Vincent’s theorem of 1836:
Overview and Future Research

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Outline of the talk

- Vincent’s theorem of 1836 and real root isolation
- The two bisection methods derived from Vincent’s theorem
- The continued fractions method derived from Vincent’s theorem

We will present Vincent’s theorem and review the various real root isolation methods derived from it (2 bisection methods and 1 Continued Fractions method).

We will concentrate on the Continued Fractions method (the fastest of them all) and show how it was recently speeded up by 40% over its initial implementation.

We will indicate new directions for future research.
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- We will indicate new directions for future research.
Descartes’ rule of signs — for the open interval \( ]0, \infty[ \)
Consider the polynomial

\[ p(x) = a_n x^n + \cdots + a_1 x + a_0, \]

where \( p(x) \in \mathbb{R}[x] \) and let \( \text{var}(p) \) represent the number of sign changes or variations (positive to negative and vice-versa) in the sequence of coefficients \( a_n, a_{n-1}, \ldots, a_0 \).
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**Theorem**

The number \( \varrho_+(p) \) of real roots — multiplicities counted — of the polynomial \( p(x) \in \mathbb{R}[x] \) in the open interval \( ]0, \infty[ \) is bounded above by \( \text{var}(p) \); that is, we have \( \text{var}(p) \geq \varrho_+(p) \).
Special Cases of Descartes’ rule of signs
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- 0 sign variations $\iff$ 0 positive roots
Special Cases of Descartes’ rule of signs

- 0 sign variations $\iff$ 0 positive roots
- 1 sign variation $\implies$ 1 positive root
Table of contents

1. Vincent’s theorem of 1836 and real root isolation
   - The Continued Fractions Version
   - The Bisection version by Alesina and Galuzzi, 2000
   - Proof of Vincent’s theorem
   - Real root isolation

2. The two bisection methods derived from Vincent’s theorem

3. The continued fractions method derived from Vincent’s theorem
Vincent’s theorem of 1836 — the Continued Fractions version

If in a polynomial, \( p(x) \), of degree \( n \), with rational coefficients and simple roots we perform sequentially replacements of the form

\[
x \leftarrow \alpha_1 + \frac{1}{x}, \quad x \leftarrow \alpha_2 + \frac{1}{x}, \quad x \leftarrow \alpha_3 + \frac{1}{x}, \ldots
\]

where \( \alpha_1 \geq 0 \) is an arbitrary non negative integer and \( \alpha_2, \alpha_3, \ldots \) are arbitrary positive integers, \( \alpha_i > 0, \ i > 1 \), then the resulting polynomial either has no sign variations or it has one sign variation. In the first case there are no positive roots whereas in the last case the equation has exactly one positive root, represented by the continued fraction

\[
\alpha_1 + \frac{1}{\alpha_2 + \frac{1}{\alpha_3 + \frac{1}{\cdots}}}
\]
<table>
<thead>
<tr>
<th>Vincent’s theorem of 1836 and real root isolation</th>
</tr>
</thead>
<tbody>
<tr>
<td>The two bisection methods derived from Vincent’s theorem</td>
</tr>
<tr>
<td>The continued fractions method derived from Vincent’s theorem</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>The Continued Fractions Version</th>
</tr>
</thead>
<tbody>
<tr>
<td>The Bisection version by Alesina and Galuzzi, 2000</td>
</tr>
<tr>
<td>Proof of Vincent’s theorem</td>
</tr>
<tr>
<td>Real root isolation</td>
</tr>
</tbody>
</table>

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- was rediscovered by Akritas in 1976 and formed the subject of his Ph.D. Thesis (1978).
Vincent’s theorem of 1836 and real root isolation
The two bisection methods derived from Vincent’s theorem
The continued fractions method derived from Vincent’s theorem

Historical note on Uspensky (1883-1947)

Uspensky never read Vincent’s original paper of 1836, which included several examples (applications). Instead, he read only a proof of Vincent’s theorem in the Russian translation of Serrets (French) book on Algebra (1866).
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- he read only a proof of Vincent’s theorem in the Russian translation of Serrets (French) book on Algebra (1866).
Let $f(z)$ be a real polynomial of degree $n$, which has only simple roots. It is possible to determine a positive quantity $\delta$ so that for every pair of positive real numbers $a$, $b$ with $|b - a| < \delta$, every transformed polynomial of the form

$$\phi(z) = (1 + z)^n f\left(\frac{a + bz}{1 + z}\right)$$

has exactly 0 or 1 variations. The second case is possible if and only if $f(z)$ has a simple root within $]a, b[$.
Proof of Vincent’s theorem

The proof by Alesina and Galuzzi (2000) is the most recent one; it uses Obreschkoff’s theorem of 1920-23 which gives the necessary condition for a polynomial with one positive root to have one sign variation!

A similar proof was presented earlier by Ostrowski (1950), who rediscovered Obreschkoff’s theorem 30 years later.
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If a real polynomial has one positive simple root $x_0$ and all the other — possibly multiple — roots lie in the sector

$$S_{\sqrt{3}} = \{x = -\alpha + i\beta \mid \alpha > 0 \text{ and } \beta^2 \leq 3\alpha^2 \}$$

then the sequence of its coefficients has exactly one sign variation.
View of Obreschkoff’s Cone and Circles

Im(y) = -s Re(y)

Im(y) = s Re(y)

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Real root isolation
Real root isolation using Vincent’s theorem

To isolate the positive roots of a polynomial $p(x)$, all we have to do is compute — for each root — the variables $a, b, c, d$ of the corresponding Möbius transformation

$$M(x) = \frac{ax + b}{cx + d}$$

that leads to a transformed polynomial

$$f(x) = (cx + d)^n p\left(\frac{ax + b}{cx + d}\right)$$

with one sign variation.
Two different ways to isolate the real roots:
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Crucial observation:

The variables $a, b, c, d$ of a Möbius transformation $M(x) = \frac{ax+b}{cx+d}$ (in Vincent’s theorem) leading to a transformed polynomial with one sign variation can be computed:
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- either by continued fractions leading to the continued fractions method developed by Vincent, Akritas and Strzeboński, which in the sequel will be called the **VAS continued fractions** method — to distinguish it from other continued fraction methods,
Two different ways to isolate the real roots:

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- either **by continued fractions** leading to the continued fractions method developed by Vincent, Akritas and Strzeboński, which in the sequel will be called the **VAS continued fractions** method — to distinguish it from other continued fraction methods,
- or, **by bisections** leading to (among others) the bisection method developed by (Vincent), Collins and Akritas, which in the sequel will be called the **VCA bisection** method — to distinguish it from Sturm’s bisection method.
Historical note on the two methods

The VCA-bisection method was developed first — in 1976 by Collins and Akritas. Its fastest implementation was developed in 2004 by Rouillier and Zimmermann.

The VAS-continued fractions method was developed later — in 1978 by Akritas and in 1993 by Akritas, Botcharov and Strzeboński. Its fastest implementation was developed in 2008 by Akritas, Strzeboński and Vigklas.
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Why bother with these methods and not use numeric ones?

Numeric methods cannot isolate just the positive roots! They isolate all the roots (real and complex). They can give wrong answers as the following example demonstrates.
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- Numeric methods cannot isolate just the positive roots! They isolate all the roots (real and complex).
- They can give wrong answers as the following example demonstrates.
Using Mma 5 or 6

Consider the polynomial
\[ f(x) = 10^{999}(x-1)^{50} - 1 \]
which has just 2 positive roots \( \neq 1 \):

▶ A numeric method using 10^{10} digits of accuracy takes 56 ms and finds all 50 roots \( = 1 \); that is, it fails!
▶ The same numeric method using 10^{20} digits of accuracy successfully finds the roots but it takes 18000 ms!
▶ The VAS Continued Fractions method, discussed below, isolates the two positive roots in 4 ms!
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Using Mma 7 (1/3 frames)

Consider again the same polynomial

\[ f(x) = 10^{999}(x-1)^{50} - 1 \]

with the 2 positive roots \( x \neq 1 \).

▶ The improved numeric method used in Mma 7 takes 12.933 seconds to find the two positive roots with 30 digits of accuracy.

Figure: Using the function NRoots in Mma 7

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Using Mma 7 (1/3 frames)

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Using Mma 7 (2/3 frames)
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Using Mma 7 (3/3 frames)

Using the function FindRoot in Mma 7 and approximates them to 30 digits of accuracy in practically no time at all!

