An Improvement on Lagrange's Quadratic Bound on the Values of the Positive Roots of Polynomials

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Abstract

Lagrange's[1] quadratic bound, `LQ`, on the values of the positive roots of polynomials consists of two parts.

In the first part of `LQ` a list is constructed containing sub-lists, which correspond to the coefficients of the given polynomial. The sub-lists corresponding to positive coefficients remain empty throughout the execution of the algorithm. By contrast, the sub-lists corresponding to negative coefficients of the polynomial may end up containing 0, 1, 2 or more radicals of the preceding positive coefficients.

In the second part of `LQ`, the function `sort` is applied to all sub-lists containing more than 2 radicals and the sum of the largest two radicals in each sub-list is a possible candidate to be returned as the bound.

The computing time of the second part as is currently implemented is $O(n \cdot log(n))$. With our improvement we reduce its computing time to $O(n)$.

1 Introduction

On p. 553 of his original paper [3][2] — or on p. 32, of his famous book [4], which constitutes the 8-th volume of *Œuvres de Lagrange*, edited by Joseph Alfred Serret [5] — Lagrange only states that given the polynomial F, where

$$-\mu y^{r-m} - \nu y^{r-n} - \bar{\omega} y^{r-p} - \cdots$$

are its "negative terms", then an upper bound for the real roots of F is given by the sum of the first two largest of the quantities

$$\sqrt[m]{\mu},\ \sqrt[n]{\nu},\ \sqrt[p]{\bar{\omega}}, \ldots$$

or "a number larger than this sum."

The interesting history of this theorem can be found elsewhere [1]. Here we simply state the theorem without its proof.

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[1]Italian mathematician Joseph-Louis Lagrange, born Giuseppe Lodovico Lagrangia, (25 January 1736 - 10 April 1813).

[2]Presented to the Berlin Academy on April 20, 1769.

Theorem 1 (**Lagrange, 1767**) *Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0,$, be a non constant monic polynomial of degree n over $\mathbb{R}$ and let $a_{n-j} : j \in J$ be the set of its negative coefficients. Then an upper bound for the positive real roots of f is given by the sum of the largest and the second largest members in the set $\left\{ \sqrt[j]{|a_{n-j}|} : j \in J \right\}$. That is,*

$$b = \max_{\{a_{n-i}, a_{n-k} \in J\}} (\sqrt[i]{|-a_{n-i}|} + \sqrt[k]{|-a_{n-k}|}). \tag{1}$$

2 Lagrange's Quadratic Algorithm `LQ`

In `LQ` we use the list (of lists) t of length $n+1$, in which initially each entry is [ ], the empty list. The list $t_{n-j} = t[n-j]$ corresponds to the coefficient a_{n-j} of the polynomial and, if $a_{n-j} > 0$, then the list $t[n-j]$ contains all the minimum values produced by $a_{n-j} > 0$ when "paired" with various negative coefficients a_{n-i}, with $i > j$.

The algorithm works as follows:

- *each* negative coefficient a_{n-i} of the polynomial is "paired" with *each one* the preceding positive coefficients a_{n-j}, $(i > j)$ and the minimum is taken of all the radicals of the form

$$\sqrt[i-j]{\frac{-a_{n-i}}{a_{n-j}}} \tag{2}$$

  as indicated in Lagrange's theorem (Theorem 1); each minimum is then appended to the corresponding list $t[n-j]$,

- we initialize a temporary bound to 0, and then for each non-empty list $t[n-j]$ we proceed as follows: (a) if the list $t[n-j]$ has a single element, and its value is greater than the temporary bound, then *it* (the single element) becomes the temporary bound and, (b) if the list $t[n-j]$ has more than one element, we sort them in increasing order and take the sum of the largest two; if the sum is greater than the temporary bound, then *it* (the sum) becomes the temporary bound; at the end the temporary bound is taken as the estimate of the bound.

An algorithmic description of Lagrange's quadratic method `LQ` is presented in Algorithm 2 below.

Algorithm 1: `LQ(f, x)`, Lagrange's Quadratic Algorithm.

```
  Input: A univariate polynomial f(x) = a_n x^n + a_{n-1} x^{n-1} + ··· + a_0 ∈ Z[x], with a_n > 0.
  Output: An upper bound on the values of the positive roots of f(x).
  // at least one sign variation (v ≥ 1)?
  cl ← [a_0, a_1, a_2, ..., a_{n-1}, a_n];                    /* list of length n + 1 */
  v ← number of sign variations in cl;
  if v = 0 then return 0;
  // initialize variables
  m ← length(cl);
  t ← [ [ ], [ ], [ ], ..., [ ], [ ] ];                        /* list of length n + 1 */
  // main loop, which is almost identical to the one in LMQ
1 for j = 0 to m − 1 step 1 do
2     if cl(j) < 0 then
          b ← +∞;
          index ← m;
          for k = j + 1 to m − 1 step 1 do
3             if cl(k) > 0 then
                  q ← (−cl[j]/cl[k])^(1/(k−j));
4                 if q < b then
                      b ← q;
                      index ← k;
                  end
              end
          end
          t[index] ← append(t[index], b);
      end
  end
  // secondary loop to process the list of lists t
  b ← 0;
5 for j = 0 to m − 1 step 1 do
      tp ← t[j];
6     if tp ≠ [] then
7         if length(tp) = 1 then
              tp ← tp[0];                            /* enumeration starts from 0 */
          else
              tp ← sort(tp);                         /* sort tp in increasing order */
              tp ← sum(tp[−2 :]);                    /* sum of the largest two values */
          end
8         if tp > b then
              b ← tp;
          end
      end
  end
  return b
```

