Bull. Math. Soc. Sci. Math. Roumanie
Tome 53(101) No. 3, 2010, 201–211

Counting the number of real roots in an interval with Vincent's theorem

by

Alkiviadis G. Akritas and Panagiotis S. Vigklas

Dedicated to the memory of Laurenţiu Panaitopol (1940-2008) on the occasion of his 70th anniversary

Abstract

It is well known that, in 1829, the French mathematician Jacques Charles François Sturm (1803-1855) solved the problem of finding the number of *real* roots of a polynomial equation $f(x) = 0$, with rational coefficients and without multiple roots, over a given interval, say $]a, b[$. As a byproduct, he also solved the related problem of isolating the real roots of $f(x)$. In 1835 Sturm published another theorem for counting the number of *complex* roots of $f(x)$; this theorem applies only to *complete* Sturm sequences and was recently extended to Sturm sequences with at least one missing term.

Less known, however, is the fact that Sturm's fellow countryman and contemporary Alexandre Joseph Hidulphe Vincent (1797-1868) also presented, in 1836, another theorem for the *isolation* (only) of the *positive* roots of $f(x)$ using continued fractions. In its latest implementation, the Vincent-Akritas-Strzeboński (`VAS`) continued fractions method for the isolation of real roots of polynomials turns out to be the fastest method derived from Vincent's theorem, by far outperformes the one by Sturm, and has been implemented in major computer algebra systems.

In this paper we use the `VAS` real root isolation method to count the number of real and complex roots of $f(x)$ as well as the number of real roots $f(x)$ has in an open interval $]a, b[$.

Key Words: Root counting, real roots, polynomial, real roots isolation, Vincent's theorem, Sturm's theorem, Sturm sequences, Sylvester's matrix.

2000 Mathematics Subject Classification: Primary 12D10, 12E05, 12E12; Secondary 26C10.

1 Introduction

The famous theorem by Sturm appeared in 1829, [12], and it solved the problem of counting the number of real roots of a polynomial equation $f(x) = 0$, over a given interval, say $]a, b[$. Using this theorem Sturm was also able to isolate the real roots of $f(x)$; that is, he was able to find open intervals such that each contains one real root and each real root is contained in some interval. Before we state this theorem, we need the following definition:

Sturm's Sequence or Chain. Let $f(x) = 0$ be a polynomial equation of degree n, with rational coefficients and without multiple roots. The *Sturm sequence* or *chain* of $f(x)$ is

$$S_{seq}(x) = \{f(x), f'(x), r_1(x), r_2(x), \ldots, r_k(x)\}, \tag{1}$$

where $f'(x)$ is the first derivative of $f(x)$ and the polynomials $r_i(x), 1 \leq i \leq k \leq n-1$, are the *negatives* of the remainders obtained by applying the Euclidean *gcd* algorithm on $f(x)$ and $f'(x)$.

That is, we have:

$$\begin{aligned} f(x) &= f'(x)q_1(x) - r_1(x) \\ f'(x) &= r_1(x)q_2(x) - r_2(x) \\ &\vdots \\ r_{k-2} &= r_{k-1}q_k(x) - r_k(x) \end{aligned}$$

When $k = n-1$, that is, when there are no missing terms, the Sturm sequence is called *complete*, and when $k < n-1$, it is called *incomplete*. The condition that $f(x)$ has only simple roots is required for the real root isolation process and it does not restrict the generality of our discussion.

The computation of the Sturm sequence poses several problems when performed in $\mathbb{Z}[x]$, since the latter is *not* a Euclidean domain. In this case we have to do *pseudo*-divisions, that is we have to pre-multiply *each* dividend times the leading coefficient of the divisor raised to some power — for the results to be in $\mathbb{Z}[x]$. This is the classical *Euclidean polynomial remainder sequence* (prs) algorithm.

However, pseudo-division leads to explosive coefficient growth, which has to be controlled. This can be done in various ways; namely, we can use one of the following:

- the *primitive prs* algorithm, which means dividing out the *content* of each pseudo-remainder before using it, [4]. Since this process involves computing the gcd of the coefficients of each remainder — we would like to avoid it.