Figure: Using the function FindRoot in Mma 7

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Using Mma 7 (3/3 frames)

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```math
ints = Last[ints];
FindRoot[f, {x, #[1], #[2]}, Method -> Brent,
  WorkingPrecision -> 30, MaxIterations -> 200] & /@ ints //
Timing
{0., {{x -> 0.99999999999999998528714519},
  {x -> 1.00000000000000000001047128548}}}
```

**Figure:** Using the function FindRoot in Mma 7
Therefore, ...
We do need the real root isolation methods derived from Vincents Theorem of 1836.
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- We do need the real root isolation methods derived from Vincent’s Theorem of 1836.
- Especially so since the method developed by Vincent in 1836 has **exponential** computing time!
Vincent’s exponential method of 1836 . . .
Vincent’s exponential method of 1836 ... uses continued fractions and is described in his original paper of 1836!
Vincent’s exponential method of 1836 . . .

- . . . uses continued fractions and is described in his original paper of 1836!,
- . . . was erroneously and unintentionally attributed by Uspensky to himself (1947).
Table of contents

1. Vincent’s theorem of 1836 and real root isolation

2. The two bisection methods derived from Vincent’s theorem
   - VCA, the first bisection method derived from Vincent’s theorem
   - B, the second bisection method derived from Vincent’s theorem

3. The continued fractions method derived from Vincent’s theorem
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- the open interval, \( [a, b] \), they bisect.
The first bisection method, VCA, ...
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- . . . was developed in 1976, in order to overcome the exponential behavior of Vincent’s method.
- It uses Uspensky’s termination test, explained below, and
- bisects the open interval ]0, 1[.
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- . . . Uspensky was the one to use it as a test, in the (also exponential) isolation method that he erroneously attributed to himself.
- Uspensky developed the test since he was not aware of Budan’s theorem of 1807, which was eclipsed by Fourier’s theorem of 1819.
Vincent’s theorem of 1836 and real root isolation

The two bisection methods derived from Vincent’s theorem

The continued fractions method derived from Vincent’s theorem

Budan’s theorem of 1807 — to be found in Vincent’s paper of 1836 and Akritas’ work

If in an equation $p(x) = 0$ we make two transformations,

$x = x' + p$

and

$x = x'' + q$, where $p < q$,

then:

▶ the transformed equation in $x' = x - p$ cannot have fewer sign variations than the equation in $x'' = x - q$;
▶ the number of real roots of the equation $p(x) = 0$, located between $p$ and $q$, can never be more than the number of sign variations lost in passing from the transformed equation in $x' = x - p$ to the transformed equation in $x'' = x - q$;
▶ when the first number is less than the second, the difference is always an even number.

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Budan’s theorem of 1807 — to be found in Vincent’s paper of 1836 and Akritas’ work

If in an equation $p(x) = 0$ we make two transformations, $x = x' + p$ and $x = x'' + q$, where $p$ and $q$ are real numbers such that $p < q$, then:

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B, the second bisection method derived from Vincent’s theorem

Vincent vs Uspensky (1/2)
Vincent knew Budan’s theorem and so he applied the transformation $x \leftarrow x + 1$ repeatedly until he detected a loss of sign variations in the transformed polynomial; in that case he knew there are real roots in the open interval $]0, 1[$. So he proceeded as indicated in the following figure:
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Vincent vs Uspensky (2/2)

On the other hand, Uspensky did not know Budan's theorem and so he had to invent a test to check whether there are real roots in the open interval \(0, 1\). That test is the transformation \(x \leftarrow \frac{1}{1 + x}\) and it was performed before Uspensky proceeded with the transformation \(x \leftarrow x + 1\). So he proceeded as indicated in the following figure:

Figure: Uspensky did not know Budan's theorem
On the other hand, Uspensky did not know Budan's theorem and so he had to **invent** a test to check whether there are real roots in the open interval \( ]0, 1[ \). That test is the transformation \( x \leftarrow \frac{1}{1+x} \) and it was performed **before** Uspensky proceeded with the transformation \( x \leftarrow x + 1 \). So he proceeded as indicated in the following figure:
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Uspenskys termination test, for the interval \([a, b] = [0, 1]\)

The number \(\varrho_{01}(p)\) of real roots in the open interval \((0, 1)\) — multiplicities counted — of the polynomial \(p(x) \in \mathbb{R}[x]\) is bounded above by the number of sign variations \(\text{var}_{01}(p)\), where \(\text{var}_{01}(p) = \text{var}((x + 1)^{\deg(p)}p(1 + x))\) and we have \(\text{var}_{01}(p) \geq \varrho_{01}(p)\).
The number \( \rho_{01}(p) \) of real roots in the open interval \( ]0, 1[ \) — multiplicities counted — of the polynomial \( p(x) \in \mathbb{R}[x] \) is bounded above by the number of sign variations \( \text{var}_{01}(p) \), where

\[
\text{var}_{01}(p) = \text{var}((x + 1)^{\text{deg}(p)} p\left(\frac{1}{x + 1}\right))
\]

and we have \( \text{var}_{01}(p) \geq \rho_{01}(p) \).
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VCA, 1976: The original version of the 1st bisection method derived from Vincent’s theorem

Input: The square-free polynomial $p(ub \cdot x) \in \mathbb{Z}[x]$, $p(0) \neq 0$, and the open interval $[a, b] = 0$, where $ub$ is an upper bound on the values of the positive roots of $p(x)$.

Output: A list of isolating intervals of the positive roots of $p(x)$.

\begin{verbatim}
var ←− the number of sign changes of $(x+1) \text{deg}(p)$;
1 if var = 0 then RETURN ∅;
2 if var = 1 then RETURN { ]a, b [};
3 p012 ←− 2 \text{deg}(p) p(1x+1);
4 m ←− a + b / 2 // Is 1/2 a root?
5 p121 ←− 2 \text{deg}(p) p(x+1/2);
6 if p12(1/2) ≠ 0 then RETURN VCA(p012, ]a, m[) ⋃ VCA(p121, ]m, b[);
7 else RETURN VCA(p012, ]a, m[) ⋃ { ]m, m[} ⋃ VCA(p121, ]m, b[);
\end{verbatim}
**VCA, 1976**: The *original* version of the 1st bisection method derived from Vincent’s theorem

**Input**: The square-free polynomial $p(ub \cdot x) \in \mathbb{Z}[x]$, $p(0) \neq 0$, and the open interval $]a, b[=]0, ub[$, where $ub$ is an upper bound on the values of the positive roots of $p(x)$.

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**Input:** The square-free polynomial \( p(ub \cdot x) \in \mathbb{Z}[x], p(0) \neq 0 \), and the open interval \([a, b[=]0, ub[\), where \( ub \) is an upper bound on the values of the positive roots of \( p(x) \).

**Output:** A list of isolating intervals of the positive roots of \( p(x) \)

1. \( \text{var} \leftarrow \) the number of sign changes of \((x + 1)^{\text{deg}(p)} p(\frac{1}{x+1});\)
2. if \( \text{var} = 0 \) then RETURN \( \emptyset \);
3. if \( \text{var} = 1 \) then RETURN \{\]a, b[\};
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Output: A list of isolating intervals of the positive roots of $p(x)$

1. $\text{var} \leftarrow$ the number of sign changes of $(x + 1)^{\deg(p)} p(\frac{1}{x+1})$;
2. if $\text{var} = 0$ then RETURN $\emptyset$;
3. if $\text{var} = 1$ then RETURN $\{]a, b[\}$;
4. $p_{0\frac{1}{2}} \leftarrow 2^{\deg(p)} p(\frac{x}{2})$ // Look for real roots in $]0, \frac{1}{2}[)$;
5. $m \leftarrow \frac{a+b}{2}$ // Is $\frac{1}{2}$ a root? ;
6. $p_{\frac{1}{2}1} \leftarrow 2^{\deg(p)} p(\frac{x+1}{2})$ // Look for real roots in $]\frac{1}{2}, 1[)$;
VCA, 1976: The original version of the 1st bisection method derived from Vincent’s theorem

Input: The square-free polynomial \( p(ub \cdot x) \in \mathbb{Z}[x], p(0) \neq 0 \), and the open interval \( ]a, b[ = ]0, ub[ \), where \( ub \) is an upper bound on the values of the positive roots of \( p(x) \).

Output: A list of isolating intervals of the positive roots of \( p(x) \)

```
1 var ← the number of sign changes of \((x + 1)^{\text{deg}(p)}p(\frac{1}{x+1})\);
2 if \( var = 0 \) then RETURN \( \emptyset \);
3 if \( var = 1 \) then RETURN \{\(]a, b[\}\};
4 \( p_{0 \frac{1}{2}} ← 2^{\text{deg}(p)}p(\frac{x}{2}) \) ; // Look for real roots in \( ]0, \frac{1}{2}[\);
5 \( m ← \frac{a+b}{2} \) ; // Is \( \frac{1}{2} \) a root? ;
6 \( p_{\frac{1}{2}1} ← 2^{\text{deg}(p)}p(\frac{x+1}{2}) \) ; // Look for real roots in \( ]\frac{1}{2}, 1[\);
7 if \( p(\frac{1}{2}) \neq 0 \) then
8     RETURN VCA\( (p_{0 \frac{1}{2}}, ]a, m[) \cup VCA(p_{\frac{1}{2}1}, ]m, b[)\)
9 else
10     RETURN VCA\( (p_{0 \frac{1}{2}}, ]a, m[) \cup \{[m, m]\} \cup VCA(p_{\frac{1}{2}1}, ]m, b[)\)
11 end
```
VCA has been implemented in maple — version 11 shown below
VCA has been implemented in maple — version 11 shown below

—and it takes 170 seconds to isolate the roots of Mignotte’s polynomial of degree 300!
VCA has been implemented in maple — version 11 shown below

— and it takes **170 seconds** to isolate the roots of Mignotte’s polynomial of degree 300!

