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ON A THEOREM BY VAN VLECK REGARDING STURM SEQUENCES

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Abstract. In 1900 E. B. Van Vleck proposed a very efficient method to compute the Sturm sequence of a polynomial $p(x) \in \mathbb{Z}[x]$ by triangularizing one of Sylvester's matrices[1] of $p(x)$ and its derivative $p'(x)$. That method works fine *only* for the case of *complete* sequences provided *no* pivots take place. In 1917, A. J. Pell[2] and R. L. Gordon pointed out this "weakness" in Van Vleck's theorem, rectified it but did not extend his method, so that it also works in the cases of: (a) complete Sturm sequences *with* pivot, and (b) *incomplete* Sturm sequences.

Despite its importance, the Pell-Gordon Theorem for polynomials in $\mathbb{Q}[x]$ has been totally forgotten and, to our knowledge, it is referenced by us for the first time in the literature.

In this paper we go over Van Vleck's theorem and method, modify slightly the formula of the Pell-Gordon Theorem and present a general triangularization method, called the VanVleck-Pell-Gordon method, that correctly computes in $\mathbb{Z}[x]$ polynomial Sturm sequences, both complete and incomplete.

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Key words: Polynomials, Real roots, Sturm sequences, Sylvester's matrices, Matrix triangularization.

[1]If $n = \deg(p)$, the dimension of the little known matrix we are talking about is $2n \times 2n$, as opposed to the widely known and used matrix of dimension $(2n-1) \times (2n-1)$.

[2]See the link `http://en.wikipedia.org/wiki/Anna_Johnson_Pell_Wheeler` for a biography of Anna Johnson Pell Wheeler.

Both methods, Van Vleck's and the extended one, have been implemented in the freely available computer algebra system Xcas and are available for use.

1. Introduction. The Sturm sequence of a polynomial $p(x) \in \mathbb{Z}[x]$ or $p(x) \in \mathbb{Q}[x]$, of degree $n \geqslant 2$, is the sequence of functions $f_0(x), f_1(x), \ldots, f_k(x)$, $k \leqslant n$, where $f_0(x) = p(x)$, $f_1(x) = p'(x)$, and, for $2 \leqslant j \leqslant k$, $f_j(x)$ is the *negative* remainder obtained on dividing $f_{j-2}(x)$ by $f_{j-1}(x)$.

In other words, the Sturm sequence of $p(x)$ results from *negating* the remainders obtained in the process of finding the greatest common divisor of $p(x)$ and $p'(x)$ using the Euclidean algorithm.

If $k = n$, the Sturm sequence is called *complete*, whereas if $k < n$, it is called *incomplete*.

We see that obtaining polynomial remainders is the major operation in computing Sturm sequences. The most widely known and commonly used methods to compute these remainders is to use either polynomial *pseudo*-divisions (explained below) in $\mathbb{Z}[x]$ or regular polynomial divisions in $\mathbb{Q}[x]$.

For example, the Sturm sequence in $\mathbb{Z}[x]$ of $p(x) = x^3 + 3x^2 - 7x + 7$ is obtained by the function `sturm` of `Xcas`[3]

```
>  sturm( x^3 + 3x^2 - 7x + 7 )[1]
```

$$[[1, 3, -7, 7], [3, 6, -7], [60, -84], -2912]$$

where to obtain the first remainder, $60x - 84$, we had to premultiply the divident times 3^2, that is, times the leading coefficient of the divisor raised to the power $\deg(p) - \deg(p') + 1$.

In $\mathbb{Q}[x]$ the sequence is obtained using the function `sturm` of `Sympy` (another freely available computer algebra system)

```
Python] import sympy
Python] x = sympy.var('x')
Python] sympy.sturm( x**3 + 3*x**2 - 7*x + 7 )
 [x**3 + 3*x**2 - 7*x + 7, 3*x**2 + 6*x - 7, 20*x/3 - 28/3, -182/25]
```

In 1840 Sylvester discovered `sylvester1`, the most widely known and used form of the two matrices that bear his name, and used it to compute in $\mathbb{Z}[x]$ the *resultant* of two polynomials $p(x), q(x)$ along with the coefficients of

[3]`TeXmacs` was used as interface thoughout this paper.

the polynomial remainders obtained by applying Euclid's algorithm on $p(x), q(x)$ [12]. The coefficients of the polynomial remainders obtained as determinants of submatrices, *subresultants*, of `sylvester1` are the *smallest possible* without introducing rationals and without computing (integer) greatest common divisors.

