzi [12], [14].

More explicitly, whereas the association of Möbius transformations of the form $M(x) = \frac{ax+b}{cx+d}$ — or, equivalently, of Vincent's theorem — with the `VAS` continued fractions method is generally acknowledged, there is a lingering perception that there is no association between the `VCA` bisection method and Vincent's theorem. In this paper we debunk this perception, eliminating thus a source of great misunderstandings and needless duplication of efforts[6].

From the crucial observation it becomes obvious that Vincent's theorem of 1836 is the origin of both the `VAS` *continued fractions* method and the `VCA` *bisection* method mentioned above and, consequently, correctly is Vincent's name included in both of them.

Please note that the title of the paper by Collins and Akritas [19] was "Polynomial real root isolation (*indirectly*) *using* Descartes' rule of signs", where the emphasis on "using" as well as the word *indirectly* have been added for clarity.

The fact that Descartes' rule of signs is used in Vincent's theorem and, hence, directly or indirectly in both methods `VAS` and `VCA`, should not, under any circumstances, be used as an excuse to call any of these two methods after Descartes; to do otherwise — as in the case of the `VCA` method— creates confusion and is the source of great misunderstandings. In other words, *there is no "Descartes' method"* and this fact is reflected in [28], and [16].

In Section 2 we deal mainly with the `VCA` bisection method and show its relation to Vincent's theorem. Since the relation between Vincent's theorem and the `VAS` continued fractions method is universally acknowledged [2], [12], we use `VAS` as a point of reference and we briefly present it below; an excellent survey of the subject can be found in the papers by Alesina and Galuzzi [12], [13] and [14].

1.2 The Continued Fractions Method Derived from Vincent's Theorem

The `VAS` continued fractions method is a *direct* implementation of Vincent's theorem. It was originally presented by Vincent in 1836 [36] in an "exponential" form; namely, Vincent computed each partial quotient a_i by a series of *unit*

[6] One cannot help but wonder why neither the work by Alesina and Galuzzi [14] nor Obreschkoff's book [26] are cited in papers that deal with "Descartes' method".

increments $a_i \leftarrow a_i + 1$, which are equivalent to substitutions of the form $x \leftarrow x + 1$.

In 1978 the method was converted into its "polynomial" form by Akritas, who in his Ph.D Thesis [1] computed each partial quotient a_i as the lower bound, ℓb, on the values of the positive roots of a polynomial — the so called "*ideal*" positive lower root bound, which computes the integer part of the smallest positive root [9]; that is, we now set $a_i \leftarrow \ell b$ or, equivalently, we perform the substitution $x \leftarrow x + \ell b$, which takes about the same time as the substitution $x \leftarrow x + 1$. For details see also [2], [3], [4] and Chapter 7 in [6].

Finally, since the ideal positive lower root bound does not exist, Strzeboński [8] introduced the substitution $x \leftarrow \ell b_{computed} \cdot x$, whenever $\ell b_{computed} > 16$ — where in general $\ell b > \ell b_{computed}$ and the value 16 was determined experimentally.

In [8] it was also shown that the `VAS` continued fractions method is faster than the fastest implementation of the `VCA` bisection method [28], a result which was independently confirmed by Tsigaridas and Emiris [33]; see also [10]. In 2007 Sharma removed the hypothesis of the ideal positive lower bound and proved that `VAS` is still polynomial in time [30], [31]!

In Algorithm 1 below we present a recursive description of the `VAS` continued fractions method. We follow [16], which pedagogically seems to be the most appropriate style of presentation:

The `VAS` continued fractions method

Input: A univariate, square-free polynomial $p(x) \in \mathbb{Z}[x], p(0) \neq 0$, and the Möbius transformation $M(x) = \frac{ax+b}{cx+d} = x,\ a, b, c, d \in \mathbb{Z}$