- the *Sylvester-Habicht subresultant prs* algorithm. Sylvester initiated this process back in 1853, [14], for complete Sturm sequnces (*reduced subresultant prs* algorithm) and Habicht wrapped it up in 1948, [9], for incomplete Sturm sequences (*subresultant prs* algorithm). In this case — without computing any gcd's — we divide out of each remainder a certain quantity, knowing "a priori" that the division will be exact!

- the *matrix-triangularization subresultant prs* algorithm which is equivalent to the one by Sylvester-Habicht. This was initiated by Van Vleck in 1900, [15], for complete Sturm sequences, and was wrapped up (within signs) by Akritas in the 1990's, [1], [2], [3], [4] for incomplete Sturm sequences; finally, the exact computation of the signs of the polynomials in the Sturm sequence was achieved by Akritas, Akritas and Malaschonok in 1995, [5], with the introduction of a new type of resultant matrix.

We can now present Sturm's theorem, which makes use of the sequence defined above:

Theorem 1. (Sturm's Theorem of 1829 for real roots) *Let $f(x) = 0$ be a polynomial equation of degree n, with rational coefficients and without multiple roots. Then the number ϱ of its real roots in the open interval $]a, b[$ satisfies the equality*

$$\varrho = \nu_a - \nu_b, \tag{2}$$

where ν_a, ν_b is the number of sign variations in the Sturm sequence $S_{seq}(a)$, $S_{seq}(b)$, respectively.

This is the theorem for which Sturm is mostly remembered for and its proof can be found in almost all texts of Numerical Analysis. To isolate the real roots of $f(x)$ Sturm suggested to first isolate the positive roots and then the negative ones (by replacing x by $-x$ in $f(x)$), not forgetting to check if 0 is a root. To isolate the positive roots, all we have to do is to: (a) compute the Sturm sequence $S_{seq}(x)$, (b) compute an upper bound ub, on the values of the positive roots of $f(x)$, and (c) bisect the interval $]a, b[=]0, ub[$ until root isolation has been accomplished.

However, there is yet another theorem by Sturm, published in 1835, [13], which deals with the number of *pairs* of complex roots. This theorem — whose proof is also in the literature — can be stated as follows:

Theorem 2. (Sturm's Theorem of 1835 for complex roots) *Let $f(x) = 0$ be a polynomial equation of degree n, with rational coefficients and without multiple roots. Then the number of pairs of complex roots of $f(x)$ is equal to the number of sign variations in the sequence of the leading coefficients of the polynomials in the tail of the Sturm sequence, where*

$$\boldsymbol{tail}(S_{seq}(x)) = \{f'(x), r_1(x), r_2(x), \ldots, r_{n-1}(x)\}. \tag{3}$$

The above theorem is true *only* when $\mathtt{tail}(S_{seq}(x))$ is *complete*. As explained in section 2, a technique was introduced by a group of Chinese mathematicians, [10], [17], [18], so that Theorem 2 can be used when the tail sequence (3) is incomplete; they have also extended Theorem 2 in such a way that it not only counts the number of complex roots of $f(x)$ but can be also used as a criterion for the number of positive or negative real roots.

It is obvious that with the two theorems by Sturm mentioned above we can:

a. compute the exact number of the real roots $f(x)$ has in an open interval $]a, b[$,

b. isolate the real roots of $f(x)$,

c. compute the number of the real and complex roots of $f(x)$.

The first two items, (a) and (b), depend *only* on Theorem 1, whereas the last item, (c), depends on *both* Theorem 1 and Theorem 2.

To count the number of complex roots using Theorem 1, all we have to do is compute ub, an upper bound on the absolute values of the roots, use (2) to evaluate the exact number of real roots in the interval $]-ub, ub[$ and subtract it from n, the degree of $f(x)$.

On the other hand, using Theorem 2 we can count the complex roots only in the case when the Sturm sequence is complete; if the sequence is incomplete, we have to resort either to the process mentioned above using Theorem 1, or to subresultants, mentioned in section 2.

The rest of the paper is structured as follows:

In section 2 we first introduce Sylvester's matrix and subresultants and then describe the technique introduced by the Chinese group of mathematicians, [17], [18] and [10], with the help of which the signs of the missing leading coefficients in (3) are "filled in" — in an easy to remember manner. In this way, incomplete tail sequences can be now handled by Theorem 2.

