Colloque AFCET, Les Mathématiques de l'Informatique, Paris, France, 1982.

#20

APPLICATIONS OF VINCENT'S THEOREM--APPROXIMATING THE REAL ROOTS OF A POLYNOMIAL EQUATION AND ISOLATING THE SMALLEST

APPLICATIONS DU THEOREME VINCENT--APPROXIMATION DES RACINES REELLES D'UNE EQUATION POLYNOMIALE ET ISOLEMENT DE LA RACINE LA PLUS PETITE

A. G. Akritas*, S. J. Chang*, and K. H. Ng**

Abstract - Previous work by the first author has shown that Vincent's theorem of 1836 forms the basis of the fastest method for the isolation of the real roots of a polynomial equation [3]--when exact integer arithmetic is used. In this paper we show how this theorem can be used to:
(a) approximate the real roots of an equation to any desired degree of accuracy--realizing thus a proposal by Lagrange [9]--, and (b) isolate only the smallest root of a given equation.

Résumé - Les ouvrages antérieurs par le premier auteur ont démontré que le théorème de Vincent de 1836 est à la base de la méthode la plus rapide pour l'isolement des racines réelles d'une équation polynomiale [3]--en utilisant une arithmetique exacte de nombres entiers. Dans cet article nous démontrons comment peut être utilisé ce théorème afin de:
(a) approximer les racines réelles d'une équation à n' importe quel degré désiré d' exactitude--réalisant ainsi une proposition presentée par Lagrange [9]--et (b) d' isoler seulement la racine la plus petite d'une équation donnée.

* Department of Computer Science, University of Kansas, Lawrence, Kansas, 66045, U.S.A.

** Present Address: Shell Oil Company, P.O. Box 991, Houston, Texas, 77001, U.S.A.

1. Introduction

Vincent's theorem of 1836 was discovered by the first author in Uspensky's Theory of Equation [13], and it formed the subject of his Ph.D. thesis [1],[15]. An extended version of this theorem is the following [2]:

Theorem 1. Let $P(x) = 0$ be a polynomial equation of degree $n > 1$, with rational coefficients and without multiple roots, and let $\Delta > 0$ be the smallest distance between any two of its roots. Let m be the smallest index such that

$$F_{m-1}\frac{\Delta}{2} > 1 \text{ and } F_{m-1}F_m\Delta > 1 + \frac{1}{\epsilon_n},$$

where F_k is the k-th member of the Fibonacci sequence

$$1,1,2,3,5,8,13,21, \ldots$$

and

$$\epsilon_n = \left(1 + \frac{1}{n}\right)^{\frac{1}{n-1}} - 1.$$

Then the transformation

$$x = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_m + \cfrac{1}{\xi}}}} \tag{1}$$

(which is equivalent to the series of successive transformations of the form $x = a_i + \frac{1}{\xi}$, $i = 1,2,\ldots,m$) presented in the form of a continued fraction with arbitrary, positive, integral elements $a_1,a_2,\ldots,a_m$, transforms the equation $P(x) = 0$ into the equation $\tilde{P}(\xi) = 0$, which has not more than one sign variation in the sequence of its coefficients.

The above theorem (whose proof can be found in [2]) can be used to isolate the real roots of a polynomial equation [4],[5],[13]. The calculation of the partial quotients $a_1,a_2,\ldots,a_m$ for the transformations of the form (1)--which lead to an equation with exactly one sign variation--constitutes the polynomial real root isolation procedure. There are two methods for doing this, one due to Vincent and one due to the first author; each of these corresponds to a different way in which the computation of the a_i's may be performed. It was shown [1],[4],[5], that Vincent's method behaves exponentially, whereas the method developed by the first author has a polynomial computing time bound; that is, the time to isolate all the real roots of a polynomial equation is

$$O(n^5 L(|P|_\infty)^3) \tag{2}$$

where n is the degree of the polynomial and $L(|P|_\infty)$ the length, in bits, of the maximum coefficient in absolute value. The computing time bound (2) is achieved by computing a particular partial quotient a_i as the lower bound b on the values of the positive roots of a polynomial. (We assume that $b = \lfloor \alpha_s \rfloor$, where α_s is the smallest positive root.) The computation of this bound b is performed with the help of the following [6],[12]:

Cauchy's Rule: Let $P(x) = x^n + c_{n-1}x^{n-1} + \ldots + c_1x + c_0 = 0$ be an integral, monic polynomial equation of positive degree n, with $c_{n-k} < 0$ for at least one k, $1 \le k \le n$, and let λ be the number of its negative coefficients. Then

$$b = \max_{\substack{1 \le k \le n \\ c_{n-k} < 0}} |\lambda c_{n-k}|^{\frac{1}{k}}$$

is an upper bound on the values of the positive roots of $P(x) = 0$.

In the sequel we examine two applications of Vincent's theorem. First we show that, after the real roots have been isolated, we can easily approximate them to any desired degree of accuracy, by basically continuing the same process used in the isolation. In this way, we realize a proposal by Lagrange, who was the first to suggest the approximation of real roots using continued fractions [9]. (We actually improved his original idea, since our method has a polynomial computing time bound. Moreover, it should be noted that Lagrange's approximation method could not work if there were more than one root in the interval $(k, k+1)$, k integer. (At that time Vincent's theorem did not exist.)) Secondly, we give an answer to the question whether it is possible to isolate just the smallest root of a polynomial equation in time less than (2).

<u>Note</u>: In both applications mentioned above we are concerned only with the positive roots; the same results hold for the negative roots, too, if we replace x by $-x$ in the original equation. Also, the coefficients of the polynomials are all integers.

2. Approximation of the Real Roots of a Polynomial Equation

Pursuing studies in the direction outlined above, it was observed that Theorem 1 can be also used to approximate the real roots to any desired degree of accuracy. This is easily achieved by extending (computing more partial quotients of) the continued fraction CF_0, which transforms the original polynomial equation into one with exactly one sign variation in the sequence of its coefficients. (The reader should notice that, now, the approximation method depends heavily on the isolation process. In other words, it can not work if it is provided only with the isolating intervals of the roots.)

In what follows we describe two ways of implementing the idea mentioned above; details can be found in [11]. It was shown ([11] p. 47) that (using exact integer arithmetic) the computation time (for the approximation of one root) for both ways is

$$O(L(\tfrac{1}{\epsilon})(n^2 L(|P|_\infty)^3 + n^3 L(|P|_\infty)^2))$$

where n is the degree of the polynomial, $L(|P|_\infty)$ is the length, in bits, of the maximum coefficient in absolute value, and ϵ is the desired degree of accuracy (tolerance).

The first way to extend the continued fraction CF_0 is to compute each additional partial quotient again with the help of Cauchy's rule (see Figure 1). However, mainly due to Cauchy's rule, this approach is inefficient as can be seen from Table 1 (at the end of this section). Actually, it is even slower than the bisection method [8], a method well-known for its slowness.

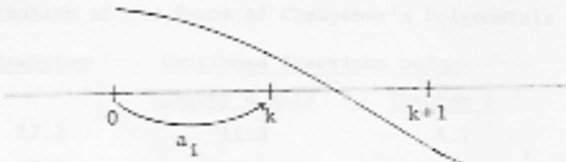


Figure 1: Here k is computed with the help of Cauchy's rule.

Trying to improve the empirical performance of our approximation method, we observed the special nature of the polynomials whose lower bounds b we are computing. They are special in the sense that they have one sign variation (and, hence, only one positive root) and consequently, they cross the x-axis only once. We then proceeded to compute each additional partial quotient of the continued fraction CF_0 by successively bisecting (and evaluating at midpoints) the interval (0,b), where b is an easily computed upper bound on the value of the positive root (see Figure 2).

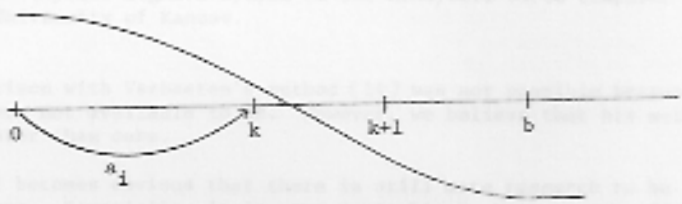


Figure 2: Here k is computed by successively bisecting the interval (0,b).