Output: A list of isolating intervals of the *positive* roots of $p(x)$

1 $var \longleftarrow$ the number of sign changes of $p(x)$;
2 **if** $var = 0$ **then RETURN** $\emptyset$;
3 **if** $var = 1$ **then RETURN** $\{]a, b[\}$ `// a = min(M(0),M(∞)), b = max(M(0),M(∞))`;
4 $\ell b \longleftarrow$ a lower bound on the positive roots of $p(x)$;
5 **if** $\ell b > 1$ **then** $\{p \longleftarrow p(x + \ell b), M \longleftarrow M(x + \ell b)\}$;
6 $p_{01} \longleftarrow (x+1)^{deg(p)} p(\frac{1}{x+1}), M_{01} \longleftarrow M(\frac{1}{x+1})$ `// Look for real roots in` $]0, 1[$;
7 $m \longleftarrow M(1)$ `// Is 1 a root?` ;
8 $p_{1\infty} \longleftarrow p(x+1), M_{1\infty} \longleftarrow M(x+1)$ `// Look for real roots in` $]1, +\infty[$;
9 **if** $p(1) \neq 0$ **then**
10 | **RETURN** $\texttt{VAS}(p_{01}, M_{01}) \bigcup \texttt{VAS}(p_{1\infty}, M_{1\infty})$
11 **else**
12 | **RETURN** $\texttt{VAS}(p_{01}, M_{01}) \bigcup \{[m, m]\} \bigcup \texttt{VAS}(p_{1\infty}, M_{1\infty})$
13 **end**

Algorithm 1: The $\texttt{VAS}(p, M)$ "*continued fractions*" algorithm, where the second argument is the Möbius transformation $M(x)$ associated with $p(x)$. For simplicity, Strzeboński's contribution is not included.

Please note the following:

VAS 1: Descartes' rule of signs is a *crucial component* of the $\texttt{VAS}(p, M)$ continued fractions algorithm — lines 1-3. Despite this fact, no one has ever called the $\texttt{VAS}(p, M)$ method after Descartes — and rightly so, since it is derived from Vincent's theorem.

VAS 2: If we remove lines 4 and 5 from $\texttt{VAS}(p, M)$ we are left with an exponential algorithm.

VAS 3: Any substitution performed on the polynomial $p(x)$ is also performed on its associated Möbius transformation $M(x)$ — lines 5, 6 and 8.

VAS 4: To isolate the real roots of $p(x)$ in the open interval $]0, +\infty[$ we proceed as follows:

- we first isolate the real roots in the interval $]0, 1[$ — lines 6 and 10 (or 12),
- we then deal with the case where 1 is a root of $p(x)$ — lines 7, 9 and 12,
- and, finally, we isolate the real roots in the interval $]1, +\infty[$ — lines 8 and 10 (or 12).

VAS 5: The isolating intervals are computed from the Möbius transformations in line 3 — except for the integer roots which are computed in lines 7 and 12.

2 One of the Bisection Methods Derived from Vincent's Theorem

In an attempt to improve the exponential behavior of Vincent's algorithm — the only one existing at that time — Collins and Akritas [19] developed in 1976 the `VCA` method, the *first* bisection method derived from Vincent's theorem; unfortunately, at that time neither of them realized the dependency of their method on Vincent's theorem.

Let $p(x)$ be the polynomial whose roots we want to isolate and let ub be an upper bound on the values of its positive roots. Then all the positive roots of $p(ub \cdot x)$ lie in the interval $]0, 1[$ and the `VCA` method isolates them by repeatedly bisecting the interval $]0, 1[$, while using in the process an appropriate "*criterion*" to make inferences about the number of positive roots certain transformed polynomials have in the interval $]0, 1[$. Finally, the isolating intervals of the roots of $p(x)$ are easily computed from the bijection:

$$\alpha_{]0,ub[} = a + \alpha_{]0,1[}(b - a), \tag{2}$$

that exists between the roots $\alpha_{]0,1[} \in]0, 1[$ of the *transformed* polynomial $p(ub \cdot x)$ and the roots $\alpha_{]0,ub[} \in]a, b[=]0, ub[$ of the *original* polynomial $p(x)$.

The appropriate criterion mentioned above is a "*test*" that determines an *upper bound* on the number of positive roots in the interval $]0, 1[$.

Please observe that Descartes' rule of signs *cannot* be used in the interval $]0, 1[$, as it applies *only* to positive roots in the interval $]0, +\infty[$. Therefore, we have to resort to a different "*rule*" if we want to avoid reinventing Sturm's method for isolating the real roots; recall that Sturm's theorem gives us the *exact* number of positive roots in any interval $]a, b[$, [12].

