A REMARK ON THE PROPOSED SYLLABUS FOR AN AMS SHORT COURSE ON COMPUTER ALGEBRA

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It was a pleasure to see that our Special Interest Group initiated contacts with the AMS and that a short course on Computer Algebra has been proposed. The Syllabus [8] for this course seems very representative of our work and I am sure that it will be of great interest to the AMS members. However, as outlined in [8], the Syllabus contains a great historical inaccuracy when referring to

"...Descartes' theorem and Uspensky's method;"

(in 4: Factorization and Zeros of Polynomials), an inaccuracy which I would like to see corrected.

To reiterate my position described in [2] and [4], I claim that there is no Uspensky's method and credit should be given to Vincent.

According to Vincent's theorem of 1836 [1], [2], [7], if an equation with rational coefficients and without multiple roots is successively transformed by substitutions of the form $x \leftarrow a_i + \frac{1}{x}$, for arbitrary, integral, positive a_i's, one eventually obtains an equation with at most one sign variation in the sequence of its coefficients. (For a correct, extended version of this theorem see [4].) This forgotten theorem can be used in order to isolate the real roots of a polynomial equation.

The calculation of the quantities $a_1, a_2, \ldots, a_m$ - for the transformations of the form $x \leftarrow a_i + \frac{1}{x}$ which lead to an equation with exactly one sign variation - constitutes the polynomial real root isolation procedure. Two methods actually result, Vincent's and ours corresponding to the two different ways in which the computation of the a_i's may be performed.

Vincent's method basically consists of computing a particular a_i by a series of unit incrementations; that is, $a_i \leftarrow a_i + 1$, which corresponds to the substitution $x \leftarrow x + 1$. This 'brute force' approach results in a method with an exponential behavior, namely, for big values of the a_i's this method will take a long time (even years in a computer) in order to isolate the real roots of an equation. Therefore, Vincent's method is of little practical importance. Examples of this approach can be found in Vincent's paper [7], and in Uspensky's book [6, pp. 129-137]. The reader should notice that in the preface of his book, Uspensky claims that he himself invented this method. A simple comparison with Vincent's paper though makes clear that what can be considered a contribution on Uspensky's part is only the fact that he used the Ruffini-Horner method [5] in order to perform the transformations $x \leftarrow x + 1$, whereas Vincent used Taylor's expansion theorem. Moreover, Uspensky seems to ignore Budan's theorem [3] and, while computing a particular a_i, he performs, after each translation $x \leftarrow x + 1$, the unnecessary transformation $x \leftarrow 1/(x + 1)$, something which Vincent avoids.

Our method [4], on the contrary, consists of immediately computing a particular a_i as the lower bound b on the values of the positive roots of a polynomial; that is, $a_i \leftarrow b$, which corresponds to the substitution $x \leftarrow x + b$ performed on the particular polynomial under consideration. It is obvious that our method is independent of how big the values of the a_i's are. (An unsuccessful treatment of the big values of the a_i's can be found in Uspensky [6, p. 136]. In this discussion it is assumed that $b = \lfloor \alpha_s \rfloor$, where α_s is the smallest positive root.) Since the substitutions $x \leftarrow x + 1$ and $x \leftarrow x + b$ can be performed in about the same time [5], we easily see that our method results in enormous savings of computing time.

From the above it becomes obvious that credit should be given to Vincent, and I, hereby, suggest that the inaccurate entry in the Syllabus be changed to:

"...Descartes' rule of signs and Vincent's forgotten theorem of 1836;"

References

1. Akritas, A. G., Vincent's Theorem in Algebraic Manipulation. Ph.D. thesis, Operations Research Program, North Carolina State University, Raleigh, NC, 1978.

2. Akritas, A. G., A Short Note on a New Method for Polynomial Real Root Isolation. ACM-SIGSAM Bulletin, Vol. 12, No. 4, Nov. 1978, 12-13.

3 Akritas, A. G., The Two Different Ways of Expressing the Budan-Fourier Theorem and Their Consequences. Lectures of the General Mathematical Seminar of the University of Patras, Vol. 5, 1979, 127-146 (in Greek).