#4

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A CORRECTION ON A THEOREM BY USPENSKY

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Uspensky's theorem is indispensable in the computing time analysis of Akritas' method for polynomial real root isolation; recent studies however, revealed that it is erroneous, both in its statement and proof. In this paper, a new, correct version of this important theorem is provided, alogn with an elegant proof.

INTRODUCTION

Quite recently, in Uspensky's *Theory of Equations* ([7] pp. 127–137), the author discovered Vincent's forgotten theorem [4], [8] . This remarkable theorem of 1836 asserts that if a univariate polynomial equation with rational coefficients and without multiple roots, is successively transformed by successive transformations of the form $x = a_i + \frac{1}{\xi}$, for arbitrary, positive, integer elements a_i , one eventually obtains an equation with at most one sign variation in the sequence of its coefficients.

Besides its theoretical interest, Vincent's theorem can be used in order to isolate the real roots of polynomial equations. Actually, two root isolation methods result, Vincent's and Akritas' , corresponding to the two differente ways of computing the quantities a_i [1], [2]. The difference between these two methods can be thought of as being analogous to the difference between the integrals of Riemann and Legesgue. It has been shown [1], [3] that Vincent's method behaves exponentially, whereas, Akritas' has a polynomial computing time bound, which in fact is the best one achieved thus far (empirical results also verify the superiority of Akritas' method over all others existing).

In order to prove the order of convergence of Akritas' method, however, it is necessary to obtain an upper bound on the number of executions of the successive transformations of the form $x = a_i + \frac{1}{\xi}$, which were mentioned earlier. One such bound is provided by Uspensky's theorem ([7] pp 298–303), which in fact is an extension of the theorem by Vincent; we discovered, however, that Uspensky's theorem is erroneous, both in its statement and proof, yielding thus an incorrect bound. In what follows we present our own, correct version of this important theorem, along with an elegant proof.

THE CORRECTED THEOREM

Before we present the main theorem, we state without proof, two lemmas, which will be needed.

Lemma 1. (Stodola, [6] p. 105). If the polynomial equation

$$P(x) = c_0 x^n + c_1 x^{n-1} + \cdots + c_n = 0 \qquad (c_0 > 0)$$

with real coefficients c_ν, $\nu = 0,\ldots,n$, has only roots with negative real parts, then all its coefficients are positive, and hence, they present no sign variation.

Lemma 2. (Akritas – Danielopoulos, [5]). Let $P(x) = 0$ be a polynomial equation of degree $n > 1$, without multiple roots, which has one positive real root $\xi \neq 0$ and $n-1$ roots, $\xi_1, \xi_2, \ldots, \xi_{n-1}$ with negative real parts, the complex roots appearing in conjugate pairs. Suppose that the roots ξ_j, $j = 1,2,\ldots,n-1$ can be expressed in the form

$$\xi_j = -(1 + \alpha_j),$$

with $|\alpha_j| < \epsilon_n$, where

$$\epsilon_n = \left(1 + \frac{1}{n}\right)^{\frac{1}{n-1}} - 1.$$

Then $P(x)$, in its expanded form, presents exactly one sign variation.

Theorem. (Vincent–Uspensky–Akritas). Let $P(x) = 0$ be a polynomial equation of degree $n > 1$, with rational coefficients and without multiple roots, and let $\Delta > 0$ be the smallest distance between any two of its root . Let m be the smallest index such that

(1) $$F_{m-1} \frac{\Delta}{2} > 1 \quad \text{and} \quad F_{m-1} F_m \Delta > 1 + \frac{1}{\epsilon_n}.$$

where F_k is the k^{th} member of the Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

and

(2) $$e_n = \left(1 + \frac{1}{n}\right)^{\frac{1}{n-1}} - 1.$$

Then the transformation

(3) $$x = a_1 + \frac{1}{a_2} + \cdots + \frac{1}{a_m} + \frac{1}{\xi}$$

(which is equivalent to the series of successive transformations of the form $x = a_i + \frac{1}{\xi}$, $i = 1,2,\ldots, m$) presented in the form of a continued fraction with arbitrary, positive integral elements $a_1, a_2, \ldots, a_m$, transforms the equation $P(x)=0$ into the equation $\widetilde{P}(\xi) = 0$, which has not more than one sign variation.