This upper bound b on the value of the positive root is easily computed with the help of the following theorem found in [11].

Theorem 2: Let $P(x) = c_n x^n + \ldots + c_{r+1}x^{r+1} - c_r x^r - \ldots - c_0$ be an integral polynomial with only one sign variation in the sequence of its coefficients. An upper bound on the (only one) positive root is given by

$$b = \frac{\max\limits_{0 \le j \le r} (|c_j|)}{\sum\limits_{i=r+1}^{n} c_i} + 1.$$

Obviously, Theorem 2 is much simpler than Cauchy's rule. As can be seen from Table 1 below, this approach resulted in great savings of computing time. (To see how much time is actually spent computing the additional partial quotients of CF_0, compare the last two columns of Table 1. The last column indicates the time needed to approximate a root, when the partial quotients are provided, that is, they were previously computed (preconditioning).)

Empirical Results

Approximation of the Roots of Chebyshev's Polynomials ($\epsilon = 10^{-15}$)

Degree	Bisection	Continued fractions using: Cauchy's Rule	Theorem 2	Preconditioning
2	17.2	11.5	6.7	5.4
3	17.9	10.3	4.9	3.8
4	42.3	38.7	15.7	10.3
5	45.8	40.0	16.4	10.8
6	83.1	99.8	46.2	29.2
7	90.9	105.1	44.6	27.2
8	146.3	257.8	93.0	50.2
9	170.6	277.6	106.2	62.2
10	243.2	524.3	202.8	116.2

Table 1: Times indicated are in seconds, and were obtained by using the SAC-1 computer algebra system on the Honeywell 66/60 computer of the University of Kansas.

A direct comparison with Verbaeten's method [14] was not possible because his programs were not available to us. However, we believe that his method is somewhat faster than ours.

From Table 1 it becomes obvious that there is still more research to be done in this area. Especially, it is very desirable to have a very efficient and inexpensive procedure to compute the bound of the positive root of a polynomial equation; this would reduce the computing time of our method even further.

3. Isolation of the Smallest Root of a Polynomial Equation

As we mentioned in the Introduction the computing time bound for the isolation of all the real roots of a polynomial equation is $O(n^5L(|P|_\infty)^3)$. The question naturally arises whether it is possible to isolate just the smallest root of a polynomial equation in time less than (2). The study of this question is of interest because in certain applications one needs to compute only the smallest root of an equation. As a result of our work we will immediately see that the answer to the above question is not always positive; we distinguish the following two cases:

(I) The roots of the polynomial equation are all real. In this case it was shown in [7] that the isolation of only the smallest root is achieved in time

$$O(n^4L(|P|_\infty)^3) \qquad (3)$$

which is better than (2).

(II) Some of the roots of the polynomial equation are complex. For this case it was shown in [7] that the isolation of only the smallest root can be done in time

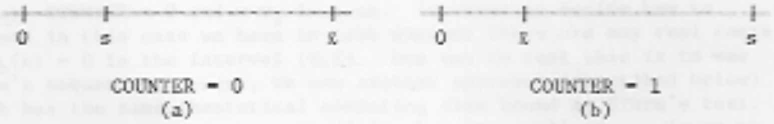
$$O(n^6L(|P|_\infty)^2) \qquad (4)$$

which is not as good as (2) (mainly due to a test that is done only in this case).

It thus becomes obvious that in the presence of complex roots it is preferable to isolate all the real roots and then choose the smallest one.

To see how Theorem 1 can be used to isolate the smallest root of a polynomial equation $P(x) = 0$, consider an infinite binary tree in which the root corresponds to $P(x) = 0$, and each node corresponds to a transformed equation resulting from the original after a series of successive transformations of the form $x = a_i + \frac{1}{y}$. The path from each node to the right descendent corresponds to the substitution $x \leftarrow 1 + x$, whereas, the path to the left descendent corresponds to the substitution $x \leftarrow \frac{1}{1+x}$. All the nodes belonging to a specific path, finite or infinite will be considered as members of disjoint sets, which can be of three types. A set of type V_0, V_1, or V_n contains nodes corresponding to polynomials with zero, one or more than one sign variation, respectively. Sets of type V_0 or V_1 are called <u>terminal sets</u>. In a terminal set, the node having the shortest path from the root of the tree will be called a <u>terminal node</u>.