In 1853 Sylvester discovered the little known matrix `sylvester2` and used it to compute in $\mathbb{Z}[x]$ the coefficients of the polynomial remainders obtained by applying Sturm's algorithm on $p(x), p'(x)$ [13], [4].[4] Again, the coefficients of the modified[5] "Euclidean" polynomial remainders obtained as determinants of submatrices, *modified subresultants*, of `sylvester2` are the *smallest possible* without introducing rationals and without computing (integer) greatest common divisors.

Sylvester's result of 1853 is valid only for complete Sturm sequences,[6] while the case of incomplete Sturm sequences remained open, since the signs of the coefficients could not be correctly computed. An analogous observation was also made by Van Vleck in 1900 and is stated as Theorem 1 in this paper.

Additionally, for complete Sturm sequences in $\mathbb{Z}[x]$, Van Vleck presented in 1900 a theorem, Theorem 2 in this paper, and a computational method for computing the coefficients of the polynomial remainders by triangularizing Sylvester's matrix `sylvester2` of $p(x)$ and $p'(x)$. In his method VanVleck cleverly takes advantage of the special form of `sylvester2` and successively triangularizes matrices of only 3 rows, thus making his method extremely fast and suitable even for computations done by hand [14].

However, Van Vleck's method computes the correct sign of the coefficients *only* for complete Sturm sequences, when no pivot occurs in the triangularization process. In all other cases the sign of the coefficients may not be correct. This was observed by Pell and Gordon ([11], p. 193) and they presented a theorem, Theorem 3 in this paper, to correctly compute the sign of the coefficients of the Sturm remainders in all cases.

To our knowledge, the Pell-Gordon paper was completely ignored and has not been cited in the literature before us.

In their work, Pell and Gordon compute in $\mathbb{Q}[x]$ the coefficients of the polynomials in complete or incomplete Sturm sequences as modified subresultants of `sylvester2` divided by appropriate powers of the leading coefficients of

[4] As stated in Van Vleck's paper ([14], p. 2), Jacobi was the first to express, in 1835, the coefficients of the polynomial remainders in a Sturm sequence as minors of a common determinant. Eighteen years later, not being aware of Jacobi's work, Sylvester obtained the same result [13].

[5] As stated in the opening paragraph, the polynomial remainders in Sturm's algorithm are the ones obtained from Euclid's algorithm appropriately *modified*.

[6] Just as his 1840 result is valid only for complete Euclidean sequences.

the remainders; a complete example demonstrating their theorem can be found elsewhere [7]. However, they did not generalize Van Vleck's triangularization method to work for incomplete sequences.

In 1988, not being aware of the 1917 paper by Pell and Gordon, Akritas extended Van Vleck's method for generalized polynomial remainder sequences [2]. He used the Dodgson[7]-Bareiss integer-preserving triangularization method but was not able to compute the exact signs of the polynomials in incomplete sequences [9, 8]. An attempt to resolve this issue was undertaken in 1994 by Akritas, E. K. Akritas and Malaschonok but the "sign problem" remained elusive [5] — despite the fact that improvements were made regarding the computational implementation [6].

In this paper we solve the "sign problem" and present a generalized triangularization method, the VanVleck-Pell-Gordon method, which exactly computes the sign of the polynomials in Sturm sequences, both complete and incomplete. Our breakthrough is due to the theorem by Pell and Gordon ([11], pp. 190, 193), and it came after Vigklas discovered their work in the scientific data bases.

The rest of the paper is organised as follows:

In Section 2 we review the theoretical background of Van Vleck's method, discuss various aspects of the triangularization method and provide a detailed example, the same one that is used in Van Vleck's paper.

Our implementation in `Xcas` can be found at the link `http://inf-server.inf.uth.gr/~akritas/publications/VanVleck_Triang_CompleteSeq`.

In Section 3 we state the Pell-Gordon Theorem ([11], pp. 190, 193) along with a modification of it and we incorporate the latter into Van Vleck's triangularization method as follows: To obtain the correct sign of each polynomial remainder in the Sturm sequence — whether complete or incomplete — and to force its coefficients to become modified subresultants — i.e., to reduce them — the leading coefficient of each remainder is computed a second time as a modified subresultant of Sylvester's matrix. An example is also presented to clarify our procedure.

Our implementation in `Xcas` can be found at the link `http://inf-server.inf.uth.gr/~akritas/publications/VanVleck_Pell_Gordon`.

Finally, in Section 4 we present our conclusions.

[7] Charles Ludwidge Dodgson (1832–1898) is the same person widely known for his writing *Alice in Wonderland* under the pseudonym Lewis Carroll.

2. Van Vleck's Theorem and the Triangularization Method for Complete Sturm Sequences in $\mathbb{Z}[x]$. For our discussion we need to introduce the notion of the resultant (and subresultants) of two polynomials; these polynomials will be $p(x)$ and its derivative $q(x) = p'(x)$, — both in $\mathbb{Z}[x]$.