Here is the test for determining an *upper bound* on the number of positive roots in the interval $]0, 1[$; as explained below, we name it after Uspensky, who was the *first* to use it.

Uspensky's test: The number $\varrho_{01}(p)$ of real roots in the open interval $]0, 1[$ — multiplicities counted — of the polynomial $p(x) \in \mathbb{R}[x]$ is bounded above by $var_{01}(p)$, where

$$var_{01}(p) = var((x+1)^{deg(p)} p(\frac{1}{x+1})), \tag{3}$$

and we have $var_{01}(p) \geq \varrho_{01}(p)$[7].

As in the case of Descartes' rule of signs if $var_{01}(p) = 0$ it follows that $\varrho_{01}(p) = 0$ and if $var_{01}(p) = 1$ it follows that $\varrho_{01}(p) = 1$.

Therefore, Uspensky's test yields the *exact* number of positive roots *only* in the two special cases mentioned above; to wit, whenever $var_{01}(p) = 0$ or $var_{01}(p) = 1$.

Please note in equation (3) that, *after* the substitution $x \longleftarrow \frac{1}{x+1}$, the positive roots of $p(x)$ that were in the interval $]0, 1[$ are now in $]0, +\infty[$, in which case Descartes' rule of signs *can* be applied.

Uspensky's test is associated with Budan's theorem [4] according to which for a given polynomial $p(x) \in \mathbb{Z}[x]$ the following two special cases hold:

- if $var(p(x)) = var(p(x+1))$, then we can conclude that there are no positive real roots of $p(x)$ in the interval $]0, 1[$,

and

- if $var(p(x)) - var(p(x+1)) = 1$, then we can conclude that there is one positive real root of $p(x)$ in the interval $]0, 1[$.

Vincent was fully aware of Budan's theorem and, consequently, the substitution $x \longleftarrow \frac{1}{x+1}$ is *never* used as a test in the `VAS` method — line 6 of Algorithm 1; it is performed *only* whenever $var(p(x)) - var(p(x+1)) \geq 2$, in which case the existence of positive roots in $]0, 1[$ *has* to be investigated.

That, however, was not the case with Uspensky. Whenever he encountered $var(p(x)) = var(p(x+1))$ — not being aware of Budan's theorem — he could

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[7] Uspensky's test is a special instance of the powerful "Vincent's test", which is based on Theorem 2 below, applies to any interval $]a, b[$ and states that: If $a \geq 0$ and $b > a$ then the number $\varrho_{ab}(p)$ of real roots in the open interval $]a, b[$ — multiplicities counted — of the polynomial $p(x) \in \mathbb{R}[x]$ is bounded above by $var_{ab}(p)$, where $var_{ab}(p) = var((1+x)^{deg(p)} p(\frac{a+bx}{1+x}))$, and we have $var_{ab}(p) = var_{ba}(p) \geq \varrho_{ab}(p)$. For applications see [11].

not conclude that there are no positive roots of $p(x)$ in the interval $]0,1[$; he would reach that conclusion *only* if $var_{01}(p) = 0$[8].

In other words, Uspensky ([34], p. 128) was the *first* to use $var_{01}(p) = 0$ *exclusively* as a test, in order to verify that there are no positive roots in the interval $]0,1[$; hence, naming the test after him seems to be very appropriate. That test was used in his unsuccessful attempt to develop a new procedure for the isolation of the real roots of polynomials [5], [12][9].

Collins and Akritas [19] used Uspensky's test in the `VCA` method — and that was the main reason they originally (and misleadingly [5]) called it "modified Uspensky's method". However, as we will show in the sequel, the `VCA` method is derived from Vincent's theorem, which we present in yet another way, due to Alessina and Galuzzi [14].

Theorem 2. *(Vincent's theorem — "bisection" version) Let $f(z)$, be a real polynomial of degree n, which has only simple roots. It is possible to determine a positive quantity δ so that for every pair of positive real numbers a, b with $|b-a| < \delta$, every transformed polynomial of the form*

$$\phi(z) = (1+z)^n f(\frac{a+bz}{1+z})$$

has exactly 0 or 1 variations. The second case is possible if and only if $f(z)$ has a simple root within $]a, b[$.