```
> with(RootFinding):
> f := x^300 - 2*(5*x - 1)^2;

f := x^{300} - 2 (5 x - 1)^2

> st := time() : Isolate(f, digits = 250) : time() - st;

170.431
```
Vincent’s theorem of 1836 and real root isolation
The two bisection methods derived from Vincent’s theorem
The continued fractions method derived from Vincent’s theorem

VCA, the first bisection method derived from Vincent’s theorem
B, the second bisection method derived from Vincent’s theorem

VCA’s other names

For the period 1976-1986 VCA was called Uspensky’s modified method.
In 1986 Akritas wrote the paper There is no Uspensky’s method.
After which the method was called Descartes’ or Collins-Akritas.
In 2007 Akritas wrote the paper There is no Descartes’ method.
By that time the correct name VCA had already been coined by Francois Boulier at the University of Lille, France.
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What's in a name?

It makes clear the existing relation with Vincent's theorem.

We can use the results by Alesina & Galuzzi (2000) to prove the termination and estimate the computing time of all methods derived from Vincent's theorem.
What’s in a name?

▶ It makes clear the existing relation with Vincent’s theorem.
What’s in a name?

▸ It makes clear the existing relation with Vincent’s theorem.

▸ We can use the results by Alesina & Galuzzi (2000) to prove the termination and estimate the computing time of all methods derived from Vincents theorem.
The computing time of the VCA bisection method is $O(n^4 \tau^2)$, where $n$ is the degree of the polynomial, and $\tau$ bounds the coefficient bitsize.
The computing time of the VCA bisection method . . .

is

\[ O(n^4 \tau^2), \]

where \( n \) is the degree of the polynomial, and \( \tau \) bounds the coefficient bitsize.
The computing time of the VCA bisection method . . .

. . . is

\[ O(n^4 \tau^2), \]

where \( n \) is the degree of the polynomial, and \( \tau \) bounds the coefficient bitsize.

▶ The fastest implementation of the VCA method, REL, was developed by Rouillier & Zimmermann in 2004!
B, 2000: The second bisection method derived from Vincent’s theorem
B, 2000: The second bisection method derived from Vincent’s theorem

Uses Vincent’s termination test, and
B, 2000: The second bisection method derived from Vincent’s theorem

- Uses Vincent’s termination test, and

- bisects the interval $]a, b[\supseteq 0, ub]$, where $ub$ is an upper bound on the values of the positive roots.
Vincent’s theorem of 1836 and real root isolation
The two bisection methods derived from Vincent’s theorem
The continued fractions method derived from Vincent’s theorem

Vincent’s termination test for the interval $]a, b[ = ]0, ub[$
If $a \geq 0$ and $b > a$ then the number $\varrho_{ab}(p)$ of real roots in the open interval $]a, b[\ —$ multiplicities counted — of the polynomial $p(x) \in \mathbb{R}[x]$ is bounded above by the number of sign variations $\text{var}_{ab}(p)$, where

$$
\text{var}_{ab}(p) = \text{var}((1 + x)^{\deg(p)} p(\frac{a + bx}{1 + x}))
$$

and we have $\text{var}_{ab}(p) = \text{var}_{ba}(p) \geq \varrho_{ab}(p)$. 

Vincent’s theorem of 1836 and real root isolation

The two bisection methods derived from Vincent’s theorem

The continued fractions method derived from Vincent’s theorem

VCA, the first bisection method derived from Vincent’s theorem

B, the second bisection method derived from Vincent’s theorem

**B, 2000**: The second bisection method derived from Vincent’s theorem

---

**Input**: The square-free polynomial \( p(x) \in \mathbb{Z}[x] \), \( p(0) \neq 0 \), and the open interval \( [a, b] = [0, ub] \), where \( ub \) is an upper bound on the values of the positive roots of \( p(x) \).

**Output**: A list of isolating intervals of the positive roots of \( p(x) \).

\[
\text{var} \leftarrow \text{the number of sign changes of } (1 + x)^{\deg(p)} p(a + bx^
\frac{1}{2})
\]

1. If \( \text{var} = 0 \) then RETURN \( \emptyset \);
2. If \( \text{var} = 1 \) then RETURN \( \{a, b\} \);
3. \( m \leftarrow a + \frac{b}{2} \) // Subdivide the interval \( [a, b] \) in two equal parts;
4. If \( p(m) \neq 0 \) then 5 RETURN \( B(p, [a, m]) \cup B(p, [m, b]) \)
5. Else 7 RETURN \( B(p, [a, m]) \cup \{[m, m]\} \cup B(p, [m, b]) \)
6. End

---

Alkiviadis G. Akritas

PCA 2009, St. Petersburg, Russia
B, 2000: The second bisection method derived from Vincent’s theorem

**Input:** The square-free polynomial $p(x) \in \mathbb{Z}[x]$, $p(0) \neq 0$, and the open interval $]a, b[=]0, ub[$, where $ub$ is an upper bound on the values of the positive roots of $p(x)$.

**Output:** A list of isolating intervals of the positive roots of $p(x)$
**B, 2000: The second bisection method derived from Vincent’s theorem**

**Input:** The square-free polynomial $p(x) \in \mathbb{Z}[x]$, $p(0) \neq 0$, and the open interval $]a, b[=]0, ub[,$ where $ub$ is an upper bound on the values of the positive roots of $p(x)$.

**Output:** A list of isolating intervals of the positive roots of $p(x)$

1. $\text{var} \leftarrow$ the number of sign changes of $(1 + x)^{\deg(p)} p\left(\frac{a+bx}{1+x}\right)$;
2. if $\text{var} = 0$ then RETURN $\emptyset$;
3. if $\text{var} = 1$ then RETURN $\{]a, b[\}$;
B, 2000: The second bisection method derived from Vincent’s theorem

**Input:** The square-free polynomial \( p(x) \in \mathbb{Z}[x], p(0) \neq 0 \), and the open interval \( ]a, b[ \), where \( ub \) is an upper bound on the values of the positive roots of \( p(x) \).

**Output:** A list of isolating intervals of the **positive** roots of \( p(x) \)

1. \( var \leftarrow \) the number of sign changes of \((1 + x)^{\deg(p)} p(\frac{a+bx}{1+x})\);
2. if \( var = 0 \) then RETURN \( \emptyset \);
3. if \( var = 1 \) then RETURN \( \{\]a, b[\} \);
4. \( m \leftarrow \frac{a+b}{2} \) // Subdivide the interval \( ]a, b[ \) in two equal parts;
B, 2000: The second bisection method derived from Vincent’s theorem

**Input:** The square-free polynomial \( p(x) \in \mathbb{Z}[x], p(0) \neq 0 \), and the open interval \( ]a, b[=]0, ub[ \), where \( ub \) is an upper bound on the values of the positive roots of \( p(x) \).

**Output:** A list of isolating intervals of the positive roots of \( p(x) \)

