Vincent's theorem of 1836:

Overview and Future Research

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- We will concentrate on the Continued Fractions method (the fastest of them all) and show how it was recently speeded up by 40% over its initial implementation.
- We will indicate new directions for future research.

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Consider the polynomial

$$p(x) = a_n x^n + \cdots + a_1 x + a_0,$$

where $p(x) \in \mathbb{R}[x]$ and let $var(p)$ represent the number of sign *changes* or *variations* (positive to negative and vice-versa) in the sequence of coefficients $a_n, a_{n-1}, \ldots, a_0$.

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Theorem

The number $\varrho_+(p)$ of real roots — multiplicities counted — of the polynomial $p(x) \in \mathbb{R}[x]$ in the open interval $]0, \infty[$ is bounded above by $var(p)$; that is, we have $var(p) \geq \varrho_+(p)$.

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- 1 sign variation $\Rightarrow$ 1 positive root

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2. The two bisection methods derived from Vincent's theorem
3. The continued fractions method derived from Vincent's theorem

Vincent's theorem of 1836 — the *C*ontinued *F*ractions version

If in a polynomial, $p(x)$, of degree n, with rational coefficients and simple roots we perform sequentially replacements of the form

$$x \leftarrow \alpha_1 + \frac{1}{x}, x \leftarrow \alpha_2 + \frac{1}{x}, x \leftarrow \alpha_3 + \frac{1}{x}, \ldots$$

where $\alpha_1 \geq 0$ is an arbitrary non negative integer and $\alpha_2, \alpha_3, \ldots$ are arbitrary positive integers, $\alpha_i > 0$, $i > 1$, then the resulting polynomial either has *no* sign variations or it has *one* sign variation. In the first case there are *no* positive roots whereas in the last case the equation has exactly one positive root, represented by the continued fraction

$$\alpha_1 + \cfrac{1}{\alpha_2 + \cfrac{1}{\alpha_3 + \cfrac{1}{\ddots}}}$$

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Vincent's theorem ...

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- was rediscovered by Akritas in 1976 and formed the subject of his Ph.D. Thesis (1978).

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- Uspensky never read Vincent's original paper of 1836, which included several examples (applications). Instead,

- he read only a proof of Vincent's theorem in the Russian translation of Serrets (French) book on Algebra (1866).

Vincent's *B*isection theorem of 2000 — by Alesina and Galuzzi

Let $f(z)$, be a real polynomial of degree n, which has only simple roots. It is possible to determine a positive quantity δ so that for every pair of positive real numbers a, b with $|b - a| < \delta$, every transformed polynomial of the form

$$\phi(z) = (1+z)^n f(\frac{a+bz}{1+z})$$

has exactly 0 or 1 variations. The second case is possible if and only if $f(z)$ has a simple root within $]a, b[$.

Proof of Vincent's theorem

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- A similar proof was presented earlier by Ostrowski (1950), who rediscovered Obreschkoff's theorem 30 years later.

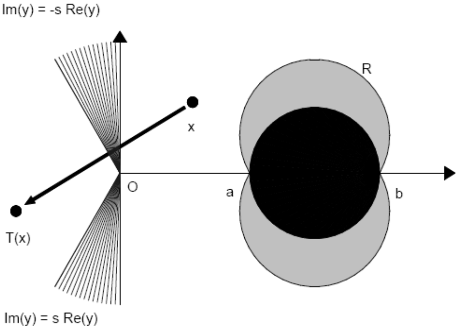
Obreschkoff's Cone or Sector Theorem

If a real polynomial has one positive simple root x_0 and all the other — possibly multiple — roots lie in the sector

$$S_{\sqrt{3}} = \{x = -\alpha + \imath\beta \mid \alpha > 0 \quad \mathit{and} \quad \beta^2 \leq 3\alpha^2\}$$

then the sequence of its coefficients has exactly one sign variation.

View of Obreschkoff's Cone and Circles



Real root isolation using Vincent's theorem

To isolate the positive roots of a polynomial $p(x)$, all we have to do is compute — for *each* root — the variables a, b, c, d of the corresponding Möbius transformation

$$M(x) = \frac{ax + b}{cx + d}$$

that leads to a transformed polynomial

$$f(x) = (cx + d)^n p(\frac{ax + b}{cx + d})$$

with one sign variation.

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Crucial observation:

The variables a, b, c, d of a Möbius transformation $M(x) = \frac{ax+b}{cx+d}$ (in Vincent's theorem) leading to a transformed polynomial with one sign variation can be computed:

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- either by continued fractions leading to the continued fractions method developed by Vincent, Akritas and Strzeboński, which in the sequel will be called the `VAS` *continued fractions* method — to distinguish it from other continued fraction methods,

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- or, by bisections leading to (among others) the bisection method developed by (Vincent), Collins and Akritas, which in the sequel will be called the `VCA` *bisection* method — to distinguish it from Sturm's bisection method.

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The VAS-continued fractions method was developed later — in 1978 by Akritas and in 1993 by Akritas, Botcharov and Strzeboński. Its fastest implementation was developed in 2008 by Akritas, Strzeboński and Vigklas.

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- Numeric methods cannot isolate just the positive roots! They isolate all the roots (real and complex).

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- Numeric methods cannot isolate just the positive roots! They isolate all the roots (real and complex).
- They can give wrong answers as the following example demonstrates.

Using Mma 5 or 6

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which has just 2 positive roots $\neq 1$:

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- The same numeric method using 1020 digits of accuracy successfully finds the roots but it takes 18000 ms!
- The VAS Continued Fractions method, discussed below, isolates the two positive roots in 4 ms!