We call this the *bisection* version of Vincent's theorem, since, besides the `VCA` bisection method, there are several other bisection methods derived from it, which are studied in detail elsewhere [11].

Below is a recursive description of the `VCA` bisection method:

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[8] On the other hand, Uspensky used correctly and to his advantage the other special case, $var_{01}(p) = 1$, as well as the case $var_{01}(p) \geq 2$.

[9] According to Professor Alexei Uteshev [35], of St. Petersburg's State University, the reason for Uspensky's unsuccessful attempt was the fact that he *never* saw Vincent's actual paper of 1836 — where Budan's theorem is stated right at the beginning. Instead, Uspensky relied on the Russian translation of J.-A. Serret's *Cours d'Algèbre Supérieur* [29]. Indeed, in Section 167, p.315 of Серре И.А. Курс высшей алгебры. М.- СПб., Вольф, Б.г., 573 с. we read:

"В одном из мемуаров, составляющих часть первого тома Journal de Mathématiques pures et appliques, Венсен изложил прекрасное свойство непрерывных дробей и вывел из него для вычисления вещественных корней уравнения способ, вытекающий одновременно и из способа Ньютона, и из способа Лагранжа..."

Please note that Serret presents *Fourier*'s theorem under the name "Budan".

The VCA bisection method — original version

Input: A univariate, square-free polynomial $p(ub \cdot x) \in \mathbb{Z}[x], p(0) \neq 0$, and the open interval $]a, b[=]0, ub[$, where ub is an upper bound on the values of the positive roots of $p(x)$. (The positive roots of $p(ub \cdot x)$ are all in the open interval $]0, 1[$.)
Output: A list of isolating intervals of the *positive* roots of $p(x)$

1 $var \longleftarrow$ the number of sign changes of $(x+1)^{deg(p)} p(\frac{1}{x+1})$;
2 **if** $var = 0$ **then RETURN** $\emptyset$;
3 **if** $var = 1$ **then RETURN** $\{]a, b[\}$;
4 $p_{0\frac{1}{2}} \longleftarrow 2^{deg(p)} p(\frac{x}{2})$ `// Look for real roots in` $]0, \frac{1}{2}[$;
5 $m \longleftarrow \frac{a+b}{2}$ `// Is` $\frac{1}{2}$ `a root?` ;
6 $p_{\frac{1}{2}1} \longleftarrow 2^{deg(p)} p(\frac{x+1}{2})$ `// Look for real roots in` $]\frac{1}{2}, 1[$;
7 **if** $p(\frac{1}{2}) \neq 0$ **then**
8   **RETURN** $\texttt{VCA}(p_{0\frac{1}{2}},]a, m[) \bigcup \texttt{VCA}(p_{\frac{1}{2}1},]m, b[)$
9 **else**
10   **RETURN** $\texttt{VCA}(p_{0\frac{1}{2}},]a, m[) \bigcup \{[m, m]\} \bigcup \texttt{VCA}(p_{\frac{1}{2}1},]m, b[)$
11 **end**

Algorithm 2: The *original* version of the $\texttt{VCA}(p,]a, b[)$ "*bisection*" algorithm, where the second argument is the open interval $]a, b[$ associated with $p(x)$. The isolating intervals of the roots of $p(x)$ are computed directly, without using bijection (2).

To obtain the isolating intervals of the positive roots of $p(x)$ we could have also used the interval $]a, b[=]0, 1[$ along with bijection (2). An excellent discussion of this algorithm can be found in [16]. Please note the following:

VCA 1: Uspensky's test is a *crucial component* of the $\texttt{VCA}(p,]a, b[)$ bisection algorithm — lines 1-3. In other words, Descartes' rule of signs is used *only* indirectly, and, hence, calling this method after Descartes is totally misleading; besides, see [5] and also remark **VAS 1** following Algorithm 1.

VCA 2: The substitutions in lines 4 and 6 are performed only on the polynomial $p(x)$, whereas at the same time — in line 5 — the interval $]a, b[$ is divided into two equal parts $]a, m[$ and $]m, b[$, to be used in line 8 (or 10).

