

# Vincent's Theorem of 1836: Overview and Recent Developments<sup>1</sup>

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*To the Students of the University of Thessaly*

**Abstract** In this paper we first present the two different versions of Vincent's theorem of 1836 and discuss the various real root isolation methods derived from them: one using continued fractions and two bisection methods — the former being the fastest real root isolation method. Subsequently we concentrate on the Continued Fractions method and show how — using a recently developed quadratic complexity bound on the values of the positive roots of polynomials — its performance has been improved by an average of 40%, over its initial implementation.

**Key Words:** Vincent's theorem, isolation of the real roots, real root isolation methods, bisection methods, continued fractions method, positive root bounds.

## 1 Introduction

Isolation of the real roots of a polynomial is the process of finding real disjoint intervals such that each contains one real root and every real root is contained in some interval.

Since the beginning of the 19-th century, and according to the French “school” of mathematics, isolation has been considered the first step in finding the real roots of a polynomial equation — the second step being the approximation of the roots to any degree of accuracy.

Sturm was the first mathematician to present a theorem, in 1829, for isolating the real roots of a polynomial using bisection, [6]. His theorem has been widely used until 1980, when it was replaced — in the major computer algebra systems — by versions of Vincent's theorem.

Vincent's theorem of 1836, [43], has a very interesting and exciting history [10], [11], [19], [20], [21] and [30]. It was almost totally forgotten until it was rediscovered, in 1976, by the author and formed the basis of his Ph.D. Thesis, [1]. Subsequently scientists from all over the world made their own contributions

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<sup>1</sup> This paper is based on the *plenary talk* the author gave at ACA 2008, the International Conference on Applications of Computer Algebra, held at RISC-Linz, Hagenberg, Austria (July 27-30, 2008).

on various aspects of it, so that we can today claim that we have a very good understanding of it.

A short biography of Vincent (in French) can be found in p. 1026, vol 31 of “La Grande Encyclopédie”, see <http://gallica.bnf.fr/ark:/12148/bpt6k24666x>, whereas in [http://www.allposters.fr/-st/Lasnier-Affiches\\_c25893\\_s88165\\_.htm](http://www.allposters.fr/-st/Lasnier-Affiches_c25893_s88165_.htm) a copy of his portrait can be seen, [24].

Vincent’s theorem depends on Descartes’ rule of signs, which gives us an *upper bound* on the number of the positive roots of a polynomial, [23]. Specifically, consider the polynomial  $p(x) \in \mathbb{R}[x]$ ,  $p(x) = a_n x^n + \dots + a_1 x + a_0$  and let  $\text{var}(p)$  represent the number of sign *variations* or *changes* (positive to negative and vice-versa) in the sequence of coefficients  $a_n, a_{n-1}, \dots, a_0$ .

**Descartes’ rule of signs:** The number  $\varrho_+(p)$  of real roots — multiplicities counted — of the polynomial  $p(x) \in \mathbb{R}[x]$  in the open interval  $]0, +\infty[$  is bounded above by  $\text{var}(p)$ ; that is, we have  $\text{var}(p) \geq \varrho_+(p)$ .

According to Descartes’ rule of signs if  $\text{var}(p) = 0$  it follows that  $\varrho_+(p) = 0$ .

Additionally, according to Descartes’ rule of signs, the mean value theorem and the fact that the polynomial functions are continuous, if  $\text{var}(p) = 1$  it follows that  $\varrho_+(p) = 1$ , [23].

Therefore, Descartes’ rule of signs yields the *exact* number of positive roots *only* in the two special cases mentioned above<sup>2</sup>.

These two special cases of Descartes’ rule are used in both versions of Vincent’s theorem of 1836, which are described below.

The rest of the paper is structured as follows:

In Section 2 we present the two versions of Vincent’s theorem and provide a sketch of one of its proofs.

In Section 3 we explain how Vincent’s theorem can be used to isolate the real roots of polynomials and describe the continued fractions method and two bisections methods derived from it.

In Section 4 we present recently developed, by us, linear and quadratic complexity bounds on the values of the positive roots of polynomials and elaborate on their impact on the performance of the continued fractions method.

Finally there is the conclusion.

## 2 The two different versions of Vincent’s theorem

We begin with Vincent’s original version, which was published in the first issue (1836) of the French Journal on *Pure and Applied Mathematics*:

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<sup>2</sup> These two special cases were known to Cardano; in other words, what Descartes did was to generalize “Cardano’s *special* rule of signs”. This detail is mentioned in [6].

**Theorem 1.** (*Vincent’s theorem — “continued fractions” version, 1836*) *If in a polynomial,  $p(x)$ , of degree  $n$ , with rational coefficients and without multiple roots we perform sequentially replacements of the form*

$$x \leftarrow \alpha_1 + \frac{1}{x}, x \leftarrow \alpha_2 + \frac{1}{x}, x \leftarrow \alpha_3 + \frac{1}{x}, \dots$$

*where  $\alpha_1 \geq 0$  is an arbitrary non negative integer and  $\alpha_2, \alpha_3, \dots$  are arbitrary positive integers,  $\alpha_i > 0, i > 1$ , then the resulting polynomial either has no sign variations or it has one sign variation. In the last case the equation has exactly one positive root, which is represented by the continued fraction*

$$\alpha_1 + \frac{1}{\alpha_2 + \frac{1}{\alpha_3 + \frac{1}{\ddots}}}$$

*whereas in the first case there are no positive roots.*

The negative roots are treated in the same way — as suggested by Sturm — after we transform them to positive with the replacement  $x \leftarrow -x$  performed on  $p(x)$ . The requirement that  $p(x)$  have no multiple roots does not restrict the generality of the theorem because in the opposite case we first apply square-free factorization and then isolate the roots of each one of the square-free factors.

This theorem was kept “alive” by J. V. Uspensky in his book of 1948, [41], where the author rediscovered it. For a detailed discussion of the theorem, its extension, the geometrical interpretation of the transformations involved and three different proofs see [19], [20] and [21]; a fourth proof is presented by Ostrowski [32], who rediscovered a special case of a previously stated theorem by Obreschkoff, ([31], p. 81). We will sketch an outline of one of the proofs after we present the second version of the theorem.

In 2000, Alesina and Galuzzi published the so called *bisection version* of Vincent’s theorem, [19], [20] and [21]. This modern version is stated as follows:

**Theorem 2.** (*Vincent’s theorem — “bisection” version, 2000*) *Let  $p(x)$ , be a real polynomial of degree  $n$ , which has only simple roots. It is possible to determine a positive quantity  $\delta$  so that for every pair of positive real numbers  $a, b$  with  $|b - a| < \delta$ , every transformed polynomial of the form*

$$f(x) = (1 + x)^n p\left(\frac{a + bx}{1 + x}\right)$$

*has exactly 0 or 1 variations. The second case is possible if and only if  $p(x)$  has a simple root within  $]a, b[$ .*

**Sketch of a proof:** The proof by Alesina and Galuzzi is the most recent and — according to the author — the most elegant of all existing proofs of Vincent’s theorem.

