Vincent's Theorem of 1836: Overview and Future Research[1]

Alkiviadis G. Akritas
Department of Computer and Communication Engineering
University of Thessaly
Greece
akritas@uth.gr

Abstract We first present the two different versions of Vincent's theorem of 1836 and discuss the various real root isolation methods derived from them: one using continued fractions and two bisection methods — the former being the fastest real root isolation method. We then concentrate on the Continued Fractions method and present: (a) a recently developed quadratic complexity bound on the values of the positive roots of polynomials, which helped improve its performance by an average of 40%, over its initial implementation, and (b) directions for future research.

Key Words: Vincent's theorem, isolation of the real roots, real root isolation methods, bisection methods, continued fractions method, positive root bounds.

1 Introduction

Isolation of the real roots of a polynomial is the process of finding real disjoint intervals such that each contains one real root and every real root is contained in some interval.

Since the beginning of the 19-th century, and according to the French "school" of mathematics, isolation has been considered the first step in finding the real roots of a polynomial equation — the second step being the approximation of the roots to any degree of accuracy.

Sturm was the first mathematician to present a theorem, in 1829, for isolating the real roots of a polynomial using bisection, [6]. His theorem has been widely used until 1980, when it was replaced — in the major computer algebra systems — by versions of Vincent's theorem.

Vincent's theorem of 1836, [38], has a very interesting and exciting history [10], [11], [18], [19], [20] and [28]. It was almost totally forgotten until it was rediscovered, in 1976, by the author and formed the basis of his Ph.D. Thesis, [1]. Subsequently scientists from all over the world made their own contributions on various aspects of it, so that we can today claim that we have a very good understanding of it.

[1] This presentation is based on the *plenary talk* the author gave at ACA 2008, the International Conference on Applications of Computer Algebra, held at RISC-Linz, Hagenberg, Austria (July 27-30, 2008).

A short biography of Vincent (in French) can be found in p. 1026, vol 31 of "La Grande Encyclopédie", see http://gallica.bnf.fr/ark:/12148/bpt6k24666x, whereas in http://www.allposters.fr/-st/Lasnier-Affiches_c25893_s88165_.htm a copy of his portrait can be seen, [23].

Vincent's theorem depends on Descartes' rule of signs, which gives us an *upper bound* on the number of the positive roots of a polynomial, [22]. Specifically, consider the polynomial $p(x) \in \mathbb{R}[x]$, $p(x) = a_n x^n + \cdots + a_1 x + a_0$ and let $var(p)$ represent the number of sign *variations* or *changes* (positive to negative and vice-versa) in the sequence of coefficients $a_n, a_{n-1}, \ldots, a_0$.

Descartes' rule of signs: The number $\varrho_+(p)$ of real roots — multiplicities counted — of the polynomial $p(x) \in \mathbb{R}[x]$ in the open interval $]0, +\infty[$ is bounded above by $var(p)$; that is, we have $var(p) \geq \varrho_+(p)$.

According to Descartes' rule of signs if $var(p) = 0$ it follows that $\varrho_+(p) = 0$.

Additionally, according to Descartes' rule of signs, the mean value theorem and the fact that the polynomial functions are continuous, if $var(p) = 1$ it follows that $\varrho_+(p) = 1$, [22].

Therefore, Descartes' rule of signs yields the *exact* number of positive roots *only* in the two special cases mentioned above[2].

These two special cases of Descartes' rule are used in both versions of Vincent's theorem of 1836, which are presented in the talk.

The rest of the presentation is structured as follows:

- We first present the two versions of Vincent's theorem and provide a sketch of one of its proofs, [1], [10], [11], [18], [19], [20], [21], [24], [28], [29], [30], [38].
- We then explain how Vincent's theorem can be used to isolate the real roots of polynomials and describe the continued fractions method and two bisections methods derived from it, [2], [3], [4], [5], [7], [12], [14], [16], [17], [25], [31], [32], [33], [36], [37].
- Subsequently, we present recently developed, linear and quadratic complexity bounds on the values of the positive roots of polynomials and elaborate on their impact on the performance of the continued fractions method, [8], [9], [13], [14], [26], [27], [34], [35].
- Finally, we indicate areas of future research: when the coefficients of the polynomial are algebraic numbers or approximate reals.