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```
f := 10^999 (x - 1)^50 - 1

Select[NRoots[f == 0, x, 30], Im[#[[2]]] == 0 &] //
 Timing

{12.933, x == 0.9999999999999999999989528714519 ||
  x == 1.0000000000000000000001047128548}
```

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```
ints = RootIntervals[f][[1]] // Timing
{5.60316×10^-16, {{0, 1}, {1, 2}}}
```

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```
ints = Last[ints];
FindRoot[f, {x, #[[1]], #[[2]]}, Method → Brent,
    WorkingPrecision → 30, MaxIterations → 200] & /@ ints //
 Timing
{0., {{x → 0.999999999999999999989528714519},
  {x → 1.00000000000000000001047128548}}}
```

Figure: Using the function FindRoot in Mma 7

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- We do need the real root isolation methods derived from Vincents Theorem of 1836.
- Especially so since the method developed by Vincent in 1836 has exponential computing time!

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- . . . was erroneously and unintentionally attributed by Uspensky to himself (1947).

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   - VCA, the first bisection method derived from Vincent's theorem
   - B, the second bisection method derived from Vincent's theorem
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- ... was developed in 1976, in order to overcome the exponential behavior of Vincent's method.
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- bisects the open interval]0, 1[.

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- . . . Uspensky was the one to use it as a test, in the (also exponential) isolation method that he erroneously attributed to himself.
- Uspensky developed the test since he was not aware of Budan's theorem of 1807, which was eclipsed by Fourier's theorem of 1819.

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- the transformed equation in $x' = x - p$ cannot have fewer sign variations than the equation in $x'' = x - q$;
- the number of real roots of the equation $p(x) = 0$, located between p and q, can never be more than the number of sign variations lost in passing from the transformed equation in $x' = x - p$ to the transformed equation in $x'' = x - q$;

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- when the first number is less than the second, the difference is always an even number.

Vincent vs Uspensky (1/2)

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- Vincent knew Budan's theorem and so he applied the transformation $x \leftarrow x + 1$ repeatedly until he detected a loss of sign variations in the transformed polynomial; in that case he knew there are real roots in the open interval $]0, 1[$. So he proceeded as indicated in the following figure:

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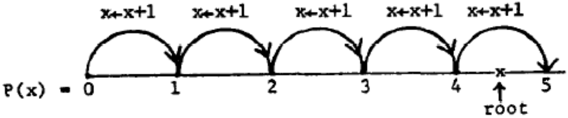

Vincent vs Uspensky (2/2)

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- On the other hand, Uspensky did not know Budan's theorem and so he had to invent a test to check whether there are real roots in the open interval $]0, 1[$. That test is the transformation $x \leftarrow \frac{1}{1+x}$ and it was performed before Uspensky proceeded with the transformation $x \leftarrow x + 1$. So he proceeded as indicated in the following figure:

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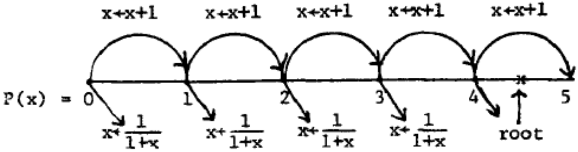

Uspenskys termination test, for the interval $]a, b[=]0, 1[$

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The number $\varrho_{01}(p)$ of real roots in the open interval $]0, 1[$ — multiplicities counted — of the polynomial $p(x) \in \mathbb{R}[x]$ is bounded above by the number of sign variations $var_{01}(p)$, where

$$var_{01}(p) = var((x+1)^{deg(p)} p(\frac{1}{x+1}))$$

and we have $var_{01}(p) \geq \varrho_{01}(p)$.

VCA, 1976: The *original* version of the 1st bisection method derived from Vincent's theorem

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Input: The square-free polynomial $p(ub \cdot x) \in \mathbb{Z}[x], p(0) \neq 0$, and the open interval $]a, b[=]0, ub[$, where ub is an upper bound on the values of the positive roots of $p(x)$.

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4 $p_{0\frac{1}{2}} \longleftarrow 2^{deg(p)} p(\frac{x}{2})$ `// Look for real roots in` $]0, \frac{1}{2}[$;
5 $m \longleftarrow \frac{a+b}{2}$ `// Is` $\frac{1}{2}$ `a root?` ;
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if $p(\frac{1}{2}) \neq 0$ **then**
  RETURN $\texttt{VCA}(p_{0\frac{1}{2}},]a, m[) \bigcup \texttt{VCA}(p_{\frac{1}{2}1},]m, b[)$
else
  RETURN $\texttt{VCA}(p_{0\frac{1}{2}},]a, m[) \bigcup \{[m, m]\} \bigcup \texttt{VCA}(p_{\frac{1}{2}1},]m, b[)$
end

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```
> with(RootFinding) :
> f := x^300 - 2(5 x - 1)^2;
                                   f := x^300 - 2 (5 x - 1)^2
> st := time( ) : Isolate(f, digits = 250) : time( ) - st;
                                        170.431
>
```

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- ...after which the method was called *Descartes'* or *Collins-Akritas*.
- In 2007 Akritas wrote the paper *There is no Descartes' method*. By that time the correct name VCA had already been coined by Francois Boulier at the University of Lille, France.

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- We can use the results by Alesina & Galuzzi (2000) to prove the termination and estimate the computing time of all methods derived from Vincents theorem.

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▶ The fastest implementation of the VCA method, REL, was developed by Rouillier & Zimmermann in 2004!

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- bisects the interval $]a, b[=]0, ub[$, where ub is an upper bound on the values of the positive roots.

Vincent's termination test for the interval $]a, b[=]0, ub[$

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$$var_{ab}(p) = var((1+x)^{deg(p)} p(\frac{a+bx}{1+x}))$$

and we have $var_{ab}(p) = var_{ba}(p) \geq \varrho_{ab}(p)$.

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$m \longleftarrow \frac{a+b}{2}$ `// Subdivide the interval` $]a, b[$ `in two equal parts ;`

B, 2000: The second bisection method derived from Vincent's theorem

Input: The square-free polynomial $p(x) \in \mathbb{Z}[x]$, $p(0) \neq 0$, and the open interval $]a, b[=]0, ub[$, where ub is an upper bound on the values of the positive roots of $p(x)$.

Output: A list of isolating intervals of the positive roots of $p(x)$

```
var ⟵ the number of sign changes of (1 + x)^{deg(p)} p((a+bx)/(1+x));
if var = 0 then RETURN ∅;
if var = 1 then RETURN {]a, b[};
m ⟵ (a+b)/2 // Subdivide the interval ]a, b[ in two equal parts ;
if p(m) ≠ 0 then
    RETURN B(p, ]a, m[) ⋃ B(p, ]m, b[)
else
    RETURN B(p, ]a, m[) ⋃ {[m, m]} ⋃ B(p, ]m, b[)
end
```

Comparison of the two bisection methods

Comparison of the two bisection methods

VCA, the method using the simpler termination test, i.e. Uspensky's test, is faster than B, which is using Vincent's more complex termination test!