As was pointed out in the literature, [17], [18], [10], subresultants along with the "Chinese" technique and extended versions of Theorem 2 are faster and better suited for polynomials with symbolic coefficients.

In section 3 we present Vincent's theorem of 1836 in its original — continued fractions — form; with this we can isolate the real roots of $f(x)$, and, hence, we can compute the number of its real and complex roots. In the sequel we present the bisection version of Vincent's theorem, which was presented in 2000 by Alesina and Galuzzi, [8]; using the transformation mentioned in this theorem we then present an algorithm to count the number of real roots of $f(x)$ in any real open interval $]a, b[$.

2 The subresultant version of Sturm's theorem and the extension of Theorem 2

As it was stated in the Introduction, it is well known that *all* the coefficients of the polynomials in the tail of the Sturm sequence, $\mathtt{tail}S_{seq}(x)$, can be computed as subresultants of Sylvester's matrix, [14], [15], [9], [2], [4] and [5]. Since our purpose is to explain the "Chinese" technique for making Theorem 2 work with incomplete sequences, we will concentrate only on the leading coefficients of the polynomials in $\texttt{tail}S_{seq}(x)$.

Given the polynomial

$$f(x) = \alpha_0 x^n + \alpha_1 x^{n-1} + \ldots + \alpha_n,$$

without multiple roots, and its derivative

$$f'(x) = 0 \cdot x^n + n \cdot \alpha_0 x^{n-1} + (n-1)\alpha_1 x^{n-2} + \ldots + \alpha_{n-1},$$

the Sylvester's matrix of $f(x)$ and $f'(x)$ (which is also referred to as the *discrimination matrix*, *Discr*(f), of $f(x)$, [10], [17], [18]) is the following $2n \times 2n$ matrix:

$$\begin{pmatrix}
\alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_n & & & \\
0 & n\alpha_0 & (n-1)\alpha_1 & \cdots & \alpha_{n-1} & & & \\
 & \alpha_0 & \alpha_1 & \cdots & \alpha_{n-1} & \alpha_n & & \\
 & 0 & n\alpha_0 & \cdots & 2\alpha_{n-2} & \alpha_{n-1} & & \\
 & & & \cdots & \cdots & & & \\
 & & & \cdots & \cdots & & & \\
 & & & & \alpha_0 & \alpha_1 & \cdots & \alpha_n \\
 & & & & 0 & n\alpha_0 & \cdots & \alpha_{n-1}
\end{pmatrix}$$

If by $d_k(f)$ we denote the determinant of the submatrix formed by the first $2k$ rows and the first $2k$ columns of Sylvester's matrix, for $k = 1, 2, \ldots, n$, then the leading coefficients of the polynomials in the tail sequence (3) are

$$\{d_1(f), d_2(f), \cdots, d_n(f)\}, \tag{4}$$

which in the sequel will be called the *leading-coefficients sequence* — our terminology differs slightly from that of our Chinese colleagues, who call it the "discriminant sequence". From (4) we form the corresponding *signs sequence*, $\{s_1, s_2, \cdots, s_n\}$, as

$$\{sign(d_1(f)), sign(d_2(f)), \cdots, sign(d_n(f))\}, \tag{5}$$

where *sign* is the known signus function:

$$sign(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

If the signs sequence (5) contains no zero, then Theorem 2 can be used to count the number of complex roots. Note that because of our assumption that

$f(x)$ contains no multiple roots, zeros *cannot* appear at the end of the signs sequence.

However, if zeros *do* appear in (5) then — in order to apply Theorem 2 — we have to construct the *revised* signs sequence, $\{\sigma_1, \sigma_2, \cdots, \sigma_n\}$, of the corresponding signs sequence $\{s_1, s_2, \cdots, s_n\}$, [10], [17], [18].

The construction of the *revised* signs sequence proceeds as follows:

- If $\{s_i, s_{i+1}, \cdots, s_{i+j}\}$ is a section of the given sequence, with $\{s_i \neq 0; s_{i+1} = s_{i+2} = \cdots = s_{i+j-1} = 0; s_{i+j} \neq 0\}$, then replace the subsection of the zero terms $\{s_{i+1}, s_{i+2}, \cdots, s_{i+j-1}\}$ by $\{-s_i, -s_i, s_i, s_i, -s_i, -s_i, \cdots,\}$
- otherwise make no changes for the other terms.