[2] These two special cases were known to Cardano; in other words, what Descartes did was to generalize "Cardano's *special* rule of signs". This detail is mentioned in [6].

References

1. Akritas, A. G.:“Vincent’s theorem in algebraic manipulation”; Ph.D. Thesis, Operations Research Program, North Carolina State University, Raleigh, NC, (1978).
2. Akritas, A. G.:“An implementation of Vincent’s Theorem”; Numerische Mathematik, 36, (1980), 53–62.
3. Akritas, A. G.:“The fastest exact algorithms for the isolation of the real roots of a polynomial equation”; Computing, 24, (1980), 299–313.
4. Akritas, A. G.:“Reflections on a pair of theorems by Budan and Fourier”; Mathematics Magazine, 55, 5, (1982), 292–298.
5. Akritas, A. G.:“There is no ‘Uspensky’s method’ ”; Proceedings of the 1986 Symposium on Symbolic and Algebraic Computation, Waterloo, Ontario, Canada, (1986), 88–90.
6. Akritas, A. G.:“Elements of Computer Algebra with Applications”; John Wiley Interscience, New York, (1989).
7. Akritas, A. G.:“There is no ‘Descartes’ method’ ”; Proceedings of the session “Computer Algebra in Education” of ACA 2007, the International Conference on Applications of Computer Algebra, Oakland University, Rochester, Michigan, USA (July 19-22), (2007).
8. Akritas, A. G.:“Linear and quadratic complexity bounds on the values of the positive roots of polynomials ”. Submitted.
9. Akritas, A. G., Argyris, A. I., Strzeboński, A. W.:“FLQ, the Fastest Quadratic Complexity Bound on the Values of Positive Roots of Polynomials.”; Serdica Journal of Computing, Vol. 2, 145–162, 2008.
10. Akritas, A. G., Danielopoulos, S. D.: “On the forgotten theorem of Mr. Vincent”; Historia Mathematica, 5, (1978), 427–435.
11. Akritas, A. G., Danielopoulos, S. D.: “An unknown theorem for the isolation of the roots of polynomials”; Ganita-Bharati (Bulletin of the Indian Society for History of Mathematics), 2, (1980), 41–49.
12. Akritas, A. G., Strzeboński , A.:“A comparative study of two real root isolation methods”; Nonlinear Analysis: Modelling and Control, 10, 4, (2005), 297–304.
13. Akritas, A. G., Vigklas, P.: “A Comparison of Various Methods for Computing Bounds for Positive Roots of Polynomials”; Journal of Universal Computer Science, 13, 4, (2007), 455–467
14. Akritas, A. G., Strzeboński , A., Vigklas, P.:“Implementations of a New Theorem for Computing Bounds for Positive Roots of Polynomials”; Computing, 78, (2006), 355–367.
15. Akritas, A. G., Strzeboński , A., Vigklas, P.:“Advances on the Continued Fractions Method Using Better Estimations of Positive Root Bounds”; Proceedings of the 10th International Workshop on Computer Algebra in Scientific Computing, CASC 2007, pp. 24 – 30, Bonn, Germany, September 16-20, 2007. LNCS 4770, Springer Verlag, Berlin. Edited by V. G. Ganzha, E. W. Mayr and E. V. Vorozhtsov..
16. Akritas, A. G., Strzeboński , A., Vigklas, P.:“On the Various Bisection Methods Derived from Vincent’s Theorem”, Serdica Journal of Computing, 2, (2008), 89–104.
17. Akritas, A. G., Strzeboński , A., Vigklas, P.: Improving the Performance of the Continued Fractions Method Using new Bounds of Positive Roots. Nonlinear Analysis: Modelling and Control, Vol. 13, No. 3, 265-279, 2008. .
18. Alesina, A., Galuzzi, M.: “A new proof of Vincent’s theorem”; L’Enseignement Mathemathique, 44, (1998), 219–256.
19. Alesina, A., Galuzzi, M.: Addentum to the paper “A new proof of Vincent’s theorem”; L’Enseignement Mathemathique, 45, (1999), 379–380.
20. Alesina, A., Galuzzi, M.: “Vincent’s Theorem from a Modern Point of View”; (Betti, R. and Lawvere W.F. (eds.)), Categorical Studies in Italy 2000, Rendiconti del Circolo Matematico di Palermo, Serie II, n. 64, (2000), 179–191.