```
1 var ← the number of sign changes of (1 + x)^deg(p)p(\frac{a+bx}{1+x});
2 if var = 0 then RETURN ∅;
3 if var = 1 then RETURN {\( ]a, b[ \);}
4 m ← \( \frac{a+b}{2} \) // Subdivide the interval \( ]a, b[ \) in two equal parts;
5 if \( p(m) \neq 0 \) then
6     RETURN B(p,]a, m[) \( \cup \) B(p,]m, b[)
7 else
8     RETURN B(p,]a, m[) \( \cup \{\]m, m]\) \( \cup \) B(p,]m, b[)
9 end
```
Comparison of the two bisection methods

VCA, the first bisection method derived from Vincent’s theorem
B, the second bisection method derived from Vincent’s theorem

Alkiviadis G. Akritas
PCA 2009, St. Petersburg, Russia
Comparison of the two bisection methods

VCA, the method using the simpler termination test, i.e. Uspensky’s test, is faster than B, which is using Vincent's more complex termination test!
Table of contents

1. Vincent’s theorem of 1836 and real root isolation

2. The two bisection methods derived from Vincent’s theorem

3. The continued fractions method derived from Vincent’s theorem
   - Improving the performance of VAS
   - Speeding up VAS with linear complexity bounds
   - Speeding up VAS with quadratic complexity bounds
The continued fractions method derived from Vincent’s theorem...
The continued fractions method derived from Vincent’s theorem . . .

. . . uses Descartes rule of signs as the termination test, and
The continued fractions method derived from Vincent’s theorem . . .

▶ . . . uses Descartes rule of signs as the termination test, and

▶ . . . relies, heavily, on the repeated estimation of lower bounds on the values of the positive roots of polynomials.
Vincent's theorem of 1836 and real root isolation
The two bisection methods derived from Vincent's theorem
The continued fractions method derived from Vincent's theorem
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VAS, 1978:

\[\text{Input} \quad \text{The square-free polynomial} \quad p(x) \in \mathbb{Z}[x], \quad p(0) \neq 0, \quad \text{and the M"obius transformation} \quad M(x) = ax + b, c x + d = x, \quad a, b, c, d \in \mathbb{Z} \]

\[\text{Output} \quad \text{A list of isolating intervals of the positive roots of} \quad p(x) \]

\[\text{var} \leftarrow \text{the number of sign changes of} \quad p(x); \]
\[1 \quad \text{if} \quad \text{var} = 0 \quad \text{then RETURN} \quad \emptyset; \]
\[2 \quad \text{if} \quad \text{var} = 1 \quad \text{then RETURN} \quad \{a, b\} \quad // \ a = \min(M(0), M(\infty)), \ b = \max(M(0), M(\infty)); \]
\[3 \quad \ell_b \leftarrow \text{a lower bound on the positive roots of} \quad p(x); \]
\[4 \quad \text{if} \quad \ell_b > 1 \quad \text{then} \]
\[\quad p \leftarrow p(x + \ell_b), \quad M \leftarrow M(x + \ell_b); \]
\[5 \quad p_{01} \leftarrow (x + 1)^{\deg(p)} p(1x + 1), \quad M_{01} \leftarrow M(1x + 1) \quad // \text{Look for real roots in } [0, 1]; \]
\[6 \quad m \leftarrow M(1) \quad // \text{Is } 1 \text{ a root?}; \]
\[7 \quad p_{1\infty} \leftarrow p(x + 1), \quad M_{1\infty} \leftarrow M(x + 1) \quad // \text{Look for real roots in } [1, +\infty]; \]
\[8 \quad \text{if} \quad p(1) \neq 0 \quad \text{then 9 \quad RETURN} \quad \text{VAS}(p_{01}, M_{01}) \cup \text{VAS}(p_{1\infty}, M_{1\infty}); \]
\[9 \quad \text{else 11 \quad RETURN} \quad \text{VAS}(p_{01}, M_{01}) \cup \{[m, m]\} \cup \text{VAS}(p_{1\infty}, M_{1\infty}); \]
\[10 \quad \text{end} \]
Vincent's theorem of 1836 and real root isolation
The two bisection methods derived from Vincent's theorem
The continued fractions method derived from Vincent's theorem

Improving the performance of VAS
Speeding up VAS with linear complexity bounds
Speeding up VAS with quadratic complexity bounds

VAS, 1978:

**Input:** The square-free polynomial $p(x) \in \mathbb{Z}[x]$, $p(0) \neq 0$, and the Möbius transformation $M(x) = \frac{ax+b}{cx+d} = x$, $a, b, c, d \in \mathbb{Z}$

**Output:** A list of isolating intervals of the positive roots of $p(x)$
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**Output:** A list of isolating intervals of the positive roots of \( p(x) \)

1. \( var \leftarrow \) the number of sign changes of \( p(x) \);
2. if \( var = 0 \) then RETURN \( \emptyset \);
3. if \( var = 1 \) then RETURN \( \{ a, b \} \) // \( a = \min(M(0),M(\infty)), \ b = \max(M(0),M(\infty)) \);
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**Output:** A list of isolating intervals of the positive roots of $p(x)$

1. $\text{var} \leftarrow$ the number of sign changes of $p(x)$;
2. If $\text{var} = 0$ then RETURN $\emptyset$;
3. If $\text{var} = 1$ then RETURN $\{a, b\}$ // $a = \min(M(0),M(\infty))$, $b = \max(M(0),M(\infty))$;
4. $\ell b \leftarrow$ a lower bound on the positive roots of $p(x)$;
5. If $\ell b > 1$ then $\{p \leftarrow p(x + \ell b), M \leftarrow M(x + \ell b)\}$;
Vincent’s theorem of 1836 and real root isolation
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VAS, 1978:

**Input:** The square-free polynomial \( p(x) \in \mathbb{Z}[x] \), \( p(0) \neq 0 \), and the Möbius transformation \( M(x) = \frac{ax+b}{cx+d} = x \), \( a,b,c,d \in \mathbb{Z} \)

**Output:** A list of isolating intervals of the positive roots of \( p(x) \)

```
1 var ←− the number of sign changes of \( p(x) \);
2 if \( \text{var} = 0 \) then RETURN \( \emptyset \);
3 if \( \text{var} = 1 \) then RETURN \{ \text{[a, b]} \} // a = \min(M(0),M(\infty)), \text{ b = max(M(0),M(\infty))} ;
4 \ell b ←− a lower bound on the positive roots of \( p(x) \);
5 if \( \ell b > 1 \) then \{ \text{p ←− } p(x + \ell b), M ←− M(x + \ell b) \} ;
6 \text{p}_{01} ←− (x + 1)^{\text{deg}(p)}p\left(\frac{1}{x+1}\right), \text{M}_{01} ←− M\left(\frac{1}{x+1}\right) // \text{Look for real roots in [0, 1[ ;}
7 m ←− M(1) // Is 1 a root? ;
8 \text{p}_{1\infty} ←− p(x + 1), \text{M}_{1\infty} ←− M(x + 1) // \text{Look for real roots in [1, +\infty[ ;}
```
Vincent’s theorem of 1836 and real root isolation
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VAS, 1978:

**Input:** The square-free polynomial \( p(x) \in \mathbb{Z}[x], p(0) \neq 0 \), and the Möbius transformation \( M(x) = \frac{ax+b}{cx+d} = x, a, b, c, d \in \mathbb{Z} \)

**Output:** A list of isolating intervals of the **positive** roots of \( p(x) \)