2.1. Polynomial Remainders with Matrix Triangularization. Van Vleck's method is based on the fact that polynomial remainders can be computed by triangularizing a special matrix. If the divident is $p(x) = a_n x^n + \ldots + a_0$, of degree n, and the divisor is $q(x) = A_m x^m + \ldots + A_0$, of degree $m, m < n$, then the dimension of the matrix M to be triangularized is $(n - m + 2) \times (n + 1)$ and its rows are listed below:

$$M = [[A_m, \ldots, A_0, 0, \ldots, 0], [0, A_m, \ldots, 0, \ldots, 0], \ldots, \\ [0, \ldots, 0, A_m, \ldots, A_0], [a_n, \ldots, a_0]].$$

In M, the first $n-m+1$ rows consist of the coefficients of $q(x)$, — shifted sequentially to the right, — and the last row consists of the coefficients of $p(x)$. After triangularization, the last row, $[a_n, \ldots, a_0]$, is transformed to the row $[0, \ldots, 0, r_k, \ldots, r_0]$ containing the coefficients of the remainder.

In the Sturm sequences we are interested in computing the polynomial remainders negated. To obtain the negated remainders we can either negate the remainder computed above or we can triangularize the matrix M after swapping its last two rows; that is, the negated remainder is obtained by triangularizing the following matrix:

$$M = [[A_m, \ldots, A_0, 0, \ldots, 0], [0, A_m, \ldots, 0, \ldots, 0], \ldots, \\ [a_n, \ldots, a_0], [0, \ldots, 0, A_m, \ldots, A_0]].$$

After triangularization of the above matrix its last row contains the coefficients of the remainder negated. This last approach of computing negated remainders is used by VanVleck in his triangularization method.

Example 1. Let $p(x) = x^3+3x^2-7x+7$ and $q(x) = p'(x) = 3x^2+6x-7$. To compute, in $\mathbb{Z}[x]$, the remainder on dividing $p(x)$ by $q(x)$ we triangularize the matrix

$$M = [[3, 6, -7, 0], [0, 3, 6, -7], [1, 3, -7, 7]].$$

In `Xcas` the triangularization is easily performed with the help of the function `pivot(M,j,j,-j)`, where $M[j, j]$ is the pivot element and $-j$ indicates that the rows above $M[j]$ do not change. In our example, two pivots are executed and the polynomial remainder is read off the last row of $M2$; that is, the remainder is $r(x) = -60x + 84$. (Note that in `Xcas` colon followed by semicolon, supresses the printing of output.)

```
> M := [ [3,6,-7,0], [0,3,6,-7], [1,3,-7,7] ]:;
> M1 := pivot( M, 0, 0 ):;
> M2 := pivot( M1, 1, 1, -1 );
```

$$\begin{pmatrix} 3 & 6 & -7 & 0 \\ 0 & 3 & 6 & -7 \\ 0 & 0 & -60 & 84 \end{pmatrix}$$

The same result can be obtained with *pseudo-division*, the process by which — in order to force the quotient and remainder to be in $\mathbb{Z}[x]$[8]— we premultiply $p(x)$, times the leading coefficient of $q(x)$ raised to the power degree (p) − degree $(q) + 1$. `Sympy` has the function `prem` that does this for us:

```
Python]  sympy.prem( x**3 + 3*x**2 - 7*x + 7, 3*x**2 + 6*x - 7)
 -60*x + 84
```

By contrast, in `Xcas` we have to use the function `rem` in which case we premultiply the divident ourselves:

```
 >  rem( 3^2 * ( [1, 3, -7, 7] ), [3, 6, -7], x)
```

$$poly1[-60, 84]$$

To compute the remainder negated, we triangularize the matrix M after swapping its last two rows.

```
> M := [ [3, 6, -7, 0], [1, 3, -7, 7], [0, 3, 6, -7]]:;
> M1 := pivot( M, 0, 0):;
> M2 := pivot( M1, 1, 1, -1);
```

$$\begin{pmatrix} 3 & 6 & -7 & 0 \\ 0 & 3 & -14 & 21 \\ 0 & 0 & 60 & -84 \end{pmatrix}$$

[8]If in `Sympy` we divide $p(x) = x^3 + 3x^2 - 7x + 7$ by $q(x) = 3x^2 + 6x - 7$ in $\mathbb{Z}[x]$ using the function, `rem`, to do the usual polynomial division, then the remainder we obtain is $p(x)$. Indeed,

```
Python] sympy.rem( x**3 + 3*x**2 - 7*x + 7, 3*x**2 + 6*x - 7, domain = sympy.ZZ )
 x**3 + 3*x**2 - 7*x + 7
```