What we are after is a V_1-terminal node associated with the smallest positive root (if it exists). However, in moving down the tree, every time we perform the transformation $x \leftarrow \frac{1}{1+x}$, the smallest root of the corresponding polynomial equation becomes the largest one of the resulting equation and vice-versa. Therefore, we need to keep track of the position of the smallest positive root. This is achieved with the help of a boolean variable COUNTER (see Figure 3), which is initialized to zero.



<u>Figure 3</u>: The positions of the <u>s</u>mallest and <u>l</u>argest positive roots when COUNTER = 0 (a) and COUNTER = 1 (b).

In what follows we give brief descriptions of the two algorithms used to isolate the smallest root of a polynomial equation for cases (I) and (II) mentioned above. Both algorithms were obtained by modifying procedure AMPRRI found in [4] (pp. 308-310); for details see [7]. We used the SAC-1 computer algebra system which is implemented on the Honeywell 66/60 computer of the University of Kansas.

(I) PROCEDURE: Isolate the Smallest Root of a Polynomial Equation (which has only real roots).

Let

$$P(x) = 0 \qquad (5)$$

be a polynomial equation with v sign variations in the sequence of its coefficients and without any multiple roots.

<u>Case (i)</u>: $v = 0$ or $v = 1$. From the Cardano-Descartes rule of signs we know that $v = 0$ implies that (5) has no positive roots, whereas, $v = 1$ indicates that (5) has exactly one positive root which is also the smallest one and $(0,\infty)$ is its isolating interval; in either case no transformation of (5) is necessary and the method terminates.

Case (ii): $v > 1$. In this case (5) has to be further investigated. We first compute the lower bound b on the values of the positive roots (we assume $b = \lfloor \alpha_s \rfloor$ where α_s is the smallest positive root--see also [6]) and then we obtain the translated equation $P_b(x) = P(b+x) = 0$ which has also v sign variations provided $P(b) \neq 0$. (If $P(b) = 0$ we have found an integer root of the original equation and v is decreased; moreover, if COUNTER = 0, this root is the smallest one we are after, and, so, we are done.) Now, obtain $P_{b1}(x) = P_b(1+x) = 0$ and let v_1 be the number of its sign variations. Obviously, $v \neq v_1$ and we consider the following possibilities:

(ii-a): COUNTER = 0. In this case, irrespective of v_1, we disregard $P_{b1}(x) = 0$; we then obtain $P_{b2}(x) = P_b(\frac{1}{1+x}) = 0$, set COUNTER = 1 and apply this procedure again with $P_{b2}(x) = 0$ in place of (5).

(ii-b): COUNTER = 1. Now we consider the following two subcases depending on the value of v_1.

(ii-b1): $v_1 = 0$. Here we disregard $P_{b1}(x) = 0$; we then obtain $P_{b2}(x) = P_b(\frac{1}{1+x}) = 0$, set COUNTER = 0 and apply this procedure again with $P_{b2}(x) = 0$ in place of (5).

(ii-b2): $v_1 \neq 0$. We now apply this procedure again with $P_{b1}(x) = 0$ in place of (5).

(II) PROCEDURE: Isolate the Smallest Root of a Polynomial Equation (some roots of which are complex).

Our approach here is similar to (i), except that now the following additional subcases have to be considered:

(II-a) COUNTER = 0 and $v-v_1$ is even. In order to decide how to proceed in this case we have to test whether there are any real roots of $P_b(x) = 0$ in the interval (0,1). One way to test this is to use Sturm's sequence; however, we use another approach (described below) which has the same theoretical computing time bound as Sturm's test. If there are some real roots in (0,1) then the smallest one is among them and we are back in case (ii-a) of (I); otherwise, we apply this procedure again with $P_{b1}(x) = 0$ in place of the equation corresponding to (5).