Example: For the polynomial $f(x) = x^9 - 7x + 7$ we obtain the leading-coefficients sequence

$$\{9, 0, 0, 0, 0, 0, 0, -15543853645824, 1556380841389577\}$$

or the signs sequence

$$\{1, 0, 0, 0, 0, 0, 0, -1, 1\}.$$

Obviously, due to the presence of zeros, Theorem 2 cannot be used. However, the revised signs sequence is

$$\{1, -1, -1, 1, 1, -1, -1, -1, 1\}$$

with 4 sign variations; hence, from Theorem 2 we deduce that $f(x)$ has 4 pairs of complex roots and $9 - 2 * 4 = 1$ real root.

Note that the Sturm sequence, $S_{seq}(x)$, of $f(x) = x^9 - 7x + 7$ is

$$\{x^9 - 7x + 7, 9x^8 - 7, 504x - 567, -1556380841389577\},$$

where the signs of the leading coefficients differ from those of the leading-coefficients sequence. As a result, we cannot use (1) to obtain the revised signs sequence!

As stated in the Introduction, when the coefficients are symbolic, it is best to use the subresultants of the Sylvester matrix as described above.

3 The two versions of Vincent's theorem

Vincent's theorem, in its original form, can be stated as follows:

Theorem 3. (Vincent's Theorem of 1836 — the continued fractions version) *If in a polynomial, $f(x)$, of degree n, with rational coefficients and without multiple roots we perform sequentially replacements of the form*

$$x \leftarrow \alpha_1 + \frac{1}{x}, x \leftarrow \alpha_2 + \frac{1}{x}, x \leftarrow \alpha_3 + \frac{1}{x}, \ldots$$

where $\alpha_1 \geq 0$ *is an arbitrary non negative integer and* $\alpha_2, \alpha_3, \ldots$ *are arbitrary positive integers,* $\alpha_i > 0$*,* $i > 1$*, then the resulting polynomial either has no sign variations or it has one sign variation. In the first case there are no positive roots whereas in the last case the equation has exactly one positive root, represented by the continued fraction*

$$\alpha_1 + \cfrac{1}{\alpha_2 + \cfrac{1}{\alpha_3 + \cfrac{1}{\ddots}}}. \tag{6}$$

In his paper of 1836, [16], Vincent proved this theorem and presented several examples to demonstrate the concepts involved. However, his theorem appeared several years *after* Sturm's theorem on counting the number of real roots in an interval and isolating them. Due to Sturm's fame and priority, Vincent's theorem was almost totally forgotten. An interesting account of the history of this theorem, along with an overview of its various proofs etc, can be found elsewhere, [6].

The termination of the process described in Vincent's theorem is guaranteed by the following theorem, [11]:

Theorem 4. (Obreschkoff's Theorem of 1920) *If a real polynomial has one positive simple root* x_0 *and all the other — possibly multiple — roots lie in the cone or sector*

$$S_{\sqrt{3}} = \{x = -\alpha + \imath\beta \ \mid \ \alpha > 0 \quad and \quad \beta^2 \leq 3\alpha^2\}$$

then the sequence of its coefficients has exactly one sign variation.

As can be seen from its statement, Theorem 3 can be used to isolate the *positive* roots of a polynomial equations $f(x) = 0$; call `VAS_positive_roots` the corresponding algorithm, [7]. For the negative roots we simply replace $x \leftarrow -x$ and repeat the process.

To isolate the positive roots Vincent computed each partial quotient a_i by a series of *unit* increments $a_i \leftarrow a_i + 1$ which are equivalent to substitutions of the form $x \leftarrow x + 1$. This approach resulted in an exponential method and for a discussion on how this problem was solved we refer the reader to the literature, [6].

From the above it becomes clear that the continued fractions version of Vincent's theorem can be used to isolate the real roots of $f(x)$ and, as a byproduct, to count the number of its real and complex roots; but it *cannot* be used to count the number of the real roots of $f(x)$ in an open interval $]a, b[$. For the latter we need the bisection version of Vincent's theorem, due to Alesina and Galuzzi, [8]:

Theorem 5. (Vincent's Theorem — the bisections version of 2000) *Let* $f(x)$*, be a polynomial of degree* n*, with rational coefficients and without multiple roots. It* *is possible to determine a positive quantity δ so that for every pair of* positive *rational numbers a, b with $|b - a| < \delta$, every transformed polynomial of the form*

$$\phi(x) = (1+x)^n f(\frac{a+bx}{1+x}) \tag{7}$$

has exactly 0 or 1 variations in the sequence of its coefficients. The second case is possible if and only if $f(x)$ has a simple root within $]a, b[$.