```plaintext
1  var ←− the number of sign changes of \( p(x) \);
2  if var = 0 then RETURN \( \emptyset \);
3  if var = 1 then RETURN \{ [a, b] \} // a = \min(M(0),M(\infty)), b = \max(M(0),M(\infty)) ;
4  \ell b ←− a lower bound on the positive roots of \( p(x) \);
5  if \( \ell b > 1 \) then \{ \( p \leftarrow p(x + \ell b), M \leftarrow M(x + \ell b) \) \};
6  \( p_{01} \leftarrow (x + 1)^{\deg(p)} p(\frac{1}{x+1}), M_{01} \leftarrow M(\frac{1}{x+1}) \) // Look for real roots in \(]0,1[\);
7  \( m \leftarrow M(1) \) // Is 1 a root? ;
8  \( p_{1\infty} \leftarrow p(x + 1), M_{1\infty} \leftarrow M(x + 1) \) // Look for real roots in \(]1,\infty[\);
9  if \( p(1) \neq 0 \) then
10     RETURN VAS(\( p_{01}, M_{01} \)) \( \cup \) VAS(\( p_{1\infty}, M_{1\infty} \))
11 else
12     RETURN VAS(\( p_{01}, M_{01} \)) \( \cup \{ [m, m] \} \cup \) VAS(\( p_{1\infty}, M_{1\infty} \))
13 end
```
Comments on the VAS-continued fractions real root isolation method:
Comments on the VAS-continued fractions real root isolation method:

- Strzebonski’s contribution is omitted for simplicity.
Comments on the VAS-continued fractions real root isolation method:

▶ Strzebonski’s contribution is omitted for simplicity.

▶ Without steps 4 and 5 it is simply Vincent’s original exponential method.
Vincent’s theorem of 1836 and real root isolation
The two bisection methods derived from Vincent’s theorem
The continued fractions method derived from Vincent’s theorem

Improving the performance of VAS
Speeding up VAS with linear complexity bounds
Speeding up VAS with quadratic complexity bounds

VAS has been implemented in *Mathematica* — version 7 shown below
Vincent’s theorem of 1836 and real root isolation

The two bisection methods derived from Vincent’s theorem

The continued fractions method derived from Vincent’s theorem

Improving the performance of VAS

Speeding up VAS with linear complexity bounds

Speeding up VAS with quadratic complexity bounds

VAS has been implemented in Mathematica — version 7 shown below

—and it takes **0.046 seconds** to isolate and approximate the roots of Mignotte’s polynomial of degree 300!
Vincent’s theorem of 1836 and real root isolation

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The continued fractions method derived from Vincent’s theorem

Improving the performance of VAS

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Speeding up VAS with quadratic complexity bounds

VAS has been implemented in Mathematica — version 7 shown below

— and it takes **0.046 seconds** to isolate and approximate the roots of Mignotte’s polynomial of degree 300!
Over the past 30 years . . .

<table>
<thead>
<tr>
<th>Vincent’s theorem of 1836 and real root isolation</th>
</tr>
</thead>
<tbody>
<tr>
<td>The two bisection methods derived from Vincent’s theorem</td>
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</tr>
</tbody>
</table>

<table>
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<th>Improving the performance of VAS</th>
</tr>
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<tr>
<td>Speeding up VAS with linear complexity bounds</td>
</tr>
<tr>
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</table>

Over the past 30 years, VAS has been using Cauchy’s bound on the values of the positive roots. For random polys, VAS has been several thousand times faster than the VCA bisection method — even up to 50000 times faster than VCA, for Mignotte polys. Only in the case of very many, (\(>50\)), very large roots, (\(10^{10}\)), had VAS been up to 4 times slower than VCA.

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Over the past 30 years . . .

VAS has been using **Cauchy’s bound** on the values of the positive roots,
Over the past 30 years . . .

- VAS has been using **Cauchy’s bound** on the values of the positive roots,

- For random polys, VAS has been **several thousand times faster** than the VCA bisection method — even up to **50000 times faster** than VCA, for Mignotte polys.
Over the past 30 years . . .

▶ VAS has been using **Cauchy’s bound** on the values of the positive roots,

▶ For random polys, VAS has been **several thousand times faster** than the VCA bisection method — even up to **50000 times faster** than VCA, for Mignotte polys.

▶ Only in the case of very many, (> 50), very large roots, (10^{100}), had VAS been up to **4 times slower** than VCA.
Computing time of the VAS continued fractions method

Using a plausible hypothesis and the fast translation algorithm by von zu Gathen, the computing time of VAS is $O(n^4 \tau^2)$, where $n$ is the degree of the polynomial, and $\tau$ bounds the coefficient bitsize. (Akritas 1978, Tsigaridas-Emiris 2005)

Without any hypotheses the computing time of VAS is $O(n^8 \tau^3)$. However, this bound does not match the performance of VAS.
Using a plausible hypothesis and the fast translation algorithm by von zu Gathen, the computing time of VAS is

\[ O(n^4 \tau^2), \]

where \( n \) is the degree of the polynomial, and \( \tau \) bounds the coefficient bit size. (Akritas 1978, Tsigaridas-Emiris 2005)
Using a plausible hypothesis and the fast translation algorithm by von zu Gathen, the computing time of VAS is

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Without any hypotheses the computing time of VAS is

\[ O(n^8 \tau^3) \]

(Sharma 2007). However, this bound does not match the performance of VAS.
To improve the performance of the VAS method even further . . .
To improve the performance of the VAS method even further...

...new bounds on the values of the positive roots of polynomials were needed.
To improve the performance of the VAS method even further...

- ...new bounds on the values of the positive roots of polynomials were needed.

- To understand the nature of these bounds we used Doru Ştefănescu’s inspirational work!
Ștefănescu’s theorem (2005): matching a positive-coefficient term with a negative-coefficient one — when the number of sign variations is even
Ștefănescu’s theorem (2005): matching a positive-coefficient term with a negative-coefficient one — when the number of sign variations is even

Let $p(x) \in \mathbb{R}[x]$ be such that the number of sign variations in the sequence of its coefficients is even. If

$$p(x) = c_1 x^{d_1} - b_1 x^{m_1} + c_2 x^{d_2} - b_2 x^{m_2} + \ldots + c_k x^{d_k} - b_k x^{m_k} + g(x),$$

with $g(x) \in \mathbb{R}_+[x], c_i > 0, b_i > 0, d_i > m_i > d_{i+1}$ for all $i$, then the number

$$ub(p) = \max \left\{ \left( \frac{b_1}{c_1} \right)^{1/(d_1-m_1)}, \ldots, \left( \frac{b_k}{c_k} \right)^{1/(d_k-m_k)} \right\}$$

is an upper bound on the values of the positive roots of the polynomial $p$ for any choice of $c_1, \ldots, c_k$. 
Remarks on Ștefănescu’s theorem:

▶ It does not work if the number of sign variations is not even.
▶ It also fails to work if a positive-coefficient term is followed by two negative-coefficient terms.
▶ The following theorem by Akritas, Strzeboński and Vigklas generalizes Ștefănescu’s theorem and works in all cases. This is achieved by breaking up a positive-coefficient term into several parts to be matched with the corresponding negative-coefficient terms.
Remarks on Ştefănescu’s theorem:

- It does not work if the number of sign variations is not even.
Remarks on Ştefănescu’s theorem:

- It does not work if the number of sign variations is not even.
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Remarks on Ștefănescu’s theorem:

- It does not work if the number of sign variations is not even.

- It also fails to work if a positive-coefficient term is followed by two negative-coefficient terms.

- The following theorem by Akritas, Strzeboński and Vigklas generalizes Ștefănescu’s theorem and works in all cases. This is achieved by breaking up a positive-coefficient term into several parts to be matched with the corresponding negative-coefficient terms.
Assumptions

Let $p(x) = \alpha_n x^n + \alpha_{n-1} x^{n-1} + \ldots + \alpha_0$, \((\alpha_n > 0)\)

be a polynomial with real coefficients and let \(d(p)\) and \(t(p)\) denote the degree and the number of its terms, respectively.

Moreover, assume that $p(x)$ can be written as $p(x) = q_1(x) - q_2(x) + q_3(x) - q_4(x) + \ldots + q_{2m-1}(x) - q_{2m}(x) + g(x)$, where all the polynomials $q_i(x)$, $i = 1, 2, \ldots, 2m$ and $g(x)$ have only positive coefficients. In addition, assume that for $i = 1, 2, \ldots, m$ we have $q_{2i-1}(x) = c_{2i-1,1} x^{e_{2i-1,1}} + \ldots + c_{2i-1,t} x^{e_{2i-1,t}}$ and $q_{2i}(x) = b_{2i,1} x^{e_{2i,1}} + \ldots + b_{2i,t} x^{e_{2i,t}}$, where $e_{2i-1,1} = d(q_{2i-1})$ and $e_{2i,1} = d(q_{2i})$ and the exponent of each term in $q_{2i-1}(x)$ is greater than the exponent of each term in $q_{2i}(x)$. 
Assumptions

Let \( p(x) \)
\[
p(x) = \alpha_n x^n + \alpha_{n-1} x^{n-1} + \ldots + \alpha_0, \quad (\alpha_n > 0)
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be a polynomial with real coefficients and let \( d(p) \) and \( t(p) \) denote the degree and the number of its terms, respectively. Moreover, assume that \( p(x) \) can be written as
\[
p(x) = q_1(x) - q_2(x) + q_3(x) - q_4(x) + \ldots + q_{2m-1}(x) - q_{2m}(x) + g(x),
\]
where all the polynomials \( q_i(x) \), \( i = 1, 2, \ldots, 2m \) and \( g(x) \) have only positive coefficients. In addition, assume that for \( i = 1, 2, \ldots, m \) we have
\[
q_{2i-1}(x) = c_{2i-1,1} x^{e_{2i-1,1}} + \ldots + c_{2i-1,t(q_{2i-1})} x^{e_{2i-1,t(q_{2i-1})}}
\]
and
\[
q_{2i}(x) = b_{2i,1} x^{e_{2i,1}} + \ldots + b_{2i,t(q_{2i})} x^{e_{2i,t(q_{2i})}},
\]
where \( e_{2i-1,1} = d(q_{2i-1}) \) and \( e_{2i,1} = d(q_{2i}) \) and the exponent of each term in \( q_{2i-1}(x) \) is greater than the exponent of each term in \( q_{2i}(x) \).