As we see, the result is $-r(x) = 60x - 84$, which is also obtained with polynomial pseudo-division.

```
Python]  -sympy.prem( x**3 + 3*x**2 - 7*x + 7, 3*x**2 + 6*x - 7 )
 60*x - 84
```

2.2. Polynomial Resultants and Sylvester's Matrices. The resultant, $\text{res}(p, q)$, of two polynomials $p(x)$ and $q(x)$ is defined as the product of all the differences between the roots of the polynomials.[9] That is, if $p(x) = a_0(x - r_1) \cdot (x - r_2) \cdots (x - r_n)$ and $q(x) = b_0(x - s_1) \cdot (x - s_2) \cdots (x - s_m)$ then

$$\text{res}(p, q) = a_0^m \cdot b_0^n \prod_{j=1}^{n} \prod_{k=1}^{m} (r_j - s_k)$$

By grouping together factors, we may also rewrite the resultant as

$$\text{res}(p, q) = a_0^m \prod_{j=1}^{n} q(r_j)$$

or

$$\text{res}(p, q) = (-1)^{m \cdot n} b_0^n \prod_{k=1}^{m} p(s_k).$$

A well known result states that the vanishing of the resultant of two polynomials is the necessary and sufficient condition for the two polynomials to have a common root.

Example 2. The resultant of the polynomials $p(x) = (x - a) \cdot (x - b) \cdot (x - c)$ and $q(x) = (x - d) \cdot (x - f)$ is the product $(a - d) \cdot (a - f) \cdot (b - d) \cdot (b - f) \cdot (c - d) \cdot (c - f)$, which can be computed by the corresponding function in `Xcas`. Note that to simplify the resulting expression we have to factor it:

```
 >  factor( resultant( (x-a)*(x-b)*(x-c), (x-d)*(x-f), x ) )
```

$$(-c + f) \cdot (d - c) \cdot (b - f) \cdot (b - d) \cdot (a - f) \cdot (a - d)$$

Closely related to the resultant are the so-called *Sylvester*'s matrices `sylvester1`, of dimension $(m + n) \times (m + n)$, and `sylvester2`, of dimension $2 \cdot \max(m, n) \times 2 \cdot \max(m, n)$.

[9] Note that $\text{res}(p, q) = (-1)^{\deg(p) \cdot \deg(q)} \text{res}(q, p)$.

In `Xcas`, the matrix `sylvester1` is constructed by the built-in function `sylvester`, and its determinant defines the resultant of two polynomials.[10] On the other hand, the matrix `sylvester2` is constructed by our own function `sylvester2`,[11] and its determinant may differ in sign from the resultant — as is the case for the two polynomials of the above example. Indeed, the determinant of `sylvester1` is identical to the resultant of the two polynomials,

```
> factor( det( sylvester1( (x-a)*(x-b)*(x-c), (x-d)*(x-f), x) ) )
```

$$(-c+f)\cdot(d-c)\cdot(b-f)\cdot(b-d)\cdot(a-f)\cdot(a-d)$$

whereas the determinant of `sylvester2` has a different sign

```
> factor( det( sylvester2( (x-a)*(x-b)*(x-c), (x-d)*(x-f), x) ) )
```

$$(c-f)\cdot(d-c)\cdot(b-f)\cdot(b-d)\cdot(a-f)\cdot(a-d)$$

Assuming $n = \text{degree}\,(p) \geqslant \text{degree}\,(q) = m$, we next describe the two Sylvester matrices.

Sylvester's matrix `sylvester1` consists of two groups of rows, the first one with m rows and the second one with n. Concatenation of the two groups yields matrix `sylvester1`.

In the first row of the first group (of m rows) are the coefficients of $p(x)$ with $m-1$ trailing zeros. The second row in this group differs from the first one in that its elements have been rotated to the right by one. A total of $m-1$ rotations are needed to construct the first group of rows.

In the first row of the second group (of n rows) are the coefficients of $q(x)$ with $n-1$ trailing zeros. The second row in this group differs from the first one in that its elements have been rotated to the right by one. A total of $n-1$ rotations are needed to construct the first group of rows.

Sylvester's matrix, `sylvester2` consists of n pairs of rows. In the first row of the *first* pair are the coefficients of $p\,(x)$ whereas in the second row of the first pair are the coefficients of $q\,(x)$; $n-m$ zeros have been prepended to $q\,(x)$ to also make it of degree n.