To prove the theorem Alesina and Galuzzi show that after a series of transformations (bisections) mentioned in Theorem 1 (Theorem 2) a polynomial with one positive root will eventually have one sign variation. To show it, they use the following theorem by Obreschkoff, of 1920-23, that gives the necessary conditions under which a polynomial with one positive root presents exactly one sign variation in the sequence of its coefficients, [31]<sup>3</sup>.

**Theorem 3.** (*Obreschkoff's Cone or Sector theorem, 1920-23*) *If a real polynomial has one simple root  $x_0$ , and all the other (possibly multiple) roots lie in the sector:*

$$S_{\sqrt{3}} = \{ x = -\alpha + i\beta \mid \alpha > 0 \text{ and } \beta^2 \leq 3\alpha^2 \},$$

*then the sequence of its coefficients has exactly one sign variation.*

### 3 Polynomial Real Root Isolation with Vincent's Theorem

By cleverly utilizing the two special cases of Descartes' rule — the case of 0 or 1 sign variation — both versions of Vincent's theorem can be used to isolate the positive roots of a given polynomial  $p(x)$ . To see this, note that if we represent by the Möbius transformation  $M(x) = \frac{ax+b}{cx+d}$  the continued fraction that leads to a transformed polynomial

$$f(x) = (cx + d)^n p\left(\frac{ax + b}{cx + d}\right), \quad (1)$$

with one sign variation, then the single positive root of  $f(x)$  — in the interval  $]0, +\infty[$  — corresponds to *that* positive root of  $p(x)$  which is located in the open interval with endpoints  $\frac{b}{d}$  and  $\frac{a}{c}$ . These endpoints are *not* ordered and correspond to  $M(0)$  and  $M(\infty)$ , respectively<sup>4</sup>.

Therefore, to isolate the positive roots of a polynomial, all we have to do is compute — for *each* root — the variables  $a, b, c, d$  of the corresponding Möbius transformation  $M(x) = \frac{ax+b}{cx+d}$  that leads to a transformed polynomial  $f(x) = (cx + d)^n p\left(\frac{ax+b}{cx+d}\right)$ , with one sign variation.

**Crucial observation 1:** As we will see in the sequel, the variables  $a, b, c, d$  of a Möbius transformation  $M(x) = \frac{ax+b}{cx+d}$  (in Vincent's theorem) leading to a transformed polynomial with one sign variation can be computed:

<sup>3</sup> This is actually Obreschkoff's theorem for the special case where the number of sign variations equals one.

<sup>4</sup> As can be seen elsewhere, [7], the endpoints may also be computed from  $M(0)$  and  $M(1)$ , if we work in the interval  $]0, 1[$ ; in that case, Descartes' rule of signs does not apply and we use Uspensky's test instead.

- *either by continued fractions*, leading to the continued fractions method developed by Akritas and Strzeboński — which is called the Vincent-Akritas-Strzeboński (VAS) *continued fractions* method<sup>5</sup>,
- *or, by bisection*, leading to 2 *bisection* methods: the first one was developed by Collins and Akritas and is called the Vincent-Collins-Akritas (VCA) *bisection* method<sup>6</sup>, whereas the second one was developed by Alesina and Galuzzi and is called the Vincent-Alesina-Galuzzi *bisection* method<sup>7</sup>.

At this point the inquisitive reader might ask why not use numeric methods instead of the symbolic ones mentioned above. The answer is twofold:

- numeric methods *cannot* isolate just the positive roots; they isolate *all* roots, both real and complex, and
- numeric methods can give wrong answers as the following example demonstrates.

**Example 1:** Consider the polynomial

$$10^{999}(x - 1)^{50} - 1,$$

whose real roots we want to isolate (it has only two, both positive and  $\neq 1$ ). In this case,

- Numeric method using 1010 digits takes 56ms and fails — finds all 50 roots equal to 1.
- Numeric method using 1020 digits successfully isolates *all fifty* roots, but takes 18000ms.
- The VAS continued fractions method discussed below takes 4ms to isolate the two real roots.

Having made clear the need for the above mentioned symbolic real root isolation methods, we proceed to their description.

<sup>5</sup> To distinguish it from other continued fraction methods such as [12], [25], [39] cited in ([44], pp. 470–478). In [36] VAS is referred to as the “Akritas’ continued fractions method”. See also [19] and [22].

<sup>6</sup> Erroneously referred to in the literature either as “modified Uspensky’s method” (1976-1986), [5], or as “Descartes’ method” (1986-2006), [7].

<sup>7</sup> To distinguish them from Sturm’s bisection method [44].

### 3.1 The Continued Fractions Method Derived from Vincent's Theorem

The VAS continued fractions method is a *direct* implementation of Vincent's theorem. It was originally presented by Vincent in 1836 [43] in an “exponential” form; namely, Vincent computed each partial quotient  $a_i$  by a series of *unit* increments  $a_i \leftarrow a_i + 1$ , which are equivalent to substitutions of the form  $x \leftarrow x + 1$ .

In 1978 the method was converted into its “polynomial” form by Akritas, who in his Ph.D Thesis [1] computed each partial quotient  $a_i$  as the lower bound,  $lb$ , on the values of the positive roots of a polynomial — the so called “*ideal*” positive lower root bound, which computes the integer part of the smallest positive root [15]; that is, we now set  $a_i \leftarrow lb$  or, equivalently, we perform the substitution  $x \leftarrow x + lb$ , which takes about the same time as the substitution  $x \leftarrow x + 1$ . For details see also [2], [3], [4] and Chapter 7 in [6].

Finally, since the ideal positive lower root bound does not exist, Strzeboński [13] introduced the substitution  $x \leftarrow lb_{computed} \cdot x$ , whenever  $lb_{computed} > 16$  — where in general  $lb > lb_{computed}$  and the value 16 was determined experimentally.

In [13] it was also shown that the VAS continued fractions method is faster than the fastest implementation of the VCA bisection method [33], a result which was independently confirmed by Tsigaridas and Emiris [40]; see also [16]. In 2007 Sharma removed the hypothesis of the ideal positive lower bound and proved that VAS is still polynomial in time, [35], [36]!