Note the function `sort` in step 7 of the algorithm above. Obviously, the computing time of the secondary loop (steps 5-8) is $O(n \cdot log(n))$.

3 Improved Lagrange's Quadratic Algorithm `LQ`

In the improved `LQ` algorithm we replaced the function `sort` by a series of instructions, with the help of which the computing time of the secondary loop becomes $O(n)$.

Algorithm 2: `LQ(f, x)`, Lagrange's Quadratic Algorithm **Improved**.

Input: A univariate polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x]$, with $a_n > 0$.

Output: An upper bound on the values of the positive roots of $f(x)$.

```
   // at least one sign variation (v ≥ 1)?
   cl ← [a_0, a_1, a_2, ..., a_{n-1}, a_n];                /* list of length n + 1 */
   v ← number of sign variations in cl;
   if v = 0 then return 0;
   // initialize variables
   m ← length(cl);
   t ← [ [ ], [ ], [ ], ..., [ ], [ ] ];                    /* list of length n + 1 */
   // main loop, which is almost identical to the one in LMQ
1  for j = 0 to m − 1 step 1 do
2      if cl(j) < 0 then
           b ← +∞;
           index ← m;
           for k = j + 1 to m − 1 step 1 do
3              if cl(k) > 0 then
                   q ← (−cl[j]/cl[k])^{1/(k−j)};
4                  if q < b then
                       b ← q;
                       index ← k;
                   end
               end
           end
           t[index] ← append(t[index], b);
       end
   end
   // secondary loop to process the list of lists t without sort
   b ← 0;
5  for j = 0 to m − 1 step 1 do
6      if tp ≠ [ ] then
7          tp ← t[j];
           ltp ← length(tp);
           if ltp = 1 then
               sc ← tp[0] ;                          /* enumeration starts from 0 */
           end
           c ← [ ];
8          if ltp > 2 then
9              if tp[0] < tp[1] then
                   c ← c[tp[0], tp[1]];
               else
                   c ← c[tp[1], tp[0]];
               end
           else
               c ← c[tp[0], tp[1]];
           end
10         for k = 2 to ltp − 1 step 1 do
11             if tp[k] > c[0] then
12                 if tp[k] > c[1] then
                       c[0] ← c[1];
                       c[1] ← tp[k];
                   else
                       c[0] ← tp[k];
                   end
               end
               k ← k + 1;
           end
           sc ← sum(c[−2 :]);                         /* sum the two values of c */
13         if sc > b then
               b ← sc;
           end
       end
   end
   return b
```

Having replaced the function `sort` the secondary loop of the algorithm (steps 5-13) is now executed in time $O(n)$.

4 Conclusions

In this paper we have presented an improvement of Lagrange's quadratic bound for computing upper bounds on the positive roots of polynomials.

The computing time of `LQ` is, of course, $O(n^2)$ but, as described in [1], the time expression has a second term; to wit, it is $O(n^2 + n \cdot log(n))$, where the second term is omitted.

In this paper we reduce the computing time of `LQ` to $O(n^2 + n)$, which can make a difference for very large n.

References

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[3] Lagrange, Joseph-Louis: "Sur la résolution des équations numériques", 1767. In: Mémoires de l' Académie Royale des Sciences et des Belle-Lettres de Berlin, (1769), **23**, 539–578. This is in volume 2 of J. A. Serret's *Œuvres de Lagrange.*[3]

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[5] Serret, J. A.: *Œuvres de Lagrange*; Gauthier-Villars, Paris, 1879.

[6] Stedall, J.: *From Cardano's great art to Lagrange's reflections: filling a gap in the history of algebra*; European Mathematical Society, (2011).

[3] The digitized form of *Œuvres de Lagrange* [5], Lagrange's collected work, can be found in the sites `http://sites.mathdoc.fr/OEUVRES/`, and `https://gdz.sub.uni-goettingen.de/`. Following Stedall's remark ([6], p. 209), Lagrange's article is listed here with two dates: the first one is the year when the paper is known to have been written — as recorded by Lagrange himself ([2], p. 384) — and the other one is the year in which the volume of papers was published.