Both rows in the first pair have $2n-(n+1)$ trailing zeros and both rows of the last pair have $2n-(n+1)$ leading zeros. The second pair of rows differs from

[10] The assignment `sylvester := sylvester1` changes the name of the built-in function to `sylvester1`, which will be used in the sequel for clarity.

[11] It can be found in the link `http://inf-server.inf.uth.gr/~akritas/publications/sylvester2`.

the first one in that the elements of both rows have been rotated to the right by one. A total of $2n - (n+1)$ rotations are needed to construct Sylvester's matrix.

Example 3. For the two polynomials $p(x) = x^3 + 3x^2 - 7x + 7$ and $q\,(x) = 3x^2 + 6x - 7$, the Sylvester matrices are:

```
> S := sylvester1( x^3 + 3x^2 - 7x + 7, 3x^2 + 6x - 7, x )
```

$$\begin{pmatrix} 1 & 3 & -7 & 7 & 0 \\ 0 & 1 & 3 & -7 & 7 \\ 3 & 6 & -7 & 0 & 0 \\ 0 & 3 & 6 & -7 & 0 \\ 0 & 0 & 3 & 6 & -7 \end{pmatrix}$$

and

```
> S := sylvester2( x^3 + 3x^2 - 7x + 7, 3x^2 + 6x - 7, x )
```

$$\begin{pmatrix} 1 & 3 & -7 & 7 & 0 & 0 \\ 0 & 3 & 6 & -7 & 0 & 0 \\ 0 & 1 & 3 & -7 & 7 & 0 \\ 0 & 0 & 3 & 6 & -7 & 0 \\ 0 & 0 & 1 & 3 & -7 & 7 \\ 0 & 0 & 0 & 3 & 6 & -7 \end{pmatrix}$$

Notice that for this example det(`sylvester1`) $= -$ det(`sylvester2`). Indeed,

```
> det( sylvester1 )
```

$$2912$$

whereas

```
> det( sylvester2 )
```

$$-2912$$

2.3. Computation of Complete Sturm Sequences in $\mathbb{Z}\,[x]$ using Sylvester's Matrix. Sylvester presented a way to exactly compute the coefficients of the polynomial remainders in complete Sturm sequences as modified subresultants of `sylvester2`. This was reiterated by Van Vleck in the following Theorem:

Theorem 1 (Van Vleck, 1900). *Consider the polynomials* $p(x) = c_n x^n + \cdots + c_0$ *and* $q(x) = d_m x^m + \cdots + d_0$*, in* $\mathbb{Z}[x]$*, with* $c_n \neq 0$*,* $d_m \neq 0, n \geqslant m$*. Then the successive polynomials that are formed from the first* $2j$ *rows,* $j = 2, \ldots, n$*, of Sylvester's matrix (*`sylvester2`*) for* $p(x), q(x)$*, constitute a Sturm sequence.*

The proof of this theorem can be found in Van Vleck's paper [14] and elsewhere ([3], p. 263).

Notice that the theorem makes *no* reference to the Sturm sequence being complete, but clearly this is what Van Vleck had in mind. Sylvester himself was aware of incomplete sequences, but did not attempt to compute the correct sign of their polynomials. As stated in the next Section, this problem was solved by Pell and Gordon in 1917 [11].

For a given j, $2 \leqslant j \leqslant n$, the $n - j + 1$ coefficients of the polynomial remainder are computed as determinants of $n - j + 1$ submatrices of the matrix s formed by the first $2j$ rows of Sylvester's matrix $S =$ `sylvester2(p,q)`. All of these submatrices have the same first $2j - 1$ columns, whereas the $2j$ th column is successively the $(2j - 1 + k)$ th column of s, where $k = 1, \ldots, n - j + 1$. The determinants of these $2j \times 2j$ submatrices are the modified subresultants.

Below we demonstrate VanVleck's theorem with two examples:

- In the first example the polynomials $p(x) = x^3 + 3x^2 - 7x + 7$ and $q(x) = p'(x) = 3x^2 + 6x - 7$ form a complete Sturm sequence and the coefficients computed as modified subresultants agree both in sign and value with the corresponding ones computed with Sturm's algorithm, i.e., with the function `sturm` of `Xcas`.
- In the second example, the polynomials $p(x) = 2x^5 - 3x^4 - 3$ and $q(x) = p'(x) = 10x^4 - 12x^3$ form an incomplete Sturm sequence in which case:
    i. the sequence we compute is somehow "wrong" in the sense that we compute 2 polynomials of degree 1,
    ii. except for the last term of the sequence, $\det(S)$, the coefficients computed as modified subresultants agree in value and sign with the corresponding ones computed with Sturm's algorithm, i.e., with the function `sturm` of `Xcas`,
    iii. $\det(S)$ does *not* agree in sign with the last term of the sequence computed with the function `sturm`. Actually, $\det(S)/2 = \operatorname{res}(p, q) = 11459232$, i.e., the determinant of S divided by 2 equals the resultant of $p(x), q(x)$.