Table of contents

1. Vincent's theorem of 1836 and real root isolation
2. The two bisection methods derived from Vincent's theorem
3. The continued fractions method derived from Vincent's theorem
   - Improving the performance of VAS
   - Speeding up VAS with linear complexity bounds
   - Speeding up VAS with quadratic complexity bounds

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The continued fractions method derived from Vincent's theorem . . .

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Input: The square-free polynomial $p(x) \in \mathbb{Z}[x], p(0) \neq 0$, and the Möbius transformation $M(x) = \frac{ax+b}{cx+d} = x,\ a, b, c, d \in \mathbb{Z}$

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2 **if** $var = 0$ **then RETURN** $\emptyset$;
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$\ell b \longleftarrow$ a lower bound on the positive roots of $p(x)$;
if $\ell b > 1$ **then** $\{p \longleftarrow p(x + \ell b), M \longleftarrow M(x + \ell b)\}$;
$p_{01} \longleftarrow (x+1)^{deg(p)} p(\frac{1}{x+1}), M_{01} \longleftarrow M(\frac{1}{x+1})$ `// Look for real roots in` $]0, 1[$;
$m \longleftarrow M(1)$ `// Is 1 a root? ;`
$p_{1\infty} \longleftarrow p(x+1), M_{1\infty} \longleftarrow M(x+1)$ `// Look for real roots in` $]1, +\infty[$;