In the sequel we will refer to the transformation mentioned above, as the $\phi(x)$-transformation. With the help of the above theorem — whose proof can be found in the literature — we can easily count the number of roots in a given interval. All we have to do is to perform the transformation and then to isolate the *positive* roots of the transformed polynomial $\phi(x)$. We are not interested in the roots themselves of $\phi(x)$; we simply want to know their cardinality.

Note, however, that *both* endpoints of the open interval $]a, b[$ have to be positive; a different approach is stated elsewhere ([18], Theorem 5, p. 145).

As Alesina and Galuzzi pointed out, [8], the substitution of x, in $f(x) = 0$, by the continued fraction (6) or, equivalently, the $\phi(x)$-transformation (7) result in all roots outside the three circles being placed in Obreschkoff's *cone* or *sector*, shown in Figure 1.

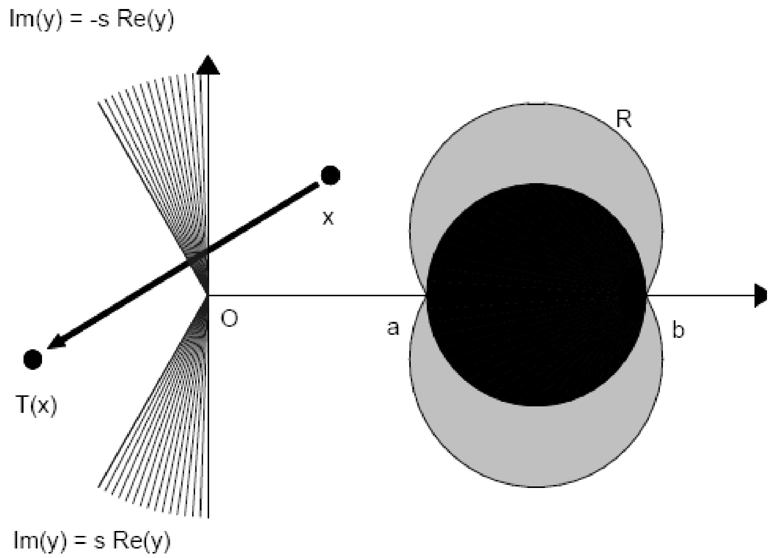

Figure 1: Obreschkoff's cone or sector.

Below is the algorithm for counting the number of real roots in *any* interval:

Algorithm to count the number of real roots in any open interval:

1. If both endpoints are positive, i.e. $a > 0$ and $b > 0$, then perform the $\phi(x)$-transformation and use `VAS_positive_roots` to isolate the positive roots of $\phi(x)$; return their cardinality.

2. If both endpoints are negative, i.e. $a < 0$ and $b < 0$, then replace $x \leftarrow -x$, in $f(x)$, along with $a \leftarrow -a$ and $b \leftarrow -b$, for the endpoints; after that, perform the $\phi(x)$-transformation and use `VAS_positive_roots` to isolate the positive roots of $\phi(x)$; return their cardinality.

3. If $a < 0$ and $b > 0$, perform the $\phi(x)$-transformation with $]a, b[\ =\ ]0, b[$ and use `VAS_positive_roots` to isolate, and count the cardinality of the positive roots of $\phi(x)$; next replace $x \leftarrow -x$, in $f(x)$, perform once again the $\phi(x)$-transformation with $]a, b[\ =\ ]0, -a[$, and use `VAS_positive_roots` to isolate, and count the cardinality of the positive roots of $\phi(x)$; return their total cardinality, adding one if 0 is a root of $f(x)$.

Obviously, the third case is the most time consuming, since it requires two $\phi(x)$-transformations.

4 Conclusions

From the above we see that in order to count the number of real roots of $f(x)$ in a given interval $]a, b[$ there are alternatives to Sturm's method. The first author was astonished to see how students exposed to the theorems by Sturm and Vincent, never think of using the latter for counting the roots. Hopefully our presentation will bring some change.