(II-b) Likewise, if COUNTER = 1 and v_1 is even we have to test whether there are any real roots of $P_b(x) = 0$, greater than 1. This test though is reduced to the previous one if we replace x by $\frac{1}{x}$ in $P_b(x) = 0$, and set COUNTER = 0. (If v_1 is odd then there is (at least) one real root.)

The test we use to decide whether there are any real roots in the interval (0,1) is the following (see also [7]):

Test: On the complex plane, the transformation $v \leftarrow \frac{1}{u} - 1$ maps every point outside the circle $|u - \frac{1}{2}| = \frac{1}{2}$ into the left half plane (i.e., into the half plane with negative real part).

See also [5] for another version of this test and its proof. When the time comes for the test to be performed, we have to make sure that all the complex roots (if any) will be outside the circle $|u - \frac{1}{2}| = \frac{1}{2}$. This is achieved with the help of Mahler's inequality (see [10]) for the minimum root separation.

$$\Delta \geq \sqrt{3} \cdot n^{-\frac{(n+2)}{2}} \cdot |P|_1^{-(n-1)} .$$

What we actually do is to compute the number $D = n^{\lceil \frac{n}{2} \rceil + 1} \cdot |P|_1^{(n-1)}$ and then obtain $P_D(x) = P_b(\frac{x}{D}) = 0$. Thus we are guaranteed that the minimum root separation of $P_D(x) = 0$ is > 1; in particular the absolute value of imaginary part of any root of $P_D(x) = 0$ is either 0 or $> \frac{1}{2}$. We then test whether there are any real roots in the interval $(0,D)$, and proceed accordingly.

In the course of our work it became obvious that what is missing in the literature is a good lower bound on the distance of two complex conjugate roots (or, equivalently, on the absolute value of their imaginary parts). Further research into this topic will improve the empirical performance of our algorithms.

References

[1] Akritas, A.G.: Vincent's theorem in algebraic manipulation. Ph.D. thesis, Operations Research Program, North Carolina State University, Raleigh, N.C., 1978.

[2] Akritas, A.G.: A correction on a theorem by Uspensky. Bulletin of the Greek Mathematical Society, Vol. 19, 278-285, 1978.

[3] Akritas, A.G.: On the solution of polynomial equations using continued fractions. Information Processing Letters, Vol. 9, No. 4, 182-184, 1979.

[4] Akritas, A.G.: The fastest exact algorithms for the isolation of the real roots of a polynomial equation. Computing, Vol. 24, 299-313, 1980.

[5] Akritas, A.G.: An implementation of Vincent's theorem. Numerische Mathematik, Vol. 36, 53-62, 1980.

[6] Akritas, A.G.: Exact algorithms for the implementation of Cauchy's rule. International Journal of Computer Mathematics, Vol. 9, 323-333, 1981.

[7] Chang, S.J.: Exact algorithms for the isolation of the smallest root of a polynomial equation. M.S. Research Report, University of Kansas, Department of Computer Science, Lawrence, Ks., 1981.

[8] Heindel, L.E.: Integer arithmetic algorithms for polynomial real zero determination. Journal of ACM, Vol. 18, 533-548, 1971.

[9] Lagrange, J.L.: Traité de la résolution des équations numériques. Paris, 1778.

[10] Mahler, K.: An inequality for the discriminant of a polynomial. Michigan Mathematical Journal, Vol. 11, 257-262, 1964.

[11] Ng, K.H.: Polynomial real root approximation using continued fractions. M.S. Research Report, University of Kansas, Department of Computer Science, Lawrence, Ks., 1980.

[12] Obreschkoff, N.: Verteilung und Berechnung der Nullstellen reeller Polynome. Berlin: VEB Deutscher Verlag der Wissenschaften, 1963.

[13] Uspensky, J.V.: Theory of equations. New York, McGraw-Hill, 1948.

[14] Verbaeten, P.: Computing real zeros of polynomial with SAC-1. ACM-SIGSAM Bulletin, Vol. 9, No. 2, 8-10, 1975.

[15] Vincent, A.J.H.: Sur la résolution des équations numériques. Journal de Mathématiques Pures et Appliquées, Vol. 1, 341-371, 1836.