Example 4 (Complete Sturm Sequence)**.** For the polynomials $p(x) = x^3+3x^2-7x+7$ and $q(x) = p'(x) = 3x^2+6x-7$ we form $S =$ `sylvester2(p,q)`:

```
> p := x^3 + 3x^2 - 7x + 7;  q := diff( p, x, 1)
```

$$x^3 + 3 \cdot x^2 - 7 \cdot x + 7, \quad 3 \cdot x^2 + 6 \cdot x - 7$$

```
> S := sylvester2( p, q, x )
```

$$\begin{pmatrix} 1 & 3 & -7 & 7 & 0 & 0 \\ 0 & 3 & 6 & -7 & 0 & 0 \\ 0 & 1 & 3 & -7 & 7 & 0 \\ 0 & 0 & 3 & 6 & -7 & 0 \\ 0 & 0 & 1 & 3 & -7 & 7 \\ 0 & 0 & 0 & 3 & 6 & -7 \end{pmatrix}$$

Following VanVleck's theorem, for $j = 2$ we take the submatrix s of S, consisting of the first $2j = 4$ rows of S and, since $n = 3$, we will compute the coefficients of the polynomial remainder of degree $n - j = 1$.

```
> s := subMat( S, 0, 0, 3, 7 )
```

$$\begin{pmatrix} 1 & 3 & -7 & 7 & 0 & 0 \\ 0 & 3 & 6 & -7 & 0 & 0 \\ 0 & 1 & 3 & -7 & 7 & 0 \\ 0 & 0 & 3 & 6 & -7 & 0 \end{pmatrix}$$

The $n - j + 1 = 2$ coefficients we are after will be computed as the determinants, the *subresultants*, of two $2j \times 2j$ submatrices of s. Both these submatrices have the same first $2j - 1$ columns, whereas the $2j$ th column is successively the $(2j - 1 + k)$-th column of s, where $k = 1, \ldots, n - j + 1$.

Since $n = 3, j = 2$, all four rows of s have at least $2n-(n+1)-j+1 = 1$ trailing zero. We now have to compute the two coefficients of the first degree polynomial as determinants of two 4×4 submatrices of s, where the first three columns stay the same, whereas the fourth one will be, respectively, the fourth and fifth column of s.

To form the first 4×4 submatrix of s, in `Xcas` we use the function `subMat(M,a,b,c,d)`, where `M[a,b]` is the upper left corner and `M[c,d]` is the lower right corner of the submatrix we want to define.

So the first coefficient is 60:

```
> det( subMat( s, 0, 0, 3, 3 ) )
```

$$60$$

For the second coefficient we have to swap the 4 th and 5 th columns of s before we form the submatrix; for this we use the function `colSwap` in `Xcas`.

So the second coefficient is -84:

```
> det( subMat( colSwap( s, 3, 4 ), 0, 0, 3, 3 ) )
```

$$-84$$

In other words we now have the first polynomial remainder of the Sturm sequence and it is $60x - 84$.

For $j = 3$ we see that the degree of the remainder is $n - j = 0$, and to compute it we take the determinant of the whole matrix S.

Hence, the constant term is -2912:

```
> det(S)
```

$$-2912$$

Therefore, the polynomial coefficients computed as subresultants agree both in sign and value with those computed below with the help of `sturm`:

```
> sturm(p)[1]
```

$$[[1, 3, -7, 7], [3, 6, -7], [60, -84], -2912]$$

Note that in this case the determinant of S is not equal to the resultant of the two polynomials

$$\det(S) \neq \mathrm{res}(x^3 + 3x^2 - 7x + 7, 3x^2 + 6x - 7) = 2912.$$

Indeed, the signs are opposite

```
> resultant( x^3 + 3x^2 - 7x + 7,  3x^2 + 6x - 7,  x)
```

$$2912$$

Example 5 (Incomplete Sturm Sequence)**.** For the polynomials $p(x) = 2x^5 - 3x^4 - 3$ and $q(x) = p'(x) = 10x^4 - 12x^3$ we form $S =$ `sylvester2(p,q)`:

```
> p := 2x^5 - 3x^4 - 3;    q := 10x^4 - 12x^3;
```

$$2 \cdot x^5 - 3 \cdot x^4 - 3, \quad 10 \cdot x^4 - 12 \cdot x^3$$

```
> S := sylvester2( p, q, x )
```

$$\begin{pmatrix} 2 & -3 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 10 & -12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -3 & 0 & 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 10 & -12 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -3 & 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 10 & -12 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -3 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 10 & -12 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -3 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 10 & -12 & 0 & 0 & 0 \end{pmatrix}$$