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var ⟵ the number of sign changes of p(x);
if var = 0 then RETURN ∅;
if var = 1 then RETURN {]a, b[} // a = min(M(0),M(∞)), b = max(M(0),M(∞));
ℓb ⟵ a lower bound on the positive roots of p(x);
if ℓb > 1 then {p ⟵ p(x + ℓb), M ⟵ M(x + ℓb)};
p01 ⟵ (x + 1)^deg(p) p(1/(x+1)), M01 ⟵ M(1/(x+1)) // Look for real roots in ]0, 1[ ;
m ⟵ M(1) // Is 1 a root? ;
p1∞ ⟵ p(x + 1), M1∞ ⟵ M(x + 1) // Look for real roots in ]1, +∞[ ;
if p(1) ≠ 0 then
    RETURN VAS(p01, M01) ⋃ VAS(p1∞, M1∞)
else
    RETURN VAS(p01, M01) ⋃ {[m, m]} ⋃ VAS(p1∞, M1∞)
end
```

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- Strzebonski's contribution is omitted for simplicity.

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- Without steps 4 and 5 it is simply Vincent's original exponential method.

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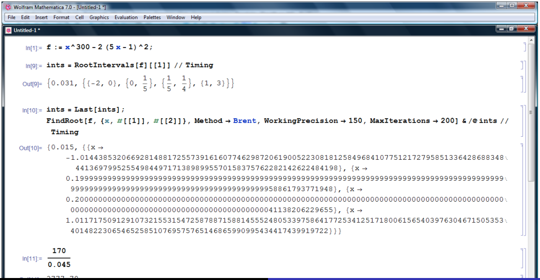

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Over the past 30 years . . .

- VAS has been using Cauchy's bound on the values of the positive roots,

- For random polys, VAS has been several thousand times faster than the VCA bisection method — even up to 50000 times faster than VCA, for Mignotte polys.

- Only in the case of very many, (> 50), very large roots, (10^{100}), had VAS been up to 4 times slower than VCA.

Computing time of the VAS continued fractions method

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- Using a plausible hypothesis and the fast translation algorithm by von zu Gathen, the computing time of VAS is

$$O(n^4\tau^2),$$

where n is the degree of the polynomial, and τ bounds the coefficient bitsize. (Akritas 1978, Tsigaridas-Emiris 2005)

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where n is the degree of the polynomial, and τ bounds the coefficient bitsize. (Akritas 1978, Tsigaridas-Emiris 2005)

- Without any hypotheses the computing time of VAS is

$$O(n^8\tau^3)$$

(Sharma 2007). However, this bound does not match the performance of VAS.

To improve the performance of the VAS method even further . . .

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- . . . new bounds on the values of the positive roots of polynomials were needed.

- To understand the nature of these bounds we used Doru Ştefănescu's inspirational work!

Ştefănescu's theorem (2005): matching a positive-coefficient term with a negative-coefficient one — when the number of sign variations is even

Ştefănescu's theorem (2005): matching a positive-coefficient term with a negative-coefficient one — when the number of sign variations is even

Let $p(x) \in \mathbb{R}[x]$ be such that the number of sign variations in the sequence of its coefficients is even. If

$$p(x) = c_1 x^{d_1} - b_1 x^{m_1} + c_2 x^{d_2} - b_2 x^{m_2} + \ldots + c_k x^{d_k} - b_k x^{m_k} + g(x),$$

with $g(x) \in \mathbb{R}_+[x]$, $c_i > 0, b_i > 0, d_i > m_i > d_{i+1}$ for all i, then the number

$$ub(p) = \max \left\{ \left(\frac{b_1}{c_1} \right)^{1/(d_1 - m_1)}, \ldots, \left(\frac{b_k}{c_k} \right)^{1/(d_k - m_k)} \right\}$$

is an upper bound on the values of the positive roots of the polynomial p for any choice of $c_1, \ldots, c_k$.

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- It does not work if the number of sign variations is not even.

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- The following theorem by Akritas, Strzeboński and Vigklas generalizes Ștefănescu's theorem and works in all cases. This is achieved by breaking up a positive-coefficient term into several parts to be matched with the corresponding negative-coefficient terms.

Generalization of Ştefănescu's theorem by Akritas, Strzeboński and Vigklas, 2006 (1/2)

Generalization of Ştefănescu's theorem by Akritas, Strzeboński and Vigklas, 2006 (1/2)

Assumptions

Let $p(x)$

$$p(x) = \alpha_n x^n + \alpha_{n-1} x^{n-1} + \ldots + \alpha_0, \quad (\alpha_n > 0)$$

be a polynomial with real coefficients and let $d(p)$ and $t(p)$ denote the degree and the number of its terms, respectively.
Moreover, assume that $p(x)$ can be written as

$$p(x) = q_1(x) - q_2(x) + q_3(x) - q_4(x) + \ldots + q_{2m-1}(x) - q_{2m}(x) + g(x),$$

where all the polynomials $q_i(x)$, $i = 1, 2, \ldots, 2m$ and $g(x)$ have only positive coefficients. In addition, assume that for $i = 1, 2, \ldots, m$ we have

$$q_{2i-1}(x) = c_{2i-1,1} x^{e_{2i-1,1}} + \ldots + c_{2i-1,t(q_{2i-1})} x^{e_{2i-1,t(q_{2i-1})}}$$

and

$$q_{2i}(x) = b_{2i,1} x^{e_{2i,1}} + \ldots + b_{2i,t(q_{2i})} x^{e_{2i,t(q_{2i})}},$$

where $e_{2i-1,1} = d(q_{2i-1})$ and $e_{2i,1} = d(q_{2i})$ and the exponent of each term in $q_{2i-1}(x)$ is greater than the exponent of each term in $q_{2i}(x)$.

Generalization of Ștefănescu's theorem by Akritas, Strzeboński and Vigklas, 2006 (2/2)

Generalization of Ştefănescu's theorem by Akritas, Strzeboński and Vigklas, 2006 (2/2)

Theorem

If for all indices $i = 1, 2, \ldots, m$, we have

$$t(q_{2i-1}) \geq t(q_{2i}),$$

then an upper bound of the values of the positive roots of $p(x)$ is given by

$$ub = \max_{\{i=1,2,\ldots,m\}} \left\{ \left(\frac{b_{2i,1}}{c_{2i-1,1}} \right)^{\frac{1}{e_{2i-1,1}-e_{2i,1}}}, \ldots, \left(\frac{b_{2i,t(q_{2i})}}{c_{2i-1,t(q_{2i})}} \right)^{\frac{1}{e_{2i-1,t(q_{2i})}-e_{2i,t(q_{2i})}}} \right\},$$

for any permutation of the positive coefficients $c_{2i-1,j}$, $j = 1, 2, \ldots, t(q_{2i-1})$. Otherwise, for each of the indices i for which we have

$$t(q_{2i-1}) < t(q_{2i}),$$

we break up one of the coefficients of $q_{2i-1}(x)$ into $t(q_{2i}) - t(q_{2i-1}) + 1$ parts, so that now $t(q_{2i}) = t(q_{2i-1})$ and apply the same formula given above.

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- It is a general theorem from which almost all methods for computing positive bounds on the values of positive roots are derived!

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- This generality is achieved by breaking up and pairing up — in various ways — unmatched positive-coefficient terms with negative-coefficient ones of lower order!

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- It is a general theorem from which almost all methods for computing positive bounds on the values of positive roots are derived!

- This generality is achieved by breaking up and pairing up — in various ways — unmatched positive-coefficient terms with negative-coefficient ones of lower order!