In Algorithm 1 below we present a recursive description of the VAS continued fractions method. We follow [23], which pedagogically seems to be the most appropriate style of presentation:

#### The VAS Continued Fractions method

**Input:** A univariate, square-free polynomial  $p(x) \in \mathbb{Z}[x]$ ,  $p(0) \neq 0$ , and the Möbius transformation  $M(x) = \frac{ax+b}{cx+d} = x$ ,  $a, b, c, d \in \mathbb{Z}$

**Output:** A list of isolating intervals of the *positive* roots of  $p(x)$

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1 var  $\leftarrow$  the number of sign changes of  $p(x)$ ;
2 if var = 0 then RETURN  $\emptyset$ ;
3 if var = 1 then RETURN  $\{a, b\}$  // a = min(M(0), M( $\infty$ )), b = max(M(0), M( $\infty$ ));
4 lb  $\leftarrow$  a lower bound on the positive roots of  $p(x)$ ;
5 if lb > 1 then  $\{p \leftarrow p(x + lb), M \leftarrow M(x + lb)\}$ ;
6  $p_{01} \leftarrow (x + 1)^{\deg(p)} p(\frac{1}{x+1})$ ,  $M_{01} \leftarrow M(\frac{1}{x+1})$  // Look for real roots in ]0, 1[;
7 m  $\leftarrow$  M(1) // Is 1 a root?;
8  $p_{1\infty} \leftarrow p(x + 1)$ ,  $M_{1\infty} \leftarrow M(x + 1)$  // Look for real roots in ]1, + $\infty$ [;
9 if  $p(1) \neq 0$  then
10 | RETURN VAS( $p_{01}, M_{01}$ )  $\cup$  VAS( $p_{1\infty}, M_{1\infty}$ )
11 else
12 | RETURN VAS( $p_{01}, M_{01}$ )  $\cup$   $\{m, m\}$   $\cup$  VAS( $p_{1\infty}, M_{1\infty}$ )
13 end

```

**Algorithm 1:** The VAS( $p, M$ ) “*continued fractions*” algorithm, where the second argument is the Möbius transformation  $M(x)$  associated with  $p(x)$ . For simplicity, Strzeboński’s contribution is not included.

The VAS continued fractions method has been implemented in the computer algebra system *Mathematica*. Please note that it uses Descartes' rule of signs as termination test — lines 1-3 — and that it relies heavily on the *repeated* estimation of lower bounds on the values of the positive roots of polynomials — line 4. It should be emphasized that the efficiency of VAS depends on how good these estimates are — a fact that is exploited in the sequel (Section 4). Moreover, note the following:

- If we remove lines 4 and 5 from  $\text{VAS}(p, M)$  we are left with an exponential algorithm.
- Any substitution performed on the polynomial  $p(x)$  is also performed on its associated Möbius transformation  $M(x)$  — lines 5, 6 and 8.
- To isolate the real roots of  $p(x)$  in the open interval  $]0, +\infty[$  we proceed as follows:
  - we first isolate the real roots in the interval  $]0, 1[$  — lines 6 and 10 (or 12),
  - we then deal with the case where 1 is a root of  $p(x)$  — lines 7, 9 and 12,
  - and, finally, we isolate the real roots in the interval  $]1, +\infty[$  — lines 8 and 10 (or 12).
- The isolating intervals are computed from the Möbius transformations in line 3 — except for the integer roots which are computed in lines 7 and 12.

Using the plausible hypothesis of the “*ideal*” lower bound on the values of the positive roots along with the fast translation algorithm by von zur Gathen and Gerhard, [28], the computing time of the VAS continued fractions method is

$$O(n^4\tau^2),$$

where  $n$  is the degree of the polynomial and  $\tau$  bounds the coefficient bitsize, [1], [40].

However, the use of the plausible hypothesis mentioned above did *not* go unchallenged, and it was a widely held belief that if it were removed then the computing time of the VAS continued fractions method would be exponential! This resulted in people shying away from VAS ignoring that:

- in the case of random polynomials the VAS continued fractions method was several thousand times faster than (the fastest implementation, [33], of) the VCA bisection method (described below), [13], [40],

- in the case of Mignotte polynomials the **VAS** continued fractions method was about 50,000 times faster than (the fastest implementation, [33], of) the **VCA** bisection method, [13], [40],
- only in the case of very many ( $\geq 20$ ) very large ( $\simeq 10^{300}$ ) roots was the **VAS** continued fractions method 4 times slower than (the fastest implementation, [33], of) the **VCA** bisection method, [13], [40]; see Table 1 in Section 4.

And all that using Cauchy’s bound (the only one available then, but also one of the worst as we will see in Section 4) to compute the lower bounds on the values of the positive roots!

The situation changed in 2007 when Sharma proved that without any hypotheses the computing time of the **VAS** continued fractions method is

$$O(n^8 \tau^3),$$

where again  $n$  is the degree of the polynomial and  $\tau$  bounds the coefficient bitsize, [35], [36]. However, this indicates that there must still be something we do not understand about this algorithm, since the gap between the two computing time bounds is quite big, and the latter does *not* match the performance of **VAS** — the fastest real root isolation method.

### 3.2 The Two Bisection Methods Derived from Vincent’s Theorem

The two bisection methods derived from Vincent’s theorem differ mainly in: (a) the termination criterion they employ, and, (b) the interval they bisect.