Proof. In order to prove the theorem, it suffices to show that, after the m successive transformations of the form $x = a_i + \frac{1}{\xi}$, the real parts of all complex roots, as well as all real roots except for at most one, become negative.

Indeed, let $\frac{P_k}{Q_k}$ be the k^{th} convergent to the continued fraction

$$a_1 + \frac{1}{a_2} + \frac{1}{a_3} + \cdots$$

From the law of convergents we have:

$$P_{k+1} = a_{k+1} P_k + P_{k-1},$$

$$Q_{k+1} = a_{k+1} Q_k + Q_{k-1}.$$

Since $Q_1 = 1$ and $Q_2 = a_2 \geqslant 1$, it follows that $Q_k \geqslant F_k$. Further, the relation (3) can be expressed in the form

$$x = \frac{P_m \xi + P_{m-1}}{Q_m \xi + Q_{m-1}},$$

from which it follows that

(4) $$\xi = -\frac{P_{m-1} - Q_{m-1}x}{P_m - Q_m x}.$$

Clearly, if x_0 is any root of the equation $P(x) = 0$, the quantity ξ_0, determined by (4), is the corresponding root of the transformed equation $\widetilde{P}(\xi) = 0$.

(a) Assume that x_0 is a complex root of $P(x) = 0$; that is $x_0 = a \pm ib$, $b \neq 0$. In this case the real part of the corresponding root ξ_0 is

(5) $$\text{r.p.}(\xi_0) = -\frac{(P_{m-1} - Q_{m-1}a)(P_m - Q_m a) + Q_{m-1}Q_m b^2}{(P_m - Q_m a)^2 + Q_m^2 b^2}$$

This is certainly negative if

$$(P_{m-1} - Q_{m-1}a)(P_m - Q_m a) \geqslant 0.$$

If, on the contrary

$$(P_{m-1} - Q_{m-1}a)(P_m - Q_m a) < 0$$

then clearly the value of a is contained between the two consecutive convergents

$$\frac{P_{m-1}}{Q_{m-1}}, \quad \frac{P_m}{Q_m},$$

whose difference in absolute value is

$$\frac{1}{Q_{m-1}Q_m}.$$

Hence,

$$\left| \frac{P_{m-1}}{Q_{m-1}} - a \right| < \frac{1}{Q_{m-1}Q_m} \quad \text{and} \quad \left| \frac{P_m}{Q_m} - a \right| < \frac{1}{Q_{m-1}Q_m},$$

from which it follows that

(6) $$|(P_{m-1} - Q_{m-1}a)(P_m - Q_m a)| < \frac{1}{Q_{m-1}Q_m} \leqslant 1.$$

From (5) and (6) we conclude that the r.p. (ξ_0) will be negative if

$$Q_{m-1}Q_m b^2 > 1.$$

To prove that this is true in our case, first observe that, since Δ is the minimum distance between any two roots of $P(x) = 0$, we have

$$|(a + ib) - (a - ib)| = |2ib| = 2|b| \geqslant \Delta ,$$

from which we obtain $|b| \geqslant \frac{\Delta}{2}$; moreover, we know that

$Q_m \geqslant Q_{m-1} \geqslant F_{m-1}$, and, from (1), $F_{m-1} \frac{\Delta}{2} > 1$. Then clearly $F_{m-1}|b| > 1$ which implies $Q_{m-1}|b| > 1$ and $Q_m|b| > 1$. From the last two inequalities we obtain $Q_{m-1} Q_m b^2 > 1$, proving thus, that the r.p.$(\xi_0) < 0$; this is obviously true for all complex roots of the transformed equation $\widetilde{P}(\xi) = 0$.