On terminology

- For simplicity, in the sequel we will simply refer to positive coefficients being matched with negative ones of lower order terms!

On upper bounds on the values of the positive roots of polynomials:

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- In general, bounds in the literature are of linear complexity!

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On upper bounds on the values of the positive roots of polynomials:

- In general, bounds in the literature are of linear complexity!

- That is, each negative coefficient of the polynomial is paired up with only one of the preceding (unmatched) positive coefficients and the maximum of all the computed radicals is taken as the estimate of the bound.

- We present four linear complexity bounds. Of those, the last two were developed by Akritas, Strzeboński and Vigklas in 2007.

A: Cauchy's "leading coefficient" bound, (C)

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- For the polynomial $p(x) \in \mathbb{R}[x]$

$$p(x) = \alpha_n x^n + \alpha_{n-1} x^{n-1} + \ldots + \alpha_0, \quad (\alpha_n > 0),$$

with λ negative coefficients, Cauchy's method first breaks up its leading coefficient, α_n, into λ equal parts and then pairs up each part with the first unmatched negative coefficient.

That is, we have:

$$ub_C = \max_{\{1 \le k \le n : \alpha_{n-k} < 0\}} \sqrt[k]{-\frac{\alpha_{n-k}}{\frac{\alpha_n}{\lambda}}}.$$

B: Kioustelidis' "leading coefficient" bound, (K)

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$$p(x) = \alpha_n x^n + \alpha_{n-1}x^{n-1} + \ldots + \alpha_0, \quad (\alpha_n > 0),$$

Kioustelidis' method matches the coefficient $-\alpha_{n-k}$ of the term $-\alpha_{n-k}x^{n-k}$ in $p(x)$ with $\frac{\alpha_n}{2^k}$, the leading coefficient divided by 2^k.

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$$ub_K = \max_{\{1 \le k \le n : \alpha_{n-k} < 0\}} \sqrt[k]{-\frac{\alpha_{n-k}}{\frac{\alpha_n}{2^k}}}.$$

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$$ub_K = \max_{\{1 \leq k \leq n : \alpha_{n-k} < 0\}} \sqrt[k]{-\frac{\alpha_{n-k}}{\frac{\alpha_n}{2^k}}}.$$

- Kioustelidis' method differs from that by Cauchy only in that the leading coefficient is now broken up in unequal parts — by dividing it with different powers of 2.

C: "First–λ" bound, (FL)

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- For the polynomial $p(x) \in \mathbb{R}[x]$

$$p(x) = q_1(x) - q_2(x) + q_3(x) - q_4(x) + \ldots + q_{2m-1}(x) - q_{2m}(x) + g(x),$$

with λ negative coefficients we:

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with λ negative coefficients we:

- first take care of all cases for which $t(q_{2i}) > t(q_{2i-1})$, by breaking up the last coefficient $c_{2i-1,t(q_{2i})}$, of $q_{2i-1}(x)$, into $t(q_{2i}) - t(q_{2i-1}) + 1$ equal parts, and

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with λ negative coefficients we:

- first take care of all cases for which $t(q_{2i}) > t(q_{2i-1})$, by breaking up the last coefficient $c_{2i-1,t(q_{2i})}$, of $q_{2i-1}(x)$, into $t(q_{2i}) - t(q_{2i-1}) + 1$ equal parts, and

- then pair each of the first λ positive coefficients of $p(x)$, encountered as we move in non-increasing order of exponents, with the first unmatched negative coefficient.

D: "Local-Max" bound, (LM)

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- For the polynomial $p(x) \in \mathbb{R}[x]$

$$p(x) = \alpha_n x^n + \alpha_{n-1} x^{n-1} + \ldots + \alpha_0, \quad (\alpha_n > 0),$$

the coefficient $-\alpha_k$ of the term $-\alpha_k x^k$ in $p(x)$ is paired with the coefficient $\frac{\alpha_m}{2^t}$, of the term $\alpha_m x^m$, where α_m is the largest positive coefficient with $n \geq m > k$ and t indicates the number of times the coefficient α_m has been used.

Example

Consider the polynomial

$$x^3 + 10^{100}x^2 - 10^{100}x - 1,$$

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With C, Cauchy's bound, we pair the terms:

- $\{\frac{x^3}{2}, -10^{100}x\}$ and $\{\frac{x^3}{2}, -1\}$,

and taking the maximum of the radicals we obtain a bound estimate of $1.41421 * 10^{50}$.

Example

Consider the polynomial

$$x^3 + 10^{100}x^2 - 10^{100}x - 1,$$

which has one sign variation and, hence, one positive root equal to 1

With K, Kioustelidis' bound, we pair the terms:

- $\{\frac{x^3}{2^2}, -10^{100}x\}$ and $\{\frac{x^3}{2^3}, -1\}$,

and taking the maximum of the radicals we obtain a bound estimate of $2 * 10^{50}$.

Example

Consider the polynomial

$$x^3 + 10^{100}x^2 - 10^{100}x - 1,$$

which has one sign variation and, hence, one positive root equal to 1

With FL, the "First-λ" bound, we pair the terms:

- $\{x^3, -10^{100}x\}$ and $\{10^{100}x^2, -1\}$,

and taking the maximum of the radicals we obtain a bound estimate of 10^{50}.

Example

Consider the polynomial

$$x^3 + 10^{100}x^2 - 10^{100}x - 1,$$

which has one sign variation and, hence, one positive root equal to 1

With LM, the "Local Max" bound, we pair the terms:

- $\{\frac{10^{100}x^2}{2}, -10^{100}x\}$ and $\{\frac{10^{100}x^2}{2^2}, -1\}$,

and taking the maximum of the radicals we obtain a bound estimate of 2!

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- Kioustelidis' bound is, in general, better (or much better) than Cauchy's; this happens because the former breaks up the leading coefficient in unequal parts, whereas the latter breaks it up in equal parts.

Comparison of the 4 linear complexity bounds

- Empirical results have indicated the following:

- Kioustelidis' bound is, in general, better (or much better) than Cauchy's; this happens because the former breaks up the leading coefficient in unequal parts, whereas the latter breaks it up in equal parts.

- Our *First*-λ bound, as the name indicates, uses additional coefficients and, therefore, it is not surprising that it is, in general, better (or much better) than both previous bounds. In the few cases where Kioustelidis' bound is better than *first*-λ, our *Local-Max* bound takes again the lead.

Conclusions

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- Therefore, to improve the performance of VAS we used two methods — *First-λ* or *FL* and *Local-Max* or *LM* — and took their minimum as the estimated value of the bound. That is, we used $min(FL, LM)$.