The *first* bisection method, **VCA**, derived from Vincent’s theorem was developed in 1976 by Collins and Akritas, [26], in an attempt to improve the exponential behavior of Vincent’s original continued fractions algorithm. It uses Uspensky’s termination criterion (explained below) and bisects the interval  $]0, 1[$ .

Let  $p(x)$  be the polynomial whose roots we want to isolate and let  $ub$  be an upper bound on the values of its positive roots. Then all the positive roots of  $p(ub \cdot x)$  lie in the interval  $]0, 1[$  and the **VCA** method isolates them by repeatedly bisecting the interval  $]0, 1[$ , while using in the process an appropriate “*criterion*” to make inferences about the number of positive roots certain transformed polynomials have in the interval  $]0, 1[$ . Finally, the isolating intervals of the roots of  $p(x)$  are easily computed from the bijection:

$$\alpha_{]0, ub[} = a + \alpha_{]0, 1[}(b - a), \tag{2}$$

that exists between the roots  $\alpha_{]0, 1[} \in ]0, 1[$  of the *transformed* polynomial  $p(ub \cdot x)$  and the roots  $\alpha_{]0, ub[} \in ]a, b[ = ]0, ub[$  of the *original* polynomial  $p(x)$ .

The appropriate criterion mentioned above is a “*test*” that determines an *upper bound* on the number of positive roots in the interval  $]0, 1[$ .

Please observe that Descartes' rule of signs *cannot* be used in the interval  $]0, 1[$ , as it applies *only* to positive roots in the interval  $]0, +\infty[$ . Therefore, we have to resort to a different “*rule*” if we want to avoid reinventing Sturm's method for isolating the real roots; recall that Sturm's theorem gives us the *exact* number of positive roots in any interval  $]a, b[$ , [19].

Here is the test for determining an *upper bound* on the number of positive roots in the interval  $]0, 1[$ ; as explained below, it is named after Uspensky, who was the *first* to use it.

**Uspensky's test:** The number  $\varrho_{01}(p)$  of real roots in the open interval  $]0, 1[$  — multiplicities counted — of the polynomial  $p(x) \in \mathbb{R}[x]$  is bounded above by  $\text{var}_{01}(p)$ , where

$$\text{var}_{01}(p) = \text{var}\left((x+1)^{\deg(p)} p\left(\frac{1}{x+1}\right)\right), \quad (3)$$

and we have  $\text{var}_{01}(p) \geq \varrho_{01}(p)$ <sup>8</sup>.

As in the case of Descartes' rule of signs if  $\text{var}_{01}(p) = 0$  it follows that  $\varrho_{01}(p) = 0$  and if  $\text{var}_{01}(p) = 1$  it follows that  $\varrho_{01}(p) = 1$ .

Therefore, Uspensky's test yields the *exact* number of positive roots *only* in the two special cases mentioned above; to wit, whenever  $\text{var}_{01}(p) = 0$  or  $\text{var}_{01}(p) = 1$ .

Please note in equation (3) that, *after* the substitution  $x \leftarrow \frac{1}{x+1}$ , the positive roots of  $p(x)$  that were in the interval  $]0, 1[$  are now in  $]0, +\infty[$ , in which case Descartes' rule of signs *can* be applied.

Uspensky's test is associated with Budan's theorem [4] according to which for a given polynomial  $p(x) \in \mathbb{Z}[x]$  the following two special cases hold:

- if  $\text{var}(p(x)) = \text{var}(p(x+1))$ , then we can conclude that there are no positive real roots of  $p(x)$  in the interval  $]0, 1[$ ,

and

- if  $\text{var}(p(x)) - \text{var}(p(x+1)) = 1$ , then we can conclude that there is one positive real root of  $p(x)$  in the interval  $]0, 1[$ .

Vincent was fully aware of Budan's theorem and, consequently, the substitution  $x \leftarrow \frac{1}{x+1}$  is *never* used as a test in the VAS method — line 6 of Algorithm 1; it is performed *only* whenever  $\text{var}(p(x)) - \text{var}(p(x+1)) \geq 2$ , in which case the existence of positive roots in  $]0, 1[$  *has* to be investigated.

That, however, was not the case with Uspensky. Whenever he encountered  $\text{var}(p(x)) = \text{var}(p(x+1))$  — not being aware of Budan's theorem — he could

<sup>8</sup> Uspensky's test is a special instance of the powerful “Vincent's test”, which is described below.

not conclude that there are no positive roots of  $p(x)$  in the interval  $]0, 1[$ ; he would reach that conclusion *only* if  $\text{var}_{01}(p) = 0^9$ .

In other words, Uspensky ([41], p. 128) was the *first* to use  $\text{var}_{01}(p) = 0$  *exclusively* as a test, in order to verify that there are no positive roots in the interval  $]0, 1[$ ; hence, naming the test after him seems to be very appropriate. That test was used by Uspensky in his unsuccessful attempt to develop a new procedure for the isolation of the real roots of polynomials [5], [19]<sup>10</sup>.

Below is a recursive description of the VCA bisection method:

### The VCA bisection method — original version

**Input:** A univariate, square-free polynomial  $p(ub \cdot x) \in \mathbb{Z}[x]$ ,  $p(0) \neq 0$ , and the open interval  $]a, b[=]0, ub[$ , where  $ub$  is an upper bound on the values of the positive roots of  $p(x)$ . (The positive roots of  $p(ub \cdot x)$  are all in the open interval  $]0, 1[$ .)

**Output:** A list of isolating intervals of the *positive* roots of  $p(x)$

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1 var ← the number of sign changes of  $(x + 1)^{\deg(p)} p(\frac{1}{x+1})$ ;
2 if var = 0 then RETURN  $\emptyset$ ;
3 if var = 1 then RETURN  $\{]a, b[\}$ ;

4  $p_{0\frac{1}{2}} \leftarrow 2^{\deg(p)} p(\frac{x}{2})$  // Look for real roots in  $]0, \frac{1}{2}[$ ;