(b) Assume now that x_0 is a real root of $P(x) = 0$. Suppose first that for all real roots x_i,

$$(P_{m-1} - Q_{m-1} x_i)(P_m - Q_m x_i) > 0.$$

From (4) it follows that all real roots of the transformed equation $\widetilde{P}(\xi) = 0$ will be negative; moreover, we know from (a), that all the complex roots of $\widetilde{P}(\xi) = 0$ have negative real parts. Consequently, due to Lemma 1, $\widetilde{P}(\xi)$ presents no sign variation. Suppose, now, that for some real root x_0.

(7) $$(P_{m-1} - Q_{m-1} x_0)(P_m - Q_m x_0) \leqslant 0.$$

Then, clearly, x_0 is contained between the two consecutive convergents

$$\frac{P_{m-1}}{Q_{m-1}} , \frac{P_m}{Q_m}$$

and hence,

$$\left| \frac{P_m}{Q_m} - x_0 \right| < \frac{1}{Q_{m-1} Q_m} .$$

Let x_k, $k \neq 0$, be any other root, real or complex, of $P(x) = 0$, and ξ_k the corresponding root of the transformed equation.

Then, keeping in mind that

$$P_m Q_{m-1} - P_{m-1} Q_m = (-1)^m$$

it follows from (4) that

$$\xi_k + \frac{Q_{m-1}}{Q_m} = \frac{(-1)^m}{Q_m(P_m - Q_m x_k)}$$

or

$$\xi_k = -\frac{Q_{m-1}}{Q_m}\left[1 - \frac{(-1)^m}{Q_{m-1}Q_m(\frac{P_m}{Q_m} - x_k)}\right] = -\frac{Q_{m-1}}{Q_m}(1 + a_k),$$

where

$$a_k = \frac{(-1)^{m-1}}{Q_{m-1}Q_m \frac{P_m}{Q_m} - x_k)} .$$

Now

$$\left|\frac{P_m}{Q_m} - x_k\right| = \left|\frac{P_m}{Q_m} - x + x - x_k\right| \geqslant \left|x - x_k\right| - \left|\frac{P_m}{Q_m} - x\right| \geqslant \Delta - \frac{1}{Q_{m-1}Q_m} > 0$$

and consequently

$$|a_k| \leqslant \frac{1}{Q_{m-1}Q_m\Delta - 1} \leqslant \frac{1}{F_{m-1}F_m\Delta - 1} ;$$

from the last expression and the second inequality of (1) we deduce that

$$|a_k| < \epsilon_n$$

Thus, the roots ξ_k, $k = 1,2,\dots,n-1$, of the transformed equation, corresponding to the roots x_k, $k = 1,2,\dots,n-1$, of the equation $P(x) = 0$, which are all different from x_0, are of the form

$$\text{(8)} \qquad \xi_k = -\frac{Q_{m-1}}{Q_m}(1 + a_k), \quad |a_k| < \epsilon_n ;$$

that is, the roots of the transformed equation are negative and clustered together around -1. If we make the substitutions

$$\xi = \frac{Q_{m-1}}{Q_m} u, \quad \xi_k = \frac{Q_{m-1}}{Q_m}\bar{\xi}_k , \quad k = 0,1,\dots,n-1,$$

where,

$$\bar{\xi}_0 > 0 \quad \text{and} \quad \bar{\xi}_k = -(1 + a_k), \quad k = 1,2,\dots,n-1,$$

the transformed polynomial $\tilde{P}(\xi)$ can be written in the form

$$\tilde{P}(\xi) = \left(\frac{Q_{m-1}}{Q_m}\right)^n \tilde{P}(u) = c\left(\frac{Q_{m-1}}{Q_m}\right)^n (u-\bar{\xi}_0)(u-\bar{\xi}_1)\dots(u-\bar{\xi}_{n-1}).$$