Generalization of Ștefănescu’s theorem by Akritas, Strzeboński and Vigklas, 2006 (2/2)
Theorem

If for all indices \( i = 1, 2, \ldots, m \), we have

\[ t(q_{2i-1}) \geq t(q_{2i}), \]

then an upper bound of the values of the positive roots of \( p(x) \) is given by

\[
ub = \max_{\{i=1,2,\ldots,m\}} \left\{ \left( \frac{b_{2i,1}}{c_{2i-1,1}} \right)^{e_{2i-1,1}-e_{2i,1}}, \ldots, \left( \frac{b_{2i,t(q_{2i})}}{c_{2i-1,t(q_{2i})}} \right)^{e_{2i-1,t(q_{2i})}-e_{2i,t(q_{2i})}} \right\},
\]

for any permutation of the positive coefficients \( c_{2i-1,j}, j = 1, 2, \ldots, t(q_{2i-1}) \).

Otherwise, for each of the indices \( i \) for which we have

\[ t(q_{2i-1}) < t(q_{2i}), \]

we break up one of the coefficients of \( q_{2i-1}(x) \) into \( t(q_{2i}) - t(q_{2i-1}) + 1 \) parts, so that now \( t(q_{2i}) = t(q_{2i-1}) \) and apply the same formula given above.
Remarks on the theorem by Akritas-Strzeboński and Vigklas, 2006:
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▶ It is a general theorem from which almost all methods for computing positive bounds on the values of positive roots are derived!
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- This generality is achieved by breaking up and pairing up — in various ways — unmatched positive-coefficient terms with negative-coefficient ones of lower order!
Remarks on the theorem by Akritas-Strzeboński and Vigklas, 2006:

► It is a general theorem from which almost all methods for computing positive bounds on the values of positive roots are derived!

► This generality is achieved by breaking up and pairing up — in various ways — unmatched positive-coefficient terms with negative-coefficient ones of lower order!

On terminology

► For simplicity, in the sequel we will simply refer to positive coefficients being matched with negative ones of lower order terms!
On upper bounds on the values of the positive roots of polynomials:
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> In general, bounds in the literature are of linear complexity!
On upper bounds on the values of the positive roots of polynomials:

▶ In general, bounds in the literature are of **linear complexity**!

▶ That is, each negative coefficient of the polynomial is paired up with **only one** of the preceding (unmatched) positive coefficients and the maximum of all the computed radicals is taken as the estimate of the bound.
On upper bounds on the values of the positive roots of polynomials:

▶ In general, bounds in the literature are of **linear complexity**!

▶ That is, **each** negative coefficient of the polynomial is paired up with **only one** of the preceding (unmatched) positive coefficients and the maximum of all the computed radicals is taken as the estimate of the bound.

▶ We present **four** linear complexity bounds. Of those, the last two were developed by Akritas, Strzeboński and Vigklas in 2007.
A: Cauchy’s “leading coefficient” bound, (C)
A: Cauchy’s “leading coefficient” bound, (C)

For the polynomial $p(x) \in \mathbb{R}[x]$

$$p(x) = \alpha_n x^n + \alpha_{n-1} x^{n-1} + \ldots + \alpha_0, \quad (\alpha_n > 0),$$

with $\lambda$ negative coefficients, Cauchy’s method first breaks up its leading coefficient, $\alpha_n$, into $\lambda$ equal parts and then pairs up each part with the first unmatched negative coefficient.

That is, we have:

$$ub_C = \max_{\{1 \leq k \leq n: \alpha_{n-k} < 0\}} \sqrt[k]{\frac{-\alpha_{n-k}}{-\frac{\alpha_n}{\lambda}}}. $$
B: Kioustelidis’ “leading coefficient” bound, (K)

For the polynomial $p(x) \in \mathbb{R}[x]$

$$p(x) = \alpha_n x^n + \alpha_{n-1} x^{n-1} + \ldots + \alpha_0,$$

($\alpha_n > 0$), Kioustelidis’ method matches the coefficient $-\alpha_{n-k}$ of the term $-\alpha_{n-k} x^{n-k}$ in $p(x)$ with $\alpha_n/2^k$, the leading coefficient divided by $2^k$.

That is, we have

$$u^{(K)} = \max\{1 \leq k \leq n : \alpha_{n-k} < 0\} k \sqrt{-\alpha_{n-k} / \alpha_n^2}.$$
For the polynomial $p(x) \in \mathbb{R}[x]$

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That is, we have

$$ub_K = \max_{\{1 \leq k \leq n : \alpha_{n-k} < 0\}} \sqrt{k \frac{-\alpha_{n-k}}{\alpha_n}} \frac{\alpha_n}{2^k}.$$
Vincent’s theorem of 1836 and real root isolation
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That is, we have

\[
ub_K = \max_{\{1 \leq k \leq n : \alpha_{n-k} < 0\}} \sqrt[k]{-\frac{\alpha_{n-k}}{\frac{\alpha_n}{2^k}}}.
\]

Kioustelidis’ method differs from that by Cauchy only in that the leading coefficient is now broken up in unequal parts — by dividing it with different powers of 2.
Vincent's theorem of 1836 and real root isolation
The two bisection methods derived from Vincent's theorem
The continued fractions method derived from Vincent's theorem

C: “First–λ” bound, (FL)
C: “First–λ” bound, (FL)

For the polynomial $p(x) \in \mathbb{R}[x]$

\[ p(x) = q_1(x) - q_2(x) + q_3(x) - q_4(x) + \ldots + q_{2m-1}(x) - q_{2m}(x) + g(x), \]

with $\lambda$ negative coefficients we:
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with $\lambda$ negative coefficients we:

first take care of all cases for which $t(q_{2i}) > t(q_{2i-1})$, by breaking up the last coefficient $c_{2i-1}, t(q_{2i})$, of $q_{2i-1}(x)$, into $t(q_{2i}) - t(q_{2i-1}) + 1$ equal parts, and
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with $\lambda$ negative coefficients we:

- First take care of all cases for which $t(q_{2i}) > t(q_{2i-1})$, by breaking up the last coefficient $c_{2i-1, t(q_{2i})}$ of $q_{2i-1}(x)$, into $t(q_{2i}) - t(q_{2i-1}) + 1$ equal parts, and

- Then pair each of the first $\lambda$ positive coefficients of $p(x)$, encountered as we move in non-increasing order of exponents, with the first unmatched negative coefficient.
Vincent’s theorem of 1836 and real root isolation
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D: “Local-Max” bound, (LM)
D: “Local-Max” bound, (LM)

For the polynomial \( p(x) \in \mathbb{R}[x] \)

\[
p(x) = \alpha_n x^n + \alpha_{n-1} x^{n-1} + \ldots + \alpha_0, \quad (\alpha_n > 0),
\]

the coefficient \(-\alpha_k\) of the term \(-\alpha_k x^k\) in \( p(x) \) is paired with the coefficient \( \frac{\alpha_m}{2^t} \), of the term \( \alpha_m x^m \), where \( \alpha_m \) is the largest positive coefficient with \( n \geq m > k \) and \( t \) indicates the number of times the coefficient \( \alpha_m \) has been used.
Example

Consider the polynomial

\[ x^3 + 10^{100}x^2 - 10^{100}x - 1, \]

which has one sign variation and, hence, one positive root \textbf{equal to 1}. 

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\[ x^3 + 10^{100}x^2 - 10^{100}x - 1, \]

which has one sign variation and, hence, one positive root equal to 1.

With Cauchy’s bound, we pair the terms:

\[ \left\{ \frac{x^3}{2}, -10^{100}x \right\} \text{ and } \left\{ \frac{x^3}{2}, -1 \right\}, \]

and taking the maximum of the radicals we obtain a bound estimate of \( 1.41421 \times 10^{50}. \)
Example

Consider the polynomial

\[ x^3 + 10^{100}x^2 - 10^{100}x - 1, \]

which has one sign variation and, hence, one positive root **equal to 1**.

With \( K \), Kioustelidis’ bound, we pair the terms:

\[ \{\frac{x^3}{2^2}, -10^{100}x\} \text{ and } \{\frac{x^3}{2^3}, -1\}, \]

and taking the maximum of the radicals we obtain a bound estimate of \( 2 \times 10^{50} \).
Example

Consider the polynomial

$$x^3 + 10^{100}x^2 - 10^{100}x - 1,$$

which has one sign variation and, hence, one positive root equal to 1.

With FL, the “First-$$\lambda$$” bound, we pair the terms:

- \(\{x^3, -10^{100}x\}\) and \(\{10^{100}x^2, -1\}\),

and taking the maximum of the radicals we obtain a bound estimate of \(10^{50}\).
Consider the polynomial

\[ x^3 + 10^{100}x^2 - 10^{100}x - 1, \]

which has one sign variation and, hence, one positive root equal to 1.

With LM, the “Local Max” bound, we pair the terms:

- \( \left\{ \frac{10^{100}x^2}{2}, -10^{100}x \right\} \) and \( \left\{ \frac{10^{100}x^2}{2^2}, -1 \right\} \),

and taking the maximum of the radicals we obtain a bound estimate of 2!
Comparison of the 4 linear complexity bounds

Empirical results have indicated the following:

- Kioustelidis' bound is, in general, better (or much better) than Cauchy's; this happens because the former breaks up the leading coefficient in unequal parts, whereas the latter breaks it up in equal parts.
- Our First-\(\lambda\) bound, as the name indicates, uses additional coefficients and, therefore, it is, in general, better (or much better) than both previous bounds. In the few cases where Kioustelidis' bound is better than first-\(\lambda\), our Local-Max bound takes again the lead.
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- Our First-$\lambda$ bound, as the name indicates, uses additional coefficients and, therefore, it is not surprising that it is, in general, better (or much better) than both previous bounds. In the few cases where Kioustelidis’ bound is better than first-$\lambda$, our Local-Max bound takes again the lead.
Conclusions

Of the four linear complexity bounds there does not exist one that always computes best estimate values. Therefore, to improve the performance of VAS we used two methods — $\lambda$ or $FL$ and $Local-Max$ or $LM$ — and took their minimum as the estimated value of the bound. That is, we used $\min(FL, LM)$.

Using $\min(FL, LM)$, instead of using Cauchy's bound, the VAS continued fractions method was speeded up, on average, by 15% and became always faster than the VCA bisections method.