Conclusions

- Of the four linear complexity bounds there does not exist one that always computes best estimate values.

- Therefore, to improve the performance of VAS we used two methods — *First-*λ or *FL* and *Local-Max* or *LM* — and took their minimum as the estimated value of the bound. That is, we used $min(FL, LM)$.

- Using $min(FL, LM)$, instead of using Cauchy's bound, the VAS continued fractions method was speeded up, on average, by 15% and became always faster than the VCA bisections method.

The only case where VAS was slower than VCA: Products of terms $x - r$, with random integer r.

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Table: We compare the timings in seconds (s) for: (a) VAS_old, i.e. VAS using Cauchy's rule, (b) VAS_new, i.e. VAS using the new rule $min(FL + LM)$, and (c) VCA_rel. The tests were run on a laptop computer with 1.8 Ghz Pentium M processor, running a Linux virtual machine with 1.78 GB of RAM.

Roots (bit length)	Deg	VAS_old Time(s) Average (Min/Max)	VAS_new Time(s) Average (Min/Max)	VCA_rel Average (Min/Max)
10	100	0.314 (0.248/0.392)	0.253 (0.228/0.280)	0.346 (0.308/0.384)
10	200	1.74 (1.42/2.33)	1.51 (1.34/1.66)	3.90 (3.72/4.05)
10	500	17.6 (16.9/18/7)	17.4 (16.3/18.1)	129 (122/140)
1000	20	0.066 (0.040/0.084)	0.031 (0.024/0.040)	0.038 (0.028/0.044)
1000	50	1.96 (1.45/2.44)	0.633 (0.512/0.840)	1.03 (0.916/1.27)
1000	100	52.3 (36.7/81.3)	12.7 (11.3/14.6)	17.2 (16.1/18.7)

Quadratic complexity bounds (1/2)

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Justification:

- Their improved estimates should compensate for the extra time needed to compute these bounds.

Quadratic complexity bounds (2/2)

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Main idea:

- Each negative coefficient of the polynomial is paired with all the preceding positive coefficients and the minimum of the computed values is associated with this coefficient. The maximum of all those minimums is taken as the estimate of the bound.

Quadratic complexity bounds (2/2)

Main idea:

- Each negative coefficient of the polynomial is paired with all the preceding positive coefficients and the minimum of the computed values is associated with this coefficient. The maximum of all those minimums is taken as the estimate of the bound.

- We will present four quadratic complexity bounds derived from the corresponding four linear complexity bounds discussed before. Of those four, one was developed by Hong in 1998, whereas the other three — including the best and fastest — were developed by Akritas, Argyris, Strzeboński and Vigklas in 2008.

A: Cauchy's quadratic complexity bound, (CQ)

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- For the polynomial $p(x) \in \mathbb{R}[x]$

$$p(x) = \alpha_n x^n + \alpha_{n-1} x^{n-1} + \ldots + \alpha_0, \quad (\alpha_n > 0),$$

each negative coefficient $a_i < 0$ is "paired" with each one of the preceding positive coefficients a_j divided by λ_i — that is, each positive coefficient a_j is "broken up" into equal parts, as is done with *just* the leading coefficient in Cauchy's bound; λ_i is the number of negative coefficients to the right of, and including, a_i — and the minimum is taken over all j; subsequently, the maximum is taken over all i.

That is, we have:

$$ub_{CQ} = \max_{\{a_i<0\}} \min_{\{a_j>0: j>i\}} \sqrt[j-i]{-\frac{a_i}{\frac{a_j}{\lambda_i}}}.$$

B: Kioustelidis' quadratic complexity bound, (KQ),

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each negative coefficient $a_i < 0$ is "paired" with each one of the preceding positive coefficients a_j divided by 2^{j-i} — that is, each positive coefficient a_j is "broken up" into unequal parts, as is done with *just* the leading coefficient in Kioustelidis' linear bound — and the minimum is taken over all j; subsequently, the maximum is taken over all i.

That is, we have:

$$ub_{KQ} = \max_{\{a_i<0\}} \min_{\{a_j>0: j>i\}} \sqrt[j-i]{-\frac{a_i}{\frac{a_j}{2^{j-i}}}}.$$

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- For the polynomial $p(x) \in \mathbb{R}[x]$

$$p(x) = q_1(x) - q_2(x) + q_3(x) - q_4(x) + \ldots + q_{2m-1}(x) - q_{2m}(x) + g(x),$$

with λ negative coefficients we do the following:

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with λ negative coefficients we do the following:

- First we take care of all cases for which $t(q_{2\ell}) > t(q_{2\ell-1})$, by breaking up the last coefficient $c_{2\ell-1,t(q_{2\ell})}$, of $q_{2\ell-1}(x)$, into $d_{2\ell-1,t(q_{2\ell})} = t(q_{2\ell}) - t(q_{2\ell-1}) + 1$ equal parts.

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- Then each negative coefficient $a_i < 0$ is "paired" with each one of the preceding $min(i, \lambda)$ positive coefficients a_j divided by d_j and the minimum is taken over all j; subsequently, the maximum is taken over all i.

...

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- That is, we have:

$$ub_{FLQ} = \max_{\{a_i<0\}} \min_{\{a_j>0:j>min(i,\lambda):u(j)\neq 0\}} \sqrt[j-i]{-\frac{a_i}{\frac{a_j}{d_j}}},$$

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where ...

- d_j indicates the number of equal parts into which each of the preceding $min(i, \lambda)$ positive coefficients a_j is "broken up". The value of d_j is initially set to 1, for each j, and it changes only if the positive coefficient a_j is broken up into equal parts.

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- $u(j)$ indicates the number of times a_j can be used to calculate the minimum.The value of $u(j)$ is originally set equal to d_j and it decreases each time a_j is used in the computation of the minimum.

D: "Local Max" quadratic complexity bound, (LMQ)

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each negative coefficient $a_i < 0$ is "paired" with each one of the preceding positive coefficients a_j divided by 2^{t_j} — where t_j is initially set to 1 and is incremented each time the positive coefficient a_j is used — and the minimum is taken over all j; subsequently, the maximum is taken over all i.

That is, we have:

$$ub_{LMQ} = \max_{\{a_i<0\}} \min_{\{a_j>0: j>i\}} \sqrt[j-i]{-\frac{a_i}{\frac{a_j}{2^{t_j}}}},$$

D: "Local Max" quadratic complexity bound, (LMQ)