Following VanVleck's theorem, for $j = 2$ we take the submatrix s of S, consisting of the first $2j = 4$ rows of S and, since $n = 5$, we compute the coefficients of the polynomial remainder of degree $n - j = 3$.

```
> s := subMat( S, 0, 0, 3, 9 )
```

$$\begin{pmatrix} 2 & -3 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 10 & -12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -3 & 0 & 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 10 & -12 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The $n-j+1 = 4$ coefficients we are after will be computed as the determinants of four $2j \times 2j$ submatrices of s. All of these submatrices have the same first $2j - 1$ columns, whereas the $2j$-th column is successively the $(2j - 1 + k)$ th column of s, where $k = 1, \dots, n - j + 1$.

Since $n = 5, j = 2$, all four rows of s have at least $2n - (n+1) - j + 1 = 3$ trailing zeros. The first coefficient is 72:

```
> det( subMat( s, 0, 0, 3, 3 ) ) / 2
```

$$72$$

The second, third and fourth coefficients are, respectively, 0, 0 and 300:

```
> det( subMat( colSwap( s, 3, 4 ), 0, 0, 3, 3 ) ) / 2
```

$$0$$

```
> det( subMat( colSwap( s, 3, 5 ), 0, 0, 3, 3 ) ) / 2
```

$$0$$

```
> det( subMat( colSwap( s, 3, 6 ), 0, 0, 3, 3 ) ) / 2
```

$$300$$

Therefore, the first remainder of the Sturm sequence is $r_1(x) = 72x^3 + 300$.

Next, we set $j = 3$, we take the submatrix s of S, consisting of the first $2j = 6$ rows of S and, since $n = 5$, we will try to compute the coefficients of the polynomial remainder of degree $n - j = 2$.

```
> s := subMat( S, 0, 0, 5, 9 )
```

$$\begin{pmatrix} 2 & -3 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 10 & -12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -3 & 0 & 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 10 & -12 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -3 & 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 10 & -12 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The $n-j+1 = 3$ coefficients we are after will be computed as the determinants of three $2j \times 2j$ submatrices of s. All of these submatrices have the same first $2j-1$ columns, whereas the $2j$ th column is successively the $(2j-1+k)$ th column of s, where $k = 1, \dots, n-j+1$.

Since $n = 5, j = 3$, all six rows of s have at least $2n-(n+1)-j+1 = 2$ trailing zeros. The first coefficient is 0:

```
> det( subMat( s, 0, 0, 5, 5 ) ) / 2
```

$$0$$

The second, and third coefficients are, respectively, 2160 and -2592:

```
> det( subMat( colSwap( s, 5, 6 ), 0, 0, 5, 5 ) ) / 2
```

$$2160$$

```
> det( subMat( colSwap( s, 5, 7 ), 0, 0, 5, 5 ) ) / 2
```

$$-2592$$

Here, instead of a polynomial remainder of degree 2 we obtained one of degree 1; namely $r_2(x) = 2160x - 2592$. This indicates that we encountered an incomplete Sturm sequence.

Next, we set $j = 4$, we take the submatrix s of S, consisting of the first $2j = 8$ rows of S and, since $n = 5$, we compute again the coefficients of the polynomial remainder of degree $n - j = 1$.

```
> s := subMat( S, 0, 0, 7, 9 )
```

$$\begin{pmatrix} 2 & -3 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 10 & -12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -3 & 0 & 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 10 & -12 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -3 & 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 10 & -12 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -3 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 10 & -12 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The two coefficients are, respectively, -64800 and 77760:

```
> det( subMat( s, 0, 0, 7, 7 ) ) / 2
```

$$-64800$$

```
> det( subMat( colSwap( s, 7, 8 ), 0, 0, 7, 7 ) ) / 2
```

$$77760$$

and the (second) first degree remainder is $r_3(x) = -64800x + 77760$.

Finally, for $j = 5$ we evaluate the determinant of the whole matrix S to compute the constant term of the sequence. Its value is 11459232:

```
> det(S) / 2
```

$$11459232$$

Note that in this case the reduced value of the determinant of S differs in sign from the last term of the Sturm sequence as computed by the function `sturm` — their difference in value is not important:

```
> sturm( 2x^5 - 3x^4 - 3 )[1]
```

$$[[2, -3, 0, 0, 0, -3], [10, -12, 0, 0, 0], [72, 0, 0, 300], [2160, -2592], -1782139760640]$$

Therefore, we cannot compute the members of an incomplete Sturm sequence using modified subresultants.