5  $m \leftarrow \frac{a+b}{2}$  // Is  $\frac{1}{2}$  a root? ;
6  $p_{\frac{1}{2}1} \leftarrow 2^{\deg(p)} p(\frac{x+1}{2})$  // Look for real roots in  $] \frac{1}{2}, 1[$ ;

7 if  $p(\frac{1}{2}) \neq 0$  then
8   | RETURN  $\text{VCA}(p_{0\frac{1}{2}}, ]a, m]) \cup \text{VCA}(p_{\frac{1}{2}1}, ]m, b])$ 
9 else
10  | RETURN  $\text{VCA}(p_{0\frac{1}{2}}, ]a, m]) \cup \{]m, m[\} \cup \text{VCA}(p_{\frac{1}{2}1}, ]m, b])$ 
11 end

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**Algorithm 2:** The *original* version of the  $\text{VCA}(p, ]a, b[)$  “bisection” algorithm, where the second argument is the open interval  $]a, b[$  associated with  $p(x)$ . The isolating intervals of the roots of  $p(x)$  are computed directly, without using bijection (2).

The VCA bisection method has been implemented in the computer algebra system *maple*. Please note that it uses Uspensky’s test as termination criterion — lines 1-3 — and that the upper bound on the values of the positive roots is computed *only* once. An excellent discussion of this algorithm can be found in [23]; another version of the same algorithm as well as additional information can

<sup>9</sup> On the other hand, Uspensky used correctly and to his advantage the other special case,  $\text{var}_{01}(p) = 1$ , as well as the case  $\text{var}_{01}(p) \geq 2$ .

<sup>10</sup> According to Professor Alexei Uteshev [42], of St. Petersburg’s State University, the reason for Uspensky’s unsuccessful attempt was the fact that he *never* saw Vincent’s actual paper of 1836 — where Budan’s theorem is stated right at the beginning. Instead, Uspensky relied on the Russian translation of J.-A. Serret’s *Cours d’Algèbre Supérieure* [34]. Indeed, in Section 167, p.315 of Серре И.А. Курс высшей алгебры. М.- СПб., Вольф, Б.г., 573 с. we read:

“В одном из мемуаров, составляющих часть первого тома Journal de Mathématiques pures et appliquées, Венсен изложил прекрасное свойство непрерывных дробей и вывел из него для вычисления вещественных корней уравнения способ, вытекающий одновременно и из способа Ньютона, и из способа Лагранжа...”

Please note that Serret presents *Fourier’s* theorem under the name “Budan”.

be found elsewhere, [7].

Moreover, note the following:

- The substitutions in lines 4 and 6 are performed only on the polynomial  $p(x)$ , whereas at the same time — in line 5 — the interval  $]a, b[$  is divided into two equal parts  $]a, m[$  and  $]m, b[$ , to be used in line 8 (or 10).
- To isolate the real roots of  $p(x)$  in the open interval  $]0, 1[$  we proceed as follows:
  - we first isolate the real roots in the interval  $]0, \frac{1}{2}[$  — lines 4 and 8 (or 10),
  - we then deal with the case where  $\frac{1}{2}$  is a root of  $p(x)$  — lines 5, 7 and 10,
  - and, finally, we isolate the real roots in the interval  $]\frac{1}{2}, 1[$  — lines 6 and 8 (or 10).
- The isolating intervals are directly obtained from line 3 — except for those roots that happen to coincide with the midpoint of an interval that gets bisected, in which case they are computed in lines 5 and 10.

The computing time of the VCA bisection method is

$$O(n^4\tau^2),$$

where  $n$  is the degree of the polynomial and  $\tau$  bounds the coefficient bitsize, [36].

The *second* bisection method was developed by Alesina and Galuzzi in 2000, [19], [21]. It uses Vincent's powerful test as the termination criterion and bisects the interval  $]a, b[=]0, ub[$ , where  $ub$  is an upper bound on the values of the positive roots. Therefore, this method is a direct implementation of Theorem 2.

**Vincent's test:** If  $a \geq 0$  and  $b > a$  then the number  $\varrho_{ab}(p)$  of real roots in the open interval  $]a, b[$ , — multiplicities counted — of the polynomial  $p(x) \in \mathbb{R}[x]$  is bounded above by  $var_{ab}(p)$ , where

$$var_{ab}(p) = var((1+x)^{deg(p)}p(\frac{a+bx}{1+x})), \quad (4)$$

and we have  $var_{ab}(p) = var_{ba}(p) \geq \varrho_{ab}(p)$ .

Note that this test can be applied also in the case  $]a, b[=]1, 0[$ , from which we obtain Uspensky's test<sup>11</sup>.

<sup>11</sup> By comparison, Uspensky's test is rather weak as it applies *only* in the interval  $]a, b[=]0, 1[$ .

As in the case of Descartes' rule of signs if  $var_{ab}(p) = 0$  it follows that  $\varrho_{ab}(p) = 0$  and if  $var_{ab}(p) = 1$  it follows that  $\varrho_{ab}(p) = 1$ .

Therefore, Vincent's test yields the *exact* number of positive roots *only* in the two special cases mentioned above; to wit, whenever  $var_{ab}(p) = 0$  or  $var_{ab}(p) = 1$ .

Below is a recursive description of the second bisection method derived from Vincent's theorem; its simplicity is unsurpassed, but we pay for it by using a much more complicated test. Obviously, there is a trade off between simplicity of the method and complexity of the termination test.

### The Vincent-Alesina-Galuzzi bisection method: B

**Input:** A univariate, square-free polynomial  $p(x) \in \mathbb{Z}[x]$ ,  $p(0) \neq 0$ , and the open interval  $]a, b[ = ]0, ub[$ , where  $ub$  is an upper bound on the values of the positive roots of  $p(x)$ .

**Output:** A list of isolating intervals of the *positive* roots of  $p(x)$