References

[1] Alkiviadis G. Akritas, *A simple proof of the validity of the reduced prs algorithm*, Computing, **38** (1987), pp. 369–372.

[2] Alkiviadis G. Akritas, *A new method for computing polynomial greatest common divisors and polynomial remainder sequences*, Numerische Mathematik, **52** (1988), 119–127.

[3] Alkiviadis G. Akritas, *Exact algorithms for the matrix-triangularization subresultant prs method*, In Erich Kaltofen, Steven Watt (Eds), Proceedings of the Conference on Computers and Mathematics, Boston, Massachusetts, June, (1989), pp. 145–155.

[4] Alkiviadis G. Akritas, *Elements of Computer Algebra—with Applications*, Wiley, New York, NY, 1989. Available also in Russian, MIR Publishers, Moscow, 1994 (with additional material).

[5] Alkiviadis G. Akritas, Evgenia K. Akritas and Genadii I. Malaschonok, *Matrix computations of subresultant polynomial remainder sequences in integral domains*, Reliable Computing, **1** (1995), pp. 375–381.

[6] Alkiviadis G. Akritas, *Vincent's Theorem of 1836: Overview and Future Research*, In N.N. Vassiliev, A.M Vershik (Eds.) Zap. Nauchnyh Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), Teor. Predst. Din. Sist. Komb. Metody. XVII, **373** (2009), pp. 5–33. (To be published by Springer Verlag, in the *Journal of Mathematical Sciences*.)

[7] Alkiviadis G. Akritas, Adam W. Strzeboński and Panagiotis S. Vigklas, *Improving the Performance of the Continued Fractions Method Using new Bounds of Positive Roots*, Nonlinear Analysis: Modelling and Control, **13:3**, (2008), pp. 265–279.

[8] Alberto Alesina and Massimo Galuzzi, *Vincent's Theorem from a Modern Point of View*, In R. Betti and W.F. Lawvere (Eds.), Categorical Studies in Italy 2000, Rendiconti del Circolo Matematico di Palermo, Serie II, **64**, (2000), pp. 179–191.

[9] Walter Habicht, *Eine Verallgemeinerung des Sturmschen Wurzelzaelverfahrens*, Commentarii Mathematici Helvetici, **21** (1948), pp. 99–116.

[10] Songxin Liang and Jingzhong Zhang, *A complete discrimination system for polynomials with complex coefficients and its automatic generation*, Science in China (Series E), **42:2** (1999), pp. 113–128.

[11] Nikola Obreschkoff, *Verteilung und Berechnung der Nullstellen reeller Polynome*. VEB Deutscher Verlag der Wissenschaften, Berlin, (1963).

[12] Jacques Charles François Sturm, *Résolution des équations algébriques*, Bulletin des Sciences de Ferussac, **11** (1829), pp. 419–425.

[13] Jacques Charles François Sturm, *Mémoire sur la résolution des équations numériques*, Mémoires des Savants Etrangers, **6** (1835), pp. 271–318.

[14] James J. Sylvester, *On a theory of the syzygetic relations of two rational integral functions, comprising an application to the theory of Sturm's functions, and that of the greatest common measure*, Philosophical Transactions, **143** (1853), pp. 407–548.

[15] Edward Burr Van Vleck, *On the determination of a series of Sturm's functions by the calculation of a single determinant*, Annals of Mathematics, Second Series **1** (1899-1900), pp. 1–13.

[16] ALEXANDRE JOSEPH HIDULPHE VINCENT, *Sur la résolution des équations numériques*, Journal de Mathématiques Pures et Appliquées, **1** (1836), pp. 341–372 .

[17] LU YANG, XIAORONG HOU AND ZHENBING ZENG, *A complete discrimination system for polynomials*, Science in China (Series E), **39:6** (1996), pp. 628–646.

[18] LU YANG AND BICAN XIA, *Explicit Criterion to Determine the Number of Positive Roots of a Polynomial*, MM Research Preprints, **15** (1997), pp. 134–145.

Received: 06.06.2010

University of Thessaly,
Department of Computer and
Communication Engineering,
GR-38221 Volos, Greece
E-mail: `akritas@uth.gr`
E-mail: `pviglas@uth.gr`