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- Therefore, to improve the performance of VAS we used two methods — First-$\lambda$ or FL and Local-Max or LM — and took their minimum as the estimated value of the bound. That is, we used $\min(FL, LM)$.

- Using $\min(FL, LM)$, instead of using Cauchy’s bound, the VAS continued fractions method was speeded up, on average, by 15% and became always faster than the VCA bisections method.
The only case where VAS was slower than VCA: Products of terms $x - r$, with random integer $r$. 
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Table: We compare the timings in seconds (s) for: (a) VAS\_old, i.e. VAS using Cauchy’s rule, (b) VAS\_new, i.e. VAS using the new rule $\min(FL + LM)$, and (c) VCA\_rel. The tests were run on a laptop computer with 1.8 Ghz Pentium M processor, running a Linux virtual machine with 1.78 GB of RAM.

<table>
<thead>
<tr>
<th>Roots (bit length)</th>
<th>Deg</th>
<th>VAS_old Time(s) Average (Min/Max)</th>
<th>VAS_new Time(s) Average (Min/Max)</th>
<th>VCA_rel Average (Min/Max)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>100</td>
<td>0.314 (0.248/0.392)</td>
<td>0.253 (0.228/0.280)</td>
<td>0.346 (0.308/0.384)</td>
</tr>
<tr>
<td>10</td>
<td>200</td>
<td>1.74 (1.42/2.33)</td>
<td>1.51 (1.34/1.66)</td>
<td>3.90 (3.72/4.05)</td>
</tr>
<tr>
<td>10</td>
<td>500</td>
<td>17.6 (16.9/18.7)</td>
<td>17.4 (16.3/18.1)</td>
<td>129 (122/140)</td>
</tr>
<tr>
<td>1000</td>
<td>20</td>
<td>0.066 (0.040/0.084)</td>
<td>0.031 (0.024/0.040)</td>
<td>0.038 (0.028/0.044)</td>
</tr>
<tr>
<td>1000</td>
<td>50</td>
<td>1.96 (1.45/2.44)</td>
<td>0.633 (0.512/0.840)</td>
<td>1.03 (0.916/1.27)</td>
</tr>
<tr>
<td>1000</td>
<td>100</td>
<td>52.3 (36.7/81.3)</td>
<td>12.7 (11.3/14.6)</td>
<td>17.2 (16.1/18.7)</td>
</tr>
</tbody>
</table>
Quadratic complexity bounds (1/2)

To further improve the performance of the VAS continued fractions method we decided to use quadratic complexity bounds. Justification: Their improved estimates should compensate for the extra time needed to compute these bounds.
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Their improved estimates should compensate for the extra time needed to compute these bounds.
Quadratic complexity bounds (2/2)

Main idea:

▶ Each negative coefficient of the polynomial is paired with all the preceding positive coefficients and the minimum of the computed values is associated with this coefficient. The maximum of all those minimums is taken as the estimate of the bound.

▶ We will present four quadratic complexity bounds derived from the corresponding four linear complexity bounds discussed before. Of those four, one was developed by Hong in 1998, whereas the other three — including the best and fastest — were developed by Akritas, Argyris, Strzeboński and Vigklas in 2008.
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A: Cauchy’s quadratic complexity bound, (CQ)
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For the polynomial $p(x) \in \mathbb{R}[x]$

$$p(x) = \alpha_n x^n + \alpha_{n-1} x^{n-1} + \ldots + \alpha_0, \quad (\alpha_n > 0),$$

each negative coefficient $a_i < 0$ is "paired" with each one of the preceding positive coefficients $a_j$ divided by $\lambda_i$ — that is, each positive coefficient $a_j$ is "broken up" into equal parts, as is done with just the leading coefficient in Cauchy’s bound; $\lambda_i$ is the number of negative coefficients to the right of, and including, $a_i$ — and the minimum is taken over all $j$; subsequently, the maximum is taken over all $i$.

That is, we have:

$$ub_{CQ} = \max_{\{a_i < 0\}} \min_{\{a_j > 0: j > i\}} \left( j-i \sqrt{\frac{a_i}{a_j} - \frac{a_i}{\lambda_i}} \right).$$
Vincent’s theorem of 1836 and real root isolation

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B: Kioustelidis’ quadratic complexity bound, \((KQ)\),

Hong 1998

For the polynomial

\[ p(x) = \alpha_n x^n + \alpha_{n-1} x^{n-1} + \ldots + \alpha_0, \quad (\alpha_n > 0) \]

each negative coefficient \(a_i < 0\) is “paired” with each one of the preceding positive coefficients \(a_j\) divided by \(2^{j-i}\) — that is, each positive coefficient \(a_j\) is “broken up” into unequal parts, as is done with just the leading coefficient in Kioustelidis’ linear bound — and the minimum is taken over all \(j\); subsequently, the maximum is taken over all \(i\).

That is, we have:

\[ \text{ub}_{KQ} = \max \left\{ a_i < 0 \right\} \min \left\{ a_j > 0 : j > i \right\} \left( \frac{\sqrt{-a_i a_j}}{2^{j-i}} \right) \]
B: **Kioustelidis’ quadratic complexity bound, (KQ),**
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For the polynomial $p(x) \in \mathbb{R}[x]$

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each negative coefficient $a_i < 0$ is “paired” with each one of the preceding positive coefficients $a_j$ divided by $2^{j-i}$ — that is, each positive coefficient $a_j$ is “broken up” into unequal parts, as is done with just the leading coefficient in Kioustelidis’ linear bound — and the minimum is taken over all $j$; subsequently, the maximum is taken over all $i$.

That is, we have:

$$ub_{KQ} = \max_{\{a_i < 0\}} \min_{\{a_j > 0 : j > i\}} j-i \sqrt{-\frac{a_i}{a_j \cdot 2^{j-i}}}.$$
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C: “First-λ” quadratic complexity bound, (FLQ)
C: “First-$\lambda$” quadratic complexity bound, (FLQ)

For the polynomial $p(x) \in \mathbb{R}[x]$

$$p(x) = q_1(x) - q_2(x) + q_3(x) - q_4(x) + \ldots + q_{2m-1}(x) - q_{2m}(x) + g(x),$$

with $\lambda$ negative coefficients we do the following:
For the polynomial \( p(x) \in \mathbb{R}[x] \)
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\]
with \( \lambda \) negative coefficients we do the following:

First we take care of all cases for which \( t(q_{2\ell}) > t(q_{2\ell-1}) \), by breaking up the last coefficient \( c_{2\ell-1,t(q_{2\ell})} \) of \( q_{2\ell-1}(x) \), into
\[
d_{2\ell-1,t(q_{2\ell})} = t(q_{2\ell}) - t(q_{2\ell-1}) + 1 \text{ equal parts.}
\]
For the polynomial $p(x) \in \mathbb{R}[x]$

$$p(x) = q_1(x) - q_2(x) + q_3(x) - q_4(x) + \ldots + q_{2m-1}(x) - q_{2m}(x) + g(x),$$

with $\lambda$ negative coefficients we do the following:

First we take care of all cases for which $t(q_{2\ell}) > t(q_{2\ell-1})$, by breaking up the last coefficient $c_{2\ell-1, t(q_{2\ell})}$, of $q_{2\ell-1}(x)$, into $d_{2\ell-1, t(q_{2\ell})} = t(q_{2\ell}) - t(q_{2\ell-1}) + 1$ equal parts.

Then each negative coefficient $a_i < 0$ is “paired” with each one of the preceding $\min(i, \lambda)$ positive coefficients $a_j$ divided by $d_j$ and the minimum is taken over all $j$; subsequently, the maximum is taken over all $i$. 
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That is, we have:
\[
\text{ub}_{FLQ} = \max \{ a_i < 0 \} \min \{ a_j > 0 : j > \min(i, \lambda) : u(j) \neq 0 \} - i \sqrt{-a_i a_j d_j},
\]
where \(d_j\) indicates the number of equal parts into which each of the preceding \(\min(i, \lambda)\) positive coefficients \(a_j\) is “broken up”. The value of \(d_j\) is initially set to 1, for each \(j\), and it changes only if the positive coefficient \(a_j\) is broken up into equal parts.

\(u(j)\) indicates the number of times \(a_j\) can be used to calculate the minimum. The value of \(u(j)\) is originally set equal to \(d_j\) and it decreases each time \(a_j\) is used in the computation of the minimum.

Alkiviadis G. Akritas
PCA 2009, St. Petersburg, Russia
That is, we have:

\[
ub_{FLQ} = \max_{\{a_i < 0\}} \min_{\{a_j > 0: j > \min(i, \lambda): u(j) \neq 0\}} \left( j-i \sqrt{\frac{a_i}{a_j d_j}} \right),
\]

where \ldots
That is, we have:

\[ ub_{FLQ} = \max_{\{a_i < 0\}} \min_{\{a_j > 0: j > \min(i, \lambda) : u(j) \neq 0\}} j - i \sqrt{-\frac{a_i}{a_j}} \frac{\sqrt{-\frac{a_i}{a_j}}}{d_j}, \]

where . . .

- \( d_j \) indicates the number of equal parts into which each of the preceding \( \min(i, \lambda) \) positive coefficients \( a_j \) is “broken up”. The value of \( d_j \) is initially set to 1, for each \( j \), and it changes only if the positive coefficient \( a_j \) is broken up into equal parts.
That is, we have:

\[ ub_{FLQ} = \max \{a_i < 0\} \min \{a_j > 0 : j > \min(i, \lambda) : u(j) \neq 0\} \min \left( j-i, \frac{a_i}{d_j} \right) \]

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- \( d_j \) indicates the number of equal parts into which each of the preceding \( \min(i, \lambda) \) positive coefficients \( a_j \) is “broken up”. The value of \( d_j \) is initially set to 1, for each \( j \), and it changes only if the positive coefficient \( a_j \) is broken up into equal parts.

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Vincent’s theorem of 1836 and real root isolation
The two bisection methods derived from Vincent’s theorem
The continued fractions method derived from Vincent’s theorem

D: “Local Max” quadratic complexity bound, (LMQ)

For the polynomial
\[ p(x) \in \mathbb{R}[x] \]
\[ p(x) = \alpha_n x^n + \alpha_{n-1} x^{n-1} + \ldots + \alpha_0, \quad (\alpha_n > 0) \]
each negative coefficient \( a_i < 0 \) is "paired" with each one of the preceding positive coefficients \( a_j \) divided by \( 2^t_j \)— where \( t_j \) is initially set to 1 and is incremented each time the positive coefficient \( a_j \) is used — and the minimum is taken over all \( j \); subsequently, the maximum is taken over all \( i \).

That is, we have:
\[ \text{ub}_{LMQ} = \max \left\{ a_i < 0 \right\} \min \left\{ a_j > 0 : j > i \right\} \frac{j - i}{\sqrt{-a_i a_j}} \]

Each positive coefficient \( a_j \) is "broken up" into unequal parts, as is done with just the locally maximum coefficient in the linear local max bound.
For the polynomial $p(x) \in \mathbb{R}[x]$