21. Bombieri, E., van der Poorten, A. J.: "Continued fractions of algebraic numbers"; In *Computational Algebra and Number Theory*, (Sydney, 1992), Math. Appl. 325, Kluwer Academic Publishers, Dordrecht, 1995, pp. 137–152.
22. Boulier, F.: "Systèmes polynomiaux : que signifie "résoudre" ?"; Lecture Notes, Université Lille 1, 8 janvier 2007. **http://www2.lifl.fr/~boulier/RESOUDRE/SHARED/support.pdf** or **http://www.fil.univ-lille1.fr/portail/ls4/resoudre**
23. Boulier, F.: Private Communication. October 2007.
24. Cantor, D. G., Galyean, P. H., Zimmer, H. G.:"A Continued Fraction Algorithm for Real Algebraic Numbers"; Mathematics of Computation, 26 (119), (1972), 785–791.
25. Collins, G. E., Akritas, A. G.: "Polynomial real root isolation using Descartes' rule of signs"; Proceedings of the 1976 ACM Symposium on Symbolic and Algebraic Computations, Yorktown Heights, N.Y., (1976), 272–275.
26. Hong, H.: "Bounds for absolute positiveness of multivariate polynomials". J. Symb. Comput., 25 (5), (1998), 571–585.
27. Kioustelidis, B.:"Bounds for positive roots of polynomials"; J. Comput. Appl. Math., 16, 2, (1986), 241–244.
28. Lloyd, E. K.: "On the forgotten Mr, Vincent"; Historia Mathematica, 6, (1979), 448–450.
29. Obreschkoff, N.:"Verteilung und Berechnung der Nullstellen reeller Polynome"; VEB Deutscher Verlag der Wissenschaften, Berlin, (1963)[3].
30. Ostrowski, A. M.:"Note on Vincent's Theorem"; The Annals of Mathematics, 2nd Series, 52, 3, (Nov., 1950), 702–707.
31. Rouillier, F., Zimmermann, P.: "Efficient isolation of polynomial's real roots"; Journal of Computational and Applied Mathematics, 162, (2004), 33–50.
32. Sharma, V.: "Complexity of Real Root Isolation Using Continued Fractions"; ISAAC07 preprint, 2007.
33. Sharma, V.: "Complexity Analysis of Algorithms in Algebraic Computation"; Ph.D. Thesis, Department of Computer Sciences, Courant Institute of Mathematical Sciences, New York University, 2007.
34. Ştefănescu, D.:"New bounds for positive roots of polynomials"; Journal of Universal Computer Science, **11**(12), (2005), 2132–2141.
35. Ştefănescu, D.:"Bounds for Real Roots and Applications to Orthogonal Polynomials". In: V. G. Ganzha, E. W. Mayr and E. V. Vorozhtsov (Editors): *Proceedings of the 10th International Workshop on Computer Algebra in Scientific Computing*, CASC 2007, pp. 377 – 391, Bonn, Germany, September 16-20, 2007. LNCS 4770, Springer Verlag, Berlin, Heidelberg.
36. Tsigaridas, E. P., Emiris, I. Z.: "Univariate polynomial real root isolation: Continued fractions revisited"; (Y. Azar and T. Erlebach (Eds.)), ESA 2006, LNCS 4168, (2006), 817–828.
37. Uspensky, J. V.:"Theory of Equations"; McGraw-Hill, New York, (1948).
38. Vincent, A. J. H.:"Sur la resolution des équations numériques"; Journal de Mathématiques Pures et Appliquées, 1, (1836), 341–372.

[3] For an English translation of a book with similar content see: Obreschkoff, N.: "Zeros of Polynomials", Bulgarian Academic Monographs (7), Sofia, 2003.