VCA 3: To isolate the real roots of $p(x)$ in the open interval $]0, 1[$ we proceed as follows:

- we first isolate the real roots in the interval $]0, \frac{1}{2}[$ — lines 4 and 8 (or 10),
- we then deal with the case where $\frac{1}{2}$ is a root of $p(x)$ — lines 5, 7 and 10,
- and, finally, we isolate the real roots in the interval $]\frac{1}{2}, 1[$ — lines 6 and 8 (or 10).

VCA 4: The isolating intervals are directly obtained from line 3 — except for those roots that happen to coincide with the midpoint of an interval that gets bisected, in which case they are computed in lines 5 and 10.

To show that the VCA bisection method is derived from Vincent's theorem we have to find a Möbius transformation $M(x) = \frac{ax+b}{cx+d}$ that leads to a transformed polynomial as in (1) — the characteristic property of Vincent's theorem.

But alas, there is no Möbius transformation to be found in Algorithm 2. The substitutions in lines 4 and 6, respectively, $x \longleftarrow \frac{x}{2}$ and $x \longleftarrow \frac{x+1}{2}$ seem to be purely a byproduct of the bisection of the interval $]0,1[$ and unrelated to Vincent's theorem. But this is not the case!

Comparing Algorithms 1 and 2, we see that there exists a striking similarity in their structure, with the following differences:

- **d1 :** in Algorithm 1 there is a Möbius transformation $M(x) = \frac{ax+b}{cx+d}$ associated with each polynomial, whereas in Algorithm 2 there is an open interval $]a,b[$ associated with each polynomial,
- **d2 :** in line 1, Algorithm 1 uses Descartes' rule of signs, whereas Algorithm 2 uses Uspensky's test,
- **d3 :** Algorithm 1 works with the interval $]0,\infty[$, whereas Algorithm 2 works with the interval $]0,1[$.

Of the three it is only **(d1)** — the fact that in Algorithm 2 there is an open interval $]a,b[$ associated with each polynomial, — that obscures the relation of the `VCA` bisection method with Vincent's theorem. The other two differences are simply procedural ones, due to the bijection (2) that exists between the roots.

In hindsight, the choice made by Collins and Akritas to associate with each polynomial an interval $]a,b[$ has turned out to be both:

- a *boon*, because the isolating intervals are computed immediately, and because this approach eventually led to the fastest implementation of the `VCA` bisection method — developed by Rouillier and Zimmermann [28],

and

- a *bane*, because it has obscured the relation of the `VCA` bisection method with Vincent's theorem, resulting in the method being called initially after Uspensky and presently after Descartes.

  Moreover, due to that obfuscation the non-appropriate disks $D_1 = \{z \in \mathbb{C} : |z| < 1\}$ and $D_2 = \{z \in \mathbb{C} : |z-1| < 1\}$ were used in a two-circles theorem [20]; see also ([25], p. 9). On the contrary, a clear understanding of the relation between the `VCA` method and Vincent's theorem leads to the determination of the appropriate disks — see ([26], p. 87), [12] and [14].

The relation of the `VCA` bisection method with Vincent's theorem is revealed if we replace the intervals $]a,b[$ by the Möbius transformations $M(x) = \frac{ax+b}{cx+d}$

leading to transformed polynomials as in (1). This is done in the Algorithm 3 below.

The `VCA` bisection method — second version

Input: A univariate, square-free polynomial $p(ub \cdot x) \in \mathbb{Z}[x], p(0) \neq 0$, and the Möbius transformation $M(x) = \frac{ax+b}{cx+d} = ub \cdot x, \ a, b, c, d \in \mathbb{Z}$, where ub is an upper bound on the values of the positive roots of $p(x)$. (The positive roots of $p(ub \cdot x)$ are all in the open interval $]0, 1[$.)