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each negative coefficient $a_i < 0$ is "paired" with each one of the preceding positive coefficients a_j divided by 2^{t_j} — where t_j is initially set to 1 and is incremented each time the positive coefficient a_j is used — and the minimum is taken over all j; subsequently, the maximum is taken over all i.

That is, we have:

$$ub_{LMQ} = \max_{\{a_i<0\}} \min_{\{a_j>0: j>i\}} \sqrt[j-i]{-\frac{a_i}{\frac{a_j}{2^{t_j}}}},$$

- Each positive coefficient a_j is "broken up" into unequal parts, as is done with just the locally maximum coefficient in the linear local max bound.

Example

Consider again the polynomial

$$x^3 + 10^{100}x^2 - 10^{100}x - 1,$$

which has one sign variation and, hence, one positive root equal to 1

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With CQ, Cauchy's quadratic complexity bound, we compute:

- the minimum of the two radicals obtained from the pairs of terms $\{\frac{x^3}{2}, -10^{100}x\}$ and $\{\frac{10^{100}x^2}{2}, -10^{100}x\}$ which is 2, and
- the minimum of the two radicals obtained from the pairs of terms $\{x^3, -1\}$ and $\{10^{100}x^2, -1\}$ which is $\frac{1}{10^{50}}$.
- Therefore, the obtained estimate of the bound is $max\{2, \frac{1}{10^{50}}\} = 2$.

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With KQ, Kioustelidis' quadratic complexity bound, we compute:

- the minimum of the two radicals obtained from the pairs of terms $\{\frac{x^3}{2^2}, -10^{100}x\}$ and $\{\frac{10^{100}x^2}{2}, -10^{100}x\}$ which is 2, and
- the minimum of the two radicals obtained from the pairs of terms $\{\frac{x^3}{2^3}, -1\}$ and $\{\frac{10^{100}x^2}{2^2}, -1\}$ which is $\frac{2}{10^{50}}$.
- Therefore, the obtained estimate of the bound is $max\{2, \frac{2}{10^{50}}\} = 2$.

Example

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which has one sign variation and, hence, one positive root equal to 1

With FLQ, the "First-λ" quadratic bound, we compute:

- the minimum of the two radicals obtained from the pairs of terms $\{x^3, -10^{100}x\}$ and $\{10^{100}x^2, -10^{100}x\}$ which is 1, and
- the minimum of the two radicals obtained from the pairs of terms $\{x^3, -1\}$ and $\{10^{100}x^2, -1\}$ which is 1.
- Therefore, the obtained estimate of the bound is $max\{1, 1\} = 1$.
- Note: Once a term with a positive coefficient has been used in obtaining the minimum, it cannot be used again!

Example

Consider again the polynomial

$$x^3 + 10^{100}x^2 - 10^{100}x - 1,$$

which has one sign variation and, hence, one positive root equal to 1

With LMQ, the "Local Max" quadratic bound, we compute:

- the minimum of the two radicals obtained from the pairs of terms $\{\frac{x^3}{2}, -10^{100}x\}$ and $\{\frac{10^{100}x^2}{2}, -10^{100}x\}$ which is 2, and
- the minimum of the two radicals obtained from the pairs of terms $\{\frac{x^3}{2^2}, -1\}$ and $\{\frac{10^{100}x^2}{2^2}, -1\}$ which is $\frac{2}{10^{50}}$.
- Therefore, the obtained estimate of the bound is $max\{2, \frac{2}{10^{50}}\} = 2$.

Linear vs quadratic complexity bounds

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- From the example we see that the estimates of all quadratic complexity bounds are much better than those of their linear complexity counterparts.

- In general, the quadratic complexity bounds cannot perform worse than the linear complexity ones; most of the times they perform a lot better!

Comparison of the 4 quadratic complexity bounds

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- *FLQ* is faster — or quite faster — than all of them because it tests just the first $min(\iota, \lambda)$ positive coefficients. By comparison, all the other quadratic complexity bounds test every preceding positive coefficient.

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  Equality holds when there are no missing terms in the polynomial.

- Experimental results indicated that *FLQ*, *LMQ* and $min(FLQ, LMQ)$ behave equally well! Therefore, we picked pick *LMQ* to improve the performance of *VAS*.

Conclusions on the quadratic complexity bounds

Conclusions on the quadratic complexity bounds

- Using *LMQ*, the performance of the VAS real root isolation method was speeded up by an average overall factor of 40%.

Overall time spent for computing bounds

Overall time spent for computing bounds

- In the following graph the left scale shows the times in seconds (bars) needed by *VAS* to isolate the roots of a certain class of polynomials using both *LM*, the Local Max bound, and *LMQ*, its quadratic version. The right scale is associated with the two curves which show the total time spent by *VAS* in computing the bounds.

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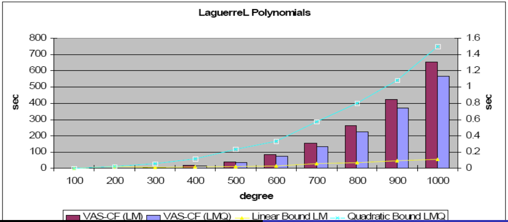

Products of terms $x - r$, with random integer r — Revisited

Products of terms $x - r$, with random integer r — Revisited

Table: We compare the timings in seconds (s) for: (a) VAS(cauchy), i.e. VAS using Cauchy's rule, (b) VAS(fl+lm), i.e. VAS using the linear complexity bound $min(FL + LM)$, and (c) VAS(lmq), i.e. VAS using the Locam Max quadratic complexity bound.The average speed-up for this table is about 35%.

Bit-length of roots	Degree	VAS(cauchy) t(s) Avg(Min/Max)	VAS(fl+lm) t(s) Avg(Min/Max)	VAS(lmq) t(s) Avg(Min/Max)
10	100	0.46 (0.28/0.94)	0.24 (0.18/0.28)	0.34 (0.30/0.41)
10	200	1.46 (1.24/1.85)	1.40 (1.28/1.69)	1.40 (1.20/1.69)
10	500	18.1 (16.5/18.9)	18.1 (16.6/18.8)	22.1 (18.7/24.2)
1000	20	0.07 (0.04/0.14)	0.02 (0.02/0.03)	0.03 (0.02/0.04)
1000	50	3.69 (2.38/6.26)	0.81 (0.60/1.28)	0.81 (0.52/1.11)
1000	100	47.8 (37.6/56.9)	13.8 (10.3/19.2)	15.8 (11.3/21.3)

Future Research

Future Research

Computing time bound

Sharma's bound on the computing time of the *VAS* continued fractions method is greatly overestimated.

Hence, theoretical research is needed to see if we can bring it down.

Future Research

Coefficients

The *VAS* continued fractions method works for integer or rational coefficients.

Hence, we need to discover new ways to deal with coefficients that are algebraic numbers or approximate reals.

Future Research

Sparse polynomials of great degree

The *VAS* continued fractions method is the fastest real root isolation method when the polynomials are not sparse and their degree is less than a few thousand.
However, *Mathematica* runs out of memory when we try to isolate the roots of a sparse polynomial of degree 100000.