```

1 var ← the number of sign changes of  $(1+x)^{deg(p)} p(\frac{a+bx}{1+x})$ ;
2 if var = 0 then RETURN  $\emptyset$ ;
3 if var = 1 then RETURN  $\{]a, b[\}$ ;
4 m ←  $\frac{a+b}{2}$  // Subdivide the interval  $]a, b[$  in two equal parts ;
5 if  $p(m) \neq 0$  then
6   | RETURN  $\mathbb{B}(p, ]a, m[) \cup \mathbb{B}(p, ]m, b[)$ 
7 else
8   | RETURN  $\mathbb{B}(p, ]a, m[) \cup \{m\} \cup \mathbb{B}(p, ]m, b[)$ 
9 end
```

**Algorithm 3:** The  $\mathbb{B}(p, ]a, b[)$  “*bisection*” algorithm, proposed by Alesina and Galuzzi [21]; the second argument is the open interval  $]a, b[$ , whose endpoints  $a, b$  are used in Vincent's test in line 1. The isolating intervals of the roots of  $p(x)$  are computed directly, without using bisection (2).

Please note the following:

- Vincent's test is a *crucial component* of the  $\mathbb{B}(p, ]a, b[)$  bisection algorithm — lines 1-3.
- In line 4 the interval  $]a, b[$  is divided into two equal parts  $]a, m[$  and  $]m, b[$ , to be used in lines 6 (or 8). Note that there are no polynomial transformations at all; only polynomial evaluations in line 1.
- To isolate the real roots of  $p(x)$  in the open interval  $]a, b[$  we proceed as follows:
  - we first isolate the real roots in the interval  $]a, \frac{a+b}{2}[$  — lines 6 (or 8),
  - we then deal with the case where  $\frac{a+b}{2}$  is a root of  $p(x)$  — lines 5, and 8,
  - and, finally, we isolate the real roots in the interval  $] \frac{a+b}{2}, b[$  — lines 6 (or 8).

- The isolating intervals are directly obtained from line 3 — except for those roots that happen to coincide with the midpoint of an interval that gets bisected, in which case they are computed in lines 5 and 8.

A complete discussion and empirical comparison between the two bisection methods can be found elsewhere, [17]. It turns out that the Vincent-Alesina-Galuzzi bisection method, despite its simplicity, is much slower than the VCA bisection method and hence of little practical importance.

#### 4 Improving the Performance of the VAS Continued Fractions Method

As was pointed out in the discussion of VAS, the efficiency of this continued fractions method depends heavily on how good are the estimates of the lower bounds on the values of the positive roots.

A *lower* bound,  $lb$ , on the values of the positive roots of a polynomial  $p(x)$ , of degree  $n$ , is found by first computing an *upper* bound,  $ub$ , on the values of the positive roots of  $x^n p(\frac{1}{x})$  and then setting  $lb = \frac{1}{ub}$ .

So, clearly, what is needed is an efficient method for computing upper bounds on the values of (just) the positive roots of polynomial equations.

In the initial implementation of VAS, in 1978, the lower bounds were computed using a theorem by Cauchy, [31]. To state it, we refer to polynomials of the type

$$p(x) = \alpha_n x^n + \alpha_{n-1} x^{n-1} + \dots + \alpha_0, \quad (\alpha_n > 0) \quad (5)$$

with real coefficients  $\alpha_n, \alpha_{n-1}, \dots, \alpha_0$  and having at least one sign variation.

**Theorem 4.** (*Cauchy's theorem*) Let  $p(x)$  be a polynomial as in Eq. (5), of degree  $n > 0$ , with  $\alpha_{n-k} < 0$  for at least one  $k$ ,  $1 \leq k \leq n$ . If  $\lambda$  is the number of negative coefficients, then an upper bound on the values of the positive roots of  $p(x)$  is given by

$$ub_C = \max_{\{1 \leq k \leq n: \alpha_{n-k} < 0\}} \sqrt[k]{-\frac{\lambda \alpha_{n-k}}{\alpha_n}}$$

Note that if  $\lambda = 0$  there are no positive roots.

Subsequently, Kioustelidis' bound appeared, [29], and was used in the SYNAPS implementation of VAS by Tsigaridas and Emiris in 2006, [40]. Kioustelidis' theorem is closely related to the one by Cauchy and is stated below:

**Theorem 5.** (*Kioustelidis' theorem*) Let  $p(x)$  be a polynomial as in Eq. (5), of degree  $n > 0$ , with  $\alpha_{n-k} < 0$  for at least one  $k$ ,  $1 \leq k \leq n$ . Then an upper bound on the values of the positive roots of  $p(x)$  is given by

$$ub_K = 2 \max_{\{1 \leq k \leq n: \alpha_{n-k} < 0\}} \sqrt[k]{-\frac{\alpha_{n-k}}{\alpha_n}}.$$

However, both implementations of the VAS continued fractions method — that is, using either Cauchy’s or Kioustelidis’ bound — showed that its “Achilles’ heel” was the case of very many very large rational roots. In this case the VAS method was up to 4 times slower than VCA (rel) — the fastest implementation of the VCA bisection method developed by Rouillier and Zimmermann, [33]. Table 1 presented below, corresponds to the last table (Table 4), found in [13].

Table 1: Products of factors (x-randomly generated integer root). All computations were done on a 850 MHz Athlon PC with 256 MB RAM; (s) stands for time in seconds and (MB) for the amount of memory used, in MBytes.

Roots (bit length)	Degree	No. of roots	VAS t(s)/M(MB)	VCA (rel) t(s)/M(MB)
10	100	100	0.8/1.82	0.61/1.92
10	200	200	2.45/2.07	10.1/2.64
10	500	500	33.9/3.34	878/8.4
1000	20	20	0.12/1.88	0.044/1.83
1000	50	50	16.7/3.18	4.27/2.86
1000	100	100	550/8.9	133/6.49

The last three lines of Table 1 demonstrate the weaker performance of VAS in the case of very many very large rational roots.

Therefore, the question was posed: can we improve the performance of VAS by discovering new bounds on the values of the positive roots?

In order to answer this question we needed to better understand the nature of these bounds, and this was achieved with the help of Ștefănescu’s theorem of 2005, [37].

**Theorem 6.** (*Ștefănescu’s theorem, 2005*) *Let  $p(x) \in R[x]$  be such that the number of variations of signs of its coefficients is even. If*

$$p(x) = c_1x^{d_1} - b_1x^{m_1} + c_2x^{d_2} - b_2x^{m_2} + \dots + c_kx^{d_k} - b_kx^{m_k} + g(x),$$

*with  $g(x) \in R_+[x], c_i > 0, b_i > 0, d_i > m_i > d_{i+1}$  for all  $i$ , the number*

$$ub_S = \max \left\{ \left( \frac{b_1}{c_1} \right)^{1/(d_1-m_1)}, \dots, \left( \frac{b_k}{c_k} \right)^{1/(d_k-m_k)} \right\}$$

is an upper bound for the positive roots of the polynomial  $p$  for any **choice** of  $c_1, \dots, c_k$ .

Ștefănescu's theorem introduces the concept of *matching* or *pairing* a positive coefficient with an unmatched negative coefficient of a lower order term; however, Ștefănescu's theorem worked *only* for polynomials with an even number of sign variations.

**Note:** More precisely, it is the *term* with the positive coefficient that is being matched to the *term* with the negative coefficient.

We generalized Ștefănescu's theorem in the sense that Theorem 7 below applies to polynomials with any number of sign variations, [14]. To accomplish this, we introduced the concept of *breaking up* a positive coefficient into several parts to be paired with negative coefficients of lower order terms<sup>12</sup>, [15].