Since $\tilde{P}(u)$ satisfies all the assumptions of Lemma 2, it presents exactly one sign variation, and, obviously, the same is true for the transformed polynomial $\tilde{P}(\xi)$. The last thing to consider now is the case when (7) holds as an equality;
that is

$$(P_{m-1} - Q_{m-1}x_0)(P_m - Q_m x_0) = 0.$$

If $P_{m-1} - Q_{m-1}x_0 = 0$ then we see, from (4), that $\xi_0 = 0$, and clearly the transformed equation $\tilde{P}(\xi) = 0$ has no sign variation (Lemma 1). In the case $P_m - Q_m x_0 = 0$ we have $\xi_0 = \infty$ and the transformed equation reduces to degree $n-1$. Since again all the roots have negative real parts, we conclude, from Lemma 1, that $\tilde{P}(\xi) = 0$ presents no sign variation. Thus we have proved the theorem completely.

This theorem gives us, clearly, m as an upper bound on the number of tranformations of the form $x = a_i + \frac{1}{\xi}$, necessary to obtain a polynomial with not more than one sign variation in the sequence of its coefficents. Comparing our theorem with Uspensky's version of it ([7] pp. 298–303), we see that his error lies in the fact that he computes m as the smallest index such that

(9) $$F_{m-1}\Delta > \frac{1}{2} \quad \text{and} \quad F_{m-1}F_m\Delta > 1 + \frac{1}{e_n}$$

In this case, however, it is not guaranteed that the real part of all the complex roots of the transformed equation $\tilde{P}(\xi) = 0$, will be negative. This follows from the fact that now the inequality (see also part (a) of our proof),

(10) $$Q_{m-1}Q_m b^2 > 1$$

is not always satisfied; indeed, given $Q_m \geqslant Q_{m-1} \geqslant F_{m-1}$ and $|b| \geqslant \frac{\Delta}{2}$, then by using $F_{m-1}\Delta > \frac{1}{2}$ we only obtain

(11) . $$Q_{m-1}Q_m b^2 > \frac{1}{16}.$$

(At this point in his proof ([7] p. 300) Uspensky erroneously claims that (10) is obtained from (9); moreover, notice that the inequality $F_{m-1}\ \Delta > \frac{1}{2}$ is stated again in the proof.) Despite the fact that (10) may be occasionally satisfied when (11) is true, in general (11) does not guarantee that $\widetilde{P}(\xi) = 0$ will have at most one sign variation in the sequence of its coefficients. As we see this ambiguity vanishes, with the way our theorem is stated.

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References

[1] Akritas, A.G.," Vincent's Theorem in Algebraic Manipulation", Ph.D.Thesis, in preparation, North Carolina State University, 1978.

[2] Akritas, A.G., " Vincent's Theorem and Akritas' Method for Polynomial Real Root Isolation" , Proceedings of the North Carolina Academy of Sciences, to appear in the *Journal of the Elisha Mitchell Society,* 1978.

[3] Akritas, A.G., " A New Method for Polynomial Real Root Isolation " ,*Proceedings of the 16th Annual Southeast Regional ACM Conference,* Atlanta, Ga., April, 1978, pp. 39–43.

[4] Akritas, A.G., and Danielopoulos, S.D., " On the Forgotten Theorem of Mr. Vincent" , *Historia Mathematica,* in press.

[5] Akritas, A.G., and Danielopoulos, S.D., " A Sufficiency Condition for the Cardano–Descartes Rule of Signs " *American Mathematical Monthly,* submitted for publication.

[6] Obreschkoff, N., *Verteilung and Berechnung der Nullstellen reeler* Polynome, VEB Deutscher Verlag der Wissenschaften, Berlin, 1963.

[7] Uspensky, J.V., *Theory of Equations,*, McGraw–Hill, New York, 1948.

[8] Vincent, "Sur la Resolution des Equations Numeriques" , *Journal de Mathematiques Pures et Appliquees,* Vol. 1, 1836, pp. 341–372.

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