$$p(x) = \alpha_n x^n + \alpha_{n-1} x^{n-1} + \ldots + \alpha_0, \quad (\alpha_n > 0),$$

each negative coefficient $a_i < 0$ is “paired” with each one of the preceding positive coefficients $a_j$ divided by $2^{t_j}$ — where $t_j$ is initially set to 1 and is incremented each time the positive coefficient $a_j$ is used — and the minimum is taken over all $j$; subsequently, the maximum is taken over all $i$.

That is, we have:

$$ub_{LMQ} = \max_{\{a_i < 0\}} \min_{\{a_j > 0; j > i\}} \sqrt{j-i} - \frac{a_i}{a_j \cdot 2^{t_j}},$$
D: “Local Max” quadratic complexity bound, (LMQ)

For the polynomial \( p(x) \in \mathbb{R}[x] \)

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p(x) = \alpha_n x^n + \alpha_{n-1} x^{n-1} + \ldots + \alpha_0, \quad (\alpha_n > 0),
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each negative coefficient \( a_i < 0 \) is “paired” with each one of the preceding positive coefficients \( a_j \) divided by \( 2^{t_j} \) — where \( t_j \) is initially set to 1 and is incremented each time the positive coefficient \( a_j \) is used — and the minimum is taken over all \( j \); subsequently, the maximum is taken over all \( i \).

That is, we have:

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ub_{LMQ} = \max_{\{a_i < 0\}} \min_{\{a_j > 0: j > i\}} \left( j - i \right) \sqrt{-\frac{a_i a_j}{a_j^2 2^{t_j}}}.
\]

Each positive coefficient \( a_j \) is “broken up” into unequal parts, as is done with just the locally maximum coefficient in the linear local max bound.
Example

Consider again the polynomial

\[ x^3 + 10^{100}x^2 - 10^{100}x - 1, \]

which has one sign variation and, hence, one positive root \textit{equal to 1}.
Example

Consider again the polynomial

$$x^3 + 10^{100}x^2 - 10^{100}x - 1,$$

which has one sign variation and, hence, one positive root equal to 1.

With \textbf{CQ}, Cauchy’s quadratic complexity bound, we compute:

- the \textbf{minimum} of the two radicals obtained from the pairs of terms \{\( \frac{x^3}{2}, -10^{100}x \)} and \{\( \frac{10^{100}x^2}{2}, -10^{100}x \)} which is 2, and
- the \textbf{minimum} of the two radicals obtained from the pairs of terms \{\( x^3, -1 \)} and \{\( 10^{100}x^2, -1 \)} which is \( \frac{1}{10^{50}} \).

Therefore, the obtained estimate of the bound is \( \max\{2, \frac{1}{10^{50}}\} = 2. \)
Consider again the polynomial

\[ x^3 + 10^{100}x^2 - 10^{100}x - 1, \]

which has one sign variation and, hence, one positive root equal to 1.

With KQ, Kioustelidis’ quadratic complexity bound, we compute:

- the minimum of the two radicals obtained from the pairs of terms \( \{\frac{x^3}{2^2}, -10^{100}x\} \) and \( \{\frac{10^{100}x^2}{2}, -10^{100}x\}\) which is 2, and
- the minimum of the two radicals obtained from the pairs of terms \( \{\frac{x^3}{2^3}, -1\} \) and \( \{\frac{10^{100}x^2}{2^2}, -1\}\) which is \( \frac{2}{10^{50}}\).
- Therefore, the obtained estimate of the bound is \( max\{2, \frac{2}{10^{50}}\} = 2. \)
Example

Consider again the polynomial

\[ x^3 + 10^{100}x^2 - 10^{100}x - 1, \]

which has one sign variation and, hence, one positive root equal to 1.

With FLQ, the “First-\(\lambda\)” quadratic bound, we compute:

- the minimum of the two radicals obtained from the pairs of terms \(\{x^3, -10^{100}x\}\) and \(\{10^{100}x^2, -10^{100}x\}\) which is 1, and
- the minimum of the two radicals obtained from the pairs of terms \(\{x^3, -1\}\) and \(\{10^{100}x^2, -1\}\) which is 1.
- Therefore, the obtained estimate of the bound is \(\max\{1, 1\} = 1\).
- Note: Once a term with a positive coefficient has been used in obtaining the minimum, it cannot be used again!
Consider again the polynomial

\[ x^3 + 10^{100}x^2 - 10^{100}x - 1, \]

which has one sign variation and, hence, one positive root equal to 1.

With LMQ, the “Local Max” quadratic bound, we compute:

- the minimum of the two radicals obtained from the pairs of terms \( \{ \frac{x^3}{2}, -10^{100}x \} \) and \( \{ \frac{10^{100}x^2}{2}, -10^{100}x \} \) which is 2,
- the minimum of the two radicals obtained from the pairs of terms \( \{ \frac{x^3}{2^2}, -1 \} \) and \( \{ \frac{10^{100}x^2}{2^2}, -1 \} \) which is \( \frac{2}{10^{50}} \).
- Therefore, the obtained estimate of the bound is \( \max\{2, \frac{2}{10^{50}}\} = 2 \).
Linear vs quadratic complexity bounds

From the example we see that the estimates of all quadratic complexity bounds are much better than those of their linear complexity counterparts.

In general, the quadratic complexity bounds cannot perform worse than the linear complexity ones; most of the times they perform a lot better!
Linear vs quadratic complexity bounds

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Comparison of the 4 quadratic complexity bounds

- FLQ is faster — or quite faster — than all of them because it tests just the first $\min(\iota, \lambda)$ positive coefficients. By comparison, all the other quadratic complexity bounds test every preceding positive coefficient.

- The estimates computed by LMQ are sharper by the factor $2^{j-i-t_j}$ than those computed by Kioustelidis' $KQ$ — because $2^{t_j} \leq 2^{j-i}$, where $i$ and $j$ are the indices realizing the max of $\min(FLQ, LMQ)$. Equality holds when there are no missing terms in the polynomial.

- Experimental results indicated that FLQ, LMQ and $\min(FLQ, LMQ)$ behave equally well! Therefore, we picked LMQ to improve the performance of VAS.
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Comparison of the 4 quadratic complexity bounds

▶ **FLQ** is faster — or quite faster — than all of them because it tests just the first \(\min(\nu, \lambda)\) positive coefficients. By comparison, all the other quadratic complexity bounds test every preceding positive coefficient.

▶ The estimates computed by **LMQ** are **sharper** by the factor 
\[
2^{\frac{j-i-t_j}{j-i}}
\]
than those computed by Kioustelidis’ **KQ** — because 
\[
2^{t_j} \leq 2^{j-i}, \text{ where } i \text{ and } j \text{ are the indices realizing the } \max \text{ of } \min.
\]
Equality holds when there are no missing terms in the polynomial.

▶ Experimental results indicated that **FLQ**, **LMQ** and 
\[
\min(\text{FLQ}, \text{LMQ})
\]
behave equally well! Therefore, we picked **LMQ** to improve the performance of **VAS**.
Conclusions on the quadratic complexity bounds
Using \textit{LMQ}, the performance of the VAS real root isolation method was speeded up by an average overall factor of 40\%.
Vincent’s theorem of 1836 and real root isolation
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Overall time spent for computing bounds
Overall time spent for computing bounds

In the following graph the left scale shows the times in seconds (bars) needed by VAS to isolate the roots of a certain class of polynomials using both LM, the Local Max bound, and LMQ, its quadratic version. The right scale is associated with the two curves which show the total time spent by VAS in computing the bounds.
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Products of terms $x - r$, with random integer $r$ — Revisited

Table: We compare the timings in seconds (s) for: (a) VAS(cauchy), i.e. VAS using Cauchy's rule, (b) VAS(fl+lm), i.e. VAS using the linear complexity bound $\min(FL + LM)$, and (c) VAS(lmq), i.e. VAS using the Locam Max quadratic complexity bound. The average speed-up for this table is about 35%.

<table>
<thead>
<tr>
<th>Bit-length</th>
<th>Degree</th>
<th>VAS(cauchy)</th>
<th>VAS(fl+lm)</th>
<th>VAS(lmq)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>100</td>
<td>0.46 (0.28/0.94)</td>
<td>0.24 (0.18/0.28)</td>
<td>0.34 (0.30/0.41)</td>
</tr>
<tr>
<td>10</td>
<td>200</td>
<td>1.46 (1.24/1.85)</td>
<td>1.40 (1.28/1.69)</td>
<td>1.40 (1.20/1.69)</td>
</tr>
<tr>
<td>10</td>
<td>500</td>
<td>18.1 (16.5/18.9)</td>
<td>18.1 (16.6/18.8)</td>
<td>22.1 (18.7/24.2)</td>
</tr>
<tr>
<td>1000</td>
<td>20</td>
<td>0.07 (0.04/0.14)</td>
<td>0.02 (0.02/0.03)</td>
<td>0.03 (0.02/0.04)</td>
</tr>
<tr>
<td>1000</td>
<td>50</td>
<td>3.69 (2.38/6.26)</td>
<td>0.81 (0.60/1.28)</td>
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<td>1000</td>
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<td>47.8 (37.6/56.9)</td>
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Future Research
Computing time bound

Sharma’s bound on the computing time of the VAS continued fractions method is greatly overestimated.

Hence, theoretical research is needed to see if we can bring it down.
Future Research

Coefficients

The VAS continued fractions method works for integer or rational coefficients.

Hence, we need to discover new ways to deal with coefficients that are algebraic numbers or approximate reals.
Future Research

Sparse polynomials of great degree

The VAS continued fractions method is the fastest real root isolation method when the polynomials are not sparse and their degree is less than a few thousand. However, Mathematica runs out of memory when we try to isolate the roots of a sparse polynomial of degree 100000.

Hence, we need to discover new ways to deal with sparse polynomials of extremely high degrees.
Future Research

Parallel implementation

Last, but not least, we need to investigate the performance of the VAS continued fractions method in a multiprocessor environment.
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References
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References


▶ Akritas, A. G.: “There is no Descartes’ method”. In M.J.Wester and M. Beaudin (Eds), Computer Algebra in Education, 2007, AullonaPress, USA, 19–35,


References


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