**Theorem 7.** (*Akritas-Strzeboński-Vigklas, 2006*) Let  $p(x)$

$$p(x) = \alpha_n x^n + \alpha_{n-1} x^{n-1} + \dots + \alpha_0, \quad (\alpha_n > 0)$$

be a polynomial with real coefficients and let  $d(p)$  and  $t(p)$  denote the degree and the number of its terms, respectively.

Moreover, assume that  $p(x)$  can be written as

$$p(x) = q_1(x) - q_2(x) + q_3(x) - q_4(x) + \dots + q_{2m-1}(x) - q_{2m}(x) + g(x), \quad (6)$$

where all the polynomials  $q_i(x)$ ,  $i = 1, 2, \dots, 2m$  and  $g(x)$  have only positive coefficients. In addition, assume that for  $i = 1, 2, \dots, m$  we have

$$q_{2i-1}(x) = c_{2i-1,1} x^{e_{2i-1,1}} + \dots + c_{2i-1,t(q_{2i-1})} x^{e_{2i-1,t(q_{2i-1})}}$$

and

$$q_{2i}(x) = b_{2i,1} x^{e_{2i,1}} + \dots + b_{2i,t(q_{2i})} x^{e_{2i,t(q_{2i})}},$$

where  $e_{2i-1,1} = d(q_{2i-1})$  and  $e_{2i,1} = d(q_{2i})$  and the exponent of each term in  $q_{2i-1}(x)$  is greater than the exponent of each term in  $q_{2i}(x)$ . If for all indices  $i = 1, 2, \dots, m$ , we have

$$t(q_{2i-1}) \geq t(q_{2i}),$$

then an upper bound of the values of the positive roots of  $p(x)$  is given by

$$ub = \max_{\{i=1,2,\dots,m\}} \left\{ \left( \frac{b_{2i,1}}{c_{2i-1,1}} \right)^{\frac{1}{e_{2i-1,1} - e_{2i,1}}}, \dots, \left( \frac{b_{2i,t(q_{2i})}}{c_{2i-1,t(q_{2i})}} \right)^{\frac{1}{e_{2i-1,t(q_{2i})} - e_{2i,t(q_{2i})}}} \right\}, \quad (7)$$

<sup>12</sup> After our work, [15], Ștefănescu also extended his Theorem 6, [38].

for any permutation of the positive coefficients  $c_{2i-1,j}$ ,  $j = 1, 2, \dots, t(q_{2i-1})$ . Otherwise, for each of the indices  $i$  for which we have

$$t(q_{2i-1}) < t(q_{2i}),$$

we **break up** one of the coefficients of  $q_{2i-1}(x)$  into  $t(q_{2i}) - t(q_{2i-1}) + 1$  parts, so that now  $t(q_{2i}) = t(q_{2i-1})$  and apply the same formula (7) given above.

For a proof of this theorem see [15]. Please note that the partial extension of Theorem 6 presented in [14] does not treat the case  $t(q_{2i-1}) < t(q_{2i})$ .

**Crucial Observation 2.** Pairing up positive with negative coefficients and breaking up a positive coefficient into the required number of parts — to match the corresponding number of negative coefficients — are the key ideas of this theorem. In general, formulae analogous to (7) hold for the cases where: (a) we pair coefficients from the non-adjacent polynomials  $q_{2l-1}(x)$  and  $q_{2i}(x)$ , for  $1 \leq l < i$ , and (b) we break up one or more positive coefficients into several parts to be paired with the negative coefficients of lower order terms.

Using Theorem 7 we obtain the following interpretation of Cauchy’s and Kioustelidis’ theorems:

- C.** Cauchy’s “leading-coefficient” implementation of Theorem 7. For a polynomial  $p(x)$ , as in Eq. (5), with  $\lambda$  negative coefficients, Cauchy method first breaks up its leading coefficient,  $\alpha_n$ , into  $\lambda$  *equal* parts and then pairs each part with the first unmatched negative coefficient.
- K.** Kioustelidis’ “leading-coefficient” implementation of Theorem 7. For a polynomial  $p(x)$ , as in Eq. (5), Kioustelidis method matches the coefficient  $-\alpha_{n-k}$  of the term  $-\alpha_{n-k}x^{n-k}$  in  $p(x)$  with  $\frac{\alpha_n}{2^k}$ , the leading coefficient divided by  $2^k$ .

Kioustelidis’ “leading-coefficient” implementation of Theorem 7, differs from that of Cauchy’s only in that the leading coefficient is now broken up in *unequal* parts, by dividing it with different powers of 2.

It turns out that all methods for computing upper bounds on the values of the positive roots of a polynomial, are derived from Theorem 7. In the sequel we present linear and quadratic complexity bounds derived from Theorem 7 and elaborate on their impact on the efficiency of VAS.

#### 4.1 Improving VAS with Linear Complexity Bounds Derived from Theorem 7

The bounds in the literature, such as Cauchy’s and Kioustelidis’, are of *linear* complexity.

**The General Idea of the Linear Complexity Bounds:** These bounds are computed as follows:

- *each* negative coefficient of the polynomial is paired with *one* of the preceding *unmatched* positive coefficients;
- the maximum of all the computed radicals is taken as the estimate of the bound.

Using Theorem 7 we developed *first- $\lambda$* , a new linear complexity method for computing an upper bound on the values of the positive roots of polynomials.

**FL.** “*first- $\lambda$* ” implementation of Theorem 7. For a polynomial  $p(x)$ , as in (6), with  $\lambda$  negative coefficients we first take care of all cases for which  $t(q_{2i}) > t(q_{2i-1})$ , by breaking up the last coefficient  $c_{2i-1, t(q_{2i})}$ , of  $q_{2i-1}(x)$ , into  $t(q_{2i}) - t(q_{2i-1}) + 1$  *equal* parts. We then pair each of the first  $\lambda$  positive coefficients of  $p(x)$ , encountered as we move in non-increasing order of exponents, with the first unmatched negative coefficient.

This is an improvement over the other two bounds by Cauchy and Kioustelidis, but as the following Example demonstrates, all three methods can fail miserably.

**Example 2:** Consider the polynomial

$$x^3 + 10^{100}x^2 - 10^{100}x - 1,$$

which has one sign variation and, hence, only one positive root = 1.

- For Cauchy’s theorem we pair the terms  $\{\frac{x^3}{2}, -10^{100}x\}$  and  $\{\frac{x^3}{2}, -1\}$ , and obtain a bound estimate of  $1.41421 * 10^{50}$ .
- For Kioustelidis’ theorem we pair the terms  $\{\frac{x^3}{2^2}, -10^{100}x\}$  and  $\{\frac{x^3}{2^3}, -1\}$ , and obtain a bound estimate of  $2 * 10^{50}$ .
- For *first- $\lambda$*  we pair the terms  $\{x^3, -10^{100}x\}$  and  $\{10^{100}x^2, -1\}$ , and obtain a bound estimate of  $10^{50}$ .

To correct this inadequacy, we developed *local-max*, yet another new linear complexity method for computing an upper bound on the values of the positive roots of polynomials, .

**LM.** “*local-max*” implementation of Theorem 7. For a polynomial  $p(x)$ , as in (5), the coefficient  $-\alpha_k$  of the term  $-\alpha_k x^k$  in  $p(x)$  — as given in Eq. (5)

— is paired with the coefficient  $\frac{\alpha_m}{2^t}$ , of the term  $\alpha_m x^m$ , where  $\alpha_m$  is the largest positive coefficient with  $n \geq m > k$  and  $t$  indicates the number of times the coefficient  $\alpha_m$  has been used.

**Example 2, continued:** For *local-max* we pair the terms  $\{\frac{10^{100}x^2}{2}, -10^{100}x\}$  and  $\{\frac{10^{100}x^2}{2^2}, -1\}$ , and obtain a bound estimate of 2.

We have tested extensively — on various classes of specific and random polynomials — all four linear complexity bounds mentioned above and the following is a summary of our findings, [15]:

- Kioustelidis’ bound is, in general, better (or much better) than Cauchy’s; this happens because the former breaks up the leading coefficient in *unequal* parts, whereas the latter breaks it up in *equal* parts.
- Our *first- $\lambda$*  bound, as the name indicates, uses additional coefficients and, therefore, it is not surprising that it is, in general, better (or much better) than both previous bounds. In the few cases where Kioustelidis’ bound is better than *first- $\lambda$* , our *local-max* bound takes again the lead.

Therefore, given their linear cost of execution,  $\min(FL, LM)$  or  $FL + LM$  is the best among the linear complexity bounds on values of the positive roots of a polynomial, [15].

In Table 2 below we recalculate the results of Table 1, and compare the timings in seconds,  $t(s)$ , for: (a) **VAS(cauchy)**, the VAS continued fractions method using Cauchy’s rule (the “*old*” method), (b) **VAS(f1+1m)**, the VAS continued fractions method using  $\min(FL, LM)$  or  $FL + LM$  (the “*new*” method), and (c) **VCA(rel)**, the fastest implementation of the VCA bisection method. (Table 2 corresponds to the last table (Table 2), found in [16].)

Due to the different computational environment the times  $t(s)$  differ substantially, but they confirm the fact that **VAS(f1+1m)** is now *always* faster than **VCA(rel)**.

Table 2: Products of terms  $x - r$  with random integer  $r$ . The tests were run on a laptop computer with 1.8 Ghz Pentium M processor, running a Linux virtual machine with 1.78 GB of RAM.

Roots (bit length)	Deg	VAS(cauchy) t(s)	VAS(f1+1m) t(s)	VCA(rel) t(s)	Memory (MB) old/new/rel
		Average (Min/Max)	Average (Min/Max)	Average (Min/Max)	
10	100	0.314 (0.248/0.392)	0.253 (0.228/0.280)	0.346 (0.308/0.384)	4.46/4.48/4.56
10	200	1.74 (1.42/2.33)	1.51 (1.34/1.66)	3.90 (3.72/4.05)	4.73/4.77/5.35
10	500	17.6 (16.9/18/7)	17.4 (16.3/18.1)	129 (122/140)	6.28/6.54/11.8
1000	20	0.066 (0.040/0.084)	0.031 (0.024/0.040)	0.038 (0.028/0.044)	4.57/4.62/4.51
1000	50	1.96 (1.45/2.44)	0.633 (0.512/0.840)	1.03 (0.916/1.27)	5.87/6.50/5.55
1000	100	52.3 (36.7/81.3)	12.7 (11.3/14.6)	17.2 (16.1/18.7)	10.4/11.7/9.17

Again, of interest are the last three lines of Table 2, where as in Table 1 the performance of `VAS(cauchy)` is worse than `VCA(re1)` — at worst 3 times slower, as the last entry indicates. However, from these same lines of Table 2 we observe that `VAS(f1+1m)` is now always faster than `VCA(re1)` — at best twice as fast, as seen in the 5-th line.

When we compare — on various classes of specific and random polynomials — the times of `VAS(cauchy)` with those of `VAS(f1+1m)` we observe an overall speed-up of 15%, [18].

## 4.2 Improving VAS with Quadratic Complexity Bounds Derived from Theorem 7

To further improve the the performance of the `VAS` continued fractions method we decided to use quadratic complexity bounds on the values of the positive real roots hoping that their improved estimates *should* compensate for the extra time needed to compute them.

**The General Idea of the Quadratic Complexity Bounds:** These bounds are computed as follows:

- *each* negative coefficient of the polynomial is paired with *all* the preceding positive coefficients and the minimum of the computed values is taken;
- the maximum of all those minimums is taken as the estimate of the bound.

In general, the estimates obtained from the quadratic complexity bounds are better than those obtained from the corresponding linear complexity bounds, as the former are computed after much greater effort and time<sup>13</sup>.

We developed several new quadratic complexity bounds by extending three (of the four) linear complexity ones, [8]; by contrast, Kioustelidis’ linear complexity bound was extended by Hong in 1998, [27]. We were able to demonstrate that one of them, *FLQ*, the “first- $\lambda$ ” quadratic complexity bound is not only one of the best to be used in `VAS`, but also the fastest, [9]. However, we decided that the following quadratic complexity bound should be used in `VAS` instead, [18]:

**LMQ. “Local-Max” Quadratic** complexity implementation of Theorem 7.

For a polynomial  $p(x)$ , as in (5), each negative coefficient  $a_i < 0$  is “paired” with *each* one of the preceding positive coefficients  $a_j$  divided by  $2^{t_j}$  — that is, *each* positive coefficient  $a_j$  is “broken up” into *unequal* parts, as is done with *just* the locally maximum coefficient in the local max bound;  $t_j$

<sup>13</sup> It should be noted that time is not so importance in our case, since — as can be seen in the description of the `VAS` algorithm, line 4 — these bounds are estimated *before* a translation of complexity at *least*  $O(n^2)$  is executed, [28].

is initially set to 1 and is incremented each time the positive coefficient  $a_j$  is used — and the minimum is taken over all  $j$ ; subsequently, the maximum is taken over all  $i$ .

That is, we have:

$$ub_{LMQ} = \max_{\{a_i < 0\}} \min_{\{a_j > 0: j > i\}} \sqrt[j-i]{-\frac{a_i}{\frac{a_j}{2^{t_j}}}}$$

Today, this is theoretically the sharpest bound on the values of the positive roots of a polynomial.

We finally present Table 3 — corresponding to Table 8 in [18] — where we demonstrate the performance of VAS using quadratic complexity bounds. This is actually the only case where the best linear complexity bound  $FL + LM$  is slightly better than  $LMQ$ .

Table 3: Products of terms  $x - r$  with random integer  $r$ . The average speed-up for this table is about 35%.

Bit-length of roots	Degree	VAS(cauchy) t(s) Avg(Min/Max)	VAS(f1+lm) t(s) Avg(Min/Max)	VAS(lmq) t(s) Avg(Min/Max)
10	100	0.46 (0.28/0.94)	0.24 (0.18/0.28)	0.34 (0.30/0.41)
10	200	1.46 (1.24/1.85)	1.40 (1.28/1.69)	1.40 (1.20/1.69)
10	500	18.1 (16.5/18.9)	18.1 (16.6/18.8)	22.1 (18.7/24.2)
1000	20	0.07 (0.04/0.14)	0.02 (0.02/0.03)	0.03 (0.02/0.04)
1000	50	3.69 (2.38/6.26)	0.81 (0.60/1.28)	0.81 (0.52/1.11)
1000	100	47.8 (37.6/56.9)	13.8 (10.3/19.2)	15.8 (11.3/21.3)

## 5 Conclusion

We have presented Vincent’s theorem of 1836, along with the various methods derived from it for isolating the real roots of polynomials with rational coefficients. It was shown that the VAS continued fractions method has always been the fastest, and recently — with the development of the quadratic complexity bounds — its performance has increased by an overall factor of 40%.

An open problem for future research is to better understand VAS and to explain the gap that exists in its computing times — in the cases when we use a plausible hypothesis and when not.

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