Output: A list of isolating intervals of the *positive* roots of $p(x)$

1 $var \longleftarrow$ the number of sign changes of $(x+1)^{deg(p)} p(\frac{1}{x+1})$;
2 **if** $var = 0$ **then RETURN** $\emptyset$;
3 **if** $var = 1$ **then RETURN** $\{]a, b[\}$ `// a = min(M(0),M(1)), b = max(M(0),M(1));`
4 $p_{0\frac{1}{2}} \longleftarrow 2^{deg(p)} p(\frac{x}{2})$, $M_{0\frac{1}{2}} \longleftarrow M(\frac{x}{2})$`// Look for real roots in` $]0, \frac{1}{2}[$;
5 $m \longleftarrow \frac{M(0)+M(1)}{2}$ `// Is` $\frac{1}{2}$ `a root?` ;
6 $p_{\frac{1}{2}1} \longleftarrow 2^{deg(p)} p(\frac{x+1}{2})$, $M_{\frac{1}{2}1} \longleftarrow M(\frac{x+1}{2})$`// Look for real roots in` $]\frac{1}{2}, 1[$;
7 **if** $p(\frac{1}{2}) \neq 0$ **then**
8 | **RETURN** `VCA`$(p_{0\frac{1}{2}}, M_{0\frac{1}{2}}) \bigcup$ `VCA`$(p_{\frac{1}{2}1}, M_{\frac{1}{2}1})$
9 **else**
10 | **RETURN** `VCA`$(p_{0\frac{1}{2}}, M_{0\frac{1}{2}}) \bigcup \{[m, m]\} \bigcup$ `VCA`$(p_{\frac{1}{2}1}, M_{\frac{1}{2}1})$
11 **end**

Algorithm 3: A second version of the "*bisection*" algorithm, `VCA`(p, M), where the second argument is the Möbius transformation $M(x)$ associated with $p(x)$. The relation to Vincent's theorem is now obvious.

To obtain the isolating intervals of the positive roots of $p(x)$ we could have also used the Möbius transformation $M(x) = x$ along with bijection (2).

Observe in line 3 of Algorithm 3 that, since we work with the interval $]0, 1[$, we now use $M(0)$ and $M(1)$ to compute the endpoints of the isolating intervals of the roots — as opposed to $M(0)$ and $M(\infty)$ used in Algorithm 1, where we work with the interval $]0, +\infty[$.

Moreover, we see in lines 4 and 6 of Algorithm 3 that any substitution performed on the polynomial $p(x)$ is also performed on its associated Möbius transformation $M(x)$ — just as in Algorithm 1.

In other words we now have the missing link, to wit Möbius transformations $M(x) = \frac{ax+b}{cx+d}$ leading to transformed polynomials as in (1) with one sign variation — the characteristic property of Vincent's theorem. Therefore, Algorithm 3 — and, hence, Algorithm 2 — is derived from Vincent's theorem.

3 Conclusion

From the above discussion we see that naming the `VCA` bisection method initially after Uspensky and presently after Descartes obscures the important relation that exists between this method and Vincent's theorem and creates great misunderstandings.

As we have seen, this obfuscation has been responsible for

– a two-circles theorem with non-appropriate disks that appeared in 1989 [20] — whereas the appropriate disks for the case of Vincent's theorem had been used by Obreschkoff back in 1952 ([26], p. 87) in his two-circles theorems[10],

and

– needless duplication of efforts, whereby published results by Alesina and Galuzzi [12], [14] — obtained for Vincent's theorem — are being rehashed for the *seemingly* "unrelated" case of "Descartes method" and presented as independent research.

Therefore, the name "Vincent-Collins-Akritas" given to the `VCA` bisection method in France [16] is the correct one; it makes existing relations transparent, does justice to Vincent and complements the name "Collins-Akritas" given to `VCA` earlier, again in France [28]. To continue calling the `VCA` method after Descartes is to willfully: (a) ignore scientific realities, (b) distort the history of mathematics, and (c) perpetuate the obfuscation.

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[10] It goes without saying that the appropriate disks were also used by Alesina and Galuzzi in 1998 and 2000 [12], [14].

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[11] For an English translation of a book with similar content see: Nikola Obreschkoff: "Zeros of Polynomials", Bulgarian Academic Monographs (7), Sofia, 2003.

[12] A short biography of Vincent (in French) can be found in p. 1026, vol 31 of "La Grande Encyclopédie", see http://gallica.bnf.fr/ark:/12148/bpt6k24666x, whereas in http://www.allposters.fr/-st/Lasnier-Affiches_c25893_s88165_.htm Vincent's portrait can be seen [17].