Hence, we need to discover new ways to deal with sparse polynomials of extremely high degrees.

Future Research

Parallel implementation

Last, but not least, we need to investigate the performance of the *VAS* continued fractions method in a multiprocessor enviroment.

References

References

- Akritas, A. G.: *Vincent's Theorem in Algebraic Manipulation*. Ph.D. Thesis, Department of Operations Research, North Carolina State University, Raleigh, NC, (1978).
- Akritas, A. G.: "An implementation of Vincent's Theorem". Numerische Mathematik 36 (1980), 53–62.
- Akritas, A. G.: "The fastest exact algorithms for the isolation of the real roots of a polynomial equation". Computing 24 (1980), 299–313.
- Akritas, A. G.: "Reflections on a pair of theorems by Budan and Fourier". Mathematics Magazine 55(5) (1982), 292–298.
- Akritas, A. G.: "There is no "Uspensky's method". Extended Abstract. Proceedings of the 1986 Symposium on Symbolic and Algebraic Computation, Waterloo, Ontario, Canada, 88–90.

References

- Akritas, A. G.:*Elements of Computer Algebra with Applications*. John Wiley Interscience, New York, (1989).
- Akritas, A. G.:"There is no Descartes' method". In M.J.Wester and M. Beaudin (Eds), Computer Algebra in Education, 2007, AullonaPress, USA, 19–35,
- Akritas, A. G.:" Linear and Quadratic Complexity Bounds on the Values of the Positive Roots of Polynomials". JUCS, to appear.
- Akritas, A. G., Argyris, A. I. & Strzeboński, A. W.:"FLQ, the Fastest Quadratic Complexity Bound on the Values of Positive Roots of Polynomials". Serdica Journal of Computing 2, (2008), 145–162.
- Akritas, A. G. & Danielopoulos, S.D.:"On the forgotten theorem of Mr. Vincent". Historia Mathematica 5, (1978), 427–435.

References

- Akritas, A. G. & Danielopoulos, S.D.: "An unknown theorem for the isolation of the roots of polynomials". Bulletin of the Indian Society for History of Mathematics, Ganita-Bharati 2, (1980) 41–49.
- Akritas, A. G. & Strzeboński, A. W.: "A comparative study of two real root isolation methods". *Nonlinear Analysis: Modelling and Control*, **10**(4), (2005), 297–304.
- Akritas, A. G. & Vigklas, P.: "A Comparison of Various Methods for Computing Bounds for Positive Roots of Polynomials". (With P. S. Vigklas). Journal of Universal Computer Science 13(4),(2007), 455–467.
- Akritas, A. G., Strzeboński, A. W. & Vigklas, P.: "Implementations of a New Theorem for Computing Bounds for Positive Roots of Polynomials"; *Computing* **78**, (2006), 355–367.

References

- Akritas, A. G., Strzeboński, A. W. & Vigklas, P.: "Advances on the Continued Fractions Method Using Better Estimations of Positive Root Bounds". In: V. G. Ganzha, E. W. Mayr and E. V. Vorozhtsov (Editors): CASC 2007, pp. 24 – 30, Bonn, Germany, September 16-20, 2007. LNCS 4770, Springer Verlag, Berlin, Heidelberg.
- Akritas, A. G., Strzeboński, A. W. & Vigklas, P.: "On the Various Bisection Methods Derived from Vincent's Theorem". Serdica Journal of Computing, 2, (2008), 89–104.
- Akritas, A. G., Strzeboński, A. W., Vigklas, P.: "Improving the Performance of the Continued Fractions Method Using New Bounds of Positive Roots."; Nonlinear Analysis: Modelling and Control 13(3), (2008), 265–279.

References

- Alesina, A., Galuzzi, M.: "A new proof of Vincent's theorem"; *L'Enseignement Mathemathique* **44**, (1998), 219–256.
- Alesina, A., Galuzzi, M.: Addentum to the paper "A new proof of Vincent's theorem"; *L'Enseignement Mathemathique* 45, (1999), 379–380.
- Alesina, A., Galuzzi, M.: "Vincent's Theorem from a Modern Point of View"; (Betti, R. and Lawvere W.F. (eds.)), Categorical Studies in Italy 2000, Rendiconti del Circolo Matematico di Palermo, Serie II, n. 64, (2000), 179–191.
- Bombieri, E., van der Poorten, A.J.: "Continued fractions of algebraic numbers"; In *Computational Algebra and Number Theory*, (Sydney, 1992), Math. Appl. 325, Kluwer Academic Publishers, Dordrecht, 1995, pp. 137–152.

References

- Boulier, F.: *Systèmes polynomiaux : que signifie "résoudre"?*. Université Lille 1, UE INFO 251, (2007).
- Cantor, D.G., Galyean, P.H., Zimmer, H.G.: "A Continued Fraction Algorithm for Real Algebraic Numbers"; Mathematics of Computation 26(119), (1972), 785–791.
- Collins, G. E. & Akritas, A. G.: "Polynomial real root isolation using Descartes' rule of signs". In: *Proceedings of the 1976 ACM Symposium on Symbolic and Algebraic Computations*, Yorktown Heights, N.Y., (1976), pp. 272–275.
- Hong, H.: "Bounds for absolute positiveness of multivariate polynomials". *J. Symb. Comput.* **25**(5), (1998), 571–585.
- Kioustelidis, I.: "Bounds for positive roots of polynomials"; *J. Comput. Appl. Math.* **16**(2), (1986), 241–244.

References

- Lloyd, E. K.: "On the forgotten Mr, Vincent", Historia Mathematica 6, (1979), 448–450.
- Obreschkoff, N.: *Verteilung und Berechnung der Nullstellen reeller Polynome*. VEB Deutscher Verlag der Wissenschaften, Berlin, (1963).
- Ostrowski, A. M.:"Note on Vincent's Theorem", The Annals of Mathematics, 2nd Series, 52(3), (Nov., 1950), 702–707.
- Rouillier, F. & Zimmermann, P.: "Efficient isolation of polynomial's real roots"; *Journal of Computational and Applied Mathematics* **162**, (2004), pp. 33–50.
- Sharma, V.: "Complexity of Real Root Isolation Using Continued Fractions"; In: *ISAAC07* preprint, 2007.

References

- Sharma, V.: *Complexity Analysis of Algorithms in Algebraic Computation*; Ph.D. Thesis, Department of Computer Sciences, Courant Institute of Mathematical Sciences, New York University, 2007.
- Ştefănescu, D.: "New bounds for positive roots of polynomials"; *Journal of Universal Computer Science* **11**(12), (2005), 2132–2141.
- Ştefănescu, D.: "Bounds for Real Roots and Applications to Orthogonal Polynomials". In: V. G. Ganzha, E. W. Mayr and E. V. Vorozhtsov (Editors): CASC 2007, pp. 377 – 391, Bonn, Germany, September 16-20, 2007. LNCS 4770, Springer Verlag, Berlin, Heidelberg.

References

- Tsigaridas, E. P., Emiris, I. Z.: "Univariate polynomial real root isolation: Continued fractions revisited". In: Y. Azar and T. Erlebach (Eds.), ESA 2006, LNCS 4168, (2006), 817–828.
- Uspensky, J. V.: "Theory of Equations"; McGraw-Hill, New York, (1948).
- Vincent, A. J. H.: "Sur la resolution des équations numériques"; Journal de Mathématiques Pures et Appliquées 1, (1836), 341–372.