In this particular example, it so happens that the reduced value of the determinant of S equals the resultant of the two polynomials

$$\det(S)/2 = \operatorname{res}(2x^5 - 3x^4 - 3, 10x^4 - 12x^3).$$

Indeed,

```
> resultant( 2x^5 - 3x^4 - 3, 10x^4 - 12 x^3, x )
```

$$11459232$$

It should be noted that — when we compute the coefficients of the remainders in an incomplete sequence using modified subresultants — the appearance of multiple remainders of the same degree is quite normal; however, of those remainders with the same degree only the first one is used. As we will see in the generalized triangularization method, the same phenomenon appears there as well, in the form of redundant rows.

2.4. Van Vleck's Triangularization Method for Computing in $\mathbb{Z}[x]$ Complete Sturm Sequences. Computing the coefficients of the polynomial remainders in a Sturm sequence by evaluating modified subresultants is quite a tedious process if carried out by hand and quite time consuming if carried out by computer.

Van Vleck realized that one does not have to compute modified subresultants of Sylvester's matrix `sylvester2` in order to find the coefficients of the polynomial remainders in the Sturm sequence. It suffices to simply triangularize `sylvester2` using integer preserving transformations, in which case the modified subresultants (the coefficients) can be read off the triangularized matrix. We have the following ([14], p. 8):

Theorem 2 (Van Vleck, 1899). *Let $p(x)$ and $q(x) = p'(x)$ be two polynomials of degree n and $n-1$ respectively and let S be their Sylvester matrix* `sylvester2(p,q)`*. If, using integer preserving transformations, we bring S into its upper triangular form, $T(S)$, then the even rows of $T(S)$ furnish the coefficients of the successive polynomial remainders of the Sturm sequence. The coefficients taken from a given row are multiplied times $(-1)^k$, where k is the number of negative elements on the principal diagonal above the row under consideration.*

Van Vleck takes advantage of the special form of Sylvester's matrix and computes $T(S)$ by updating only two rows at a time; to update these two rows he triangularizes a matrix of only three rows, a fact that makes his procedure extremely efficient. To keep the coefficients small he removes at each step the greatest common divisor (content) of the elements in both updated rows, and uses those reduced coefficients in the next three-row matrix[12].

Van Vleck's computation is justified by the fact that in Sylvester's matrix the elements (entries) of any two consecutive rows are the same as those of the two preceding rows.

Therefore, if in any row the values of the elements are changed by adding a multiple of the preceding row, exactly the same change can be made in the

[12] It turns out that computationally this is the fastest way to proceed [10].

elements of each alternate row thereafter, without altering the value of any modied subresultant that appears as a coefficient in one of the polynomials of the Sturm sequence.

In conclusion, Van Vleck presented a very efficient procedure for computing Sturm sequences in $\mathbb{Z}[x]$, and we next demonstrate it with the same example used by him ([14], pp. 8–9).

Warning 1. Although it is not stated in his paper, Van Vleck also applies the sign rule — mentioned in his theorem — to the triangularized smaller matrices of three rows. Namely, the coefficients taken from a given row are multiplied times $(-1)^k$, where k is the number of negative elements on the principal diagonal above the row under consideration.

Example 6. To compute the Sturm sequence of $p(x) = x^6 + x^5 - x^4 - x^3 + x^2 - x + 1$ we form the matrix $S =$ `sylvester2`(p, p'):

```
> S:=sylvester2(x^6+x^5-x^4-x^3+x^2-x+1, 6x^5+5x^4-4x^3-3x^2+2x-1,x)
```

$$\begin{pmatrix}
1 & 1 & -1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 6 & 5 & -4 & -3 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 6 & 5 & -4 & -3 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 6 & 5 & -4 & -3 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 6 & 5 & -4 & -3 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 6 & 5 & -4 & -3 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 1 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 6 & 5 & -4 & -3 & 2 & -1
\end{pmatrix}$$

The first two rows stay the same, since there is no element to be eliminated in the first column.

In the second column there is one element to be eliminated. So, we form the 3×12 matrix M consisting of the second, third and fourth rows of S.

```
> M := subMat( S, 1, 0, 3, 11 )
```

$$\begin{pmatrix}
0 & 6 & 5 & -4 & -3 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 6 & 5 & -4 & -3 & 2 & -1 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Using the function `pivot` of `Xcas` we obtain, in two steps, matrix $M2$, the triangularized version of M.

```
> M1 := pivot( M, 0, 1 )
```

$$\begin{pmatrix} 0 & 6 & 5 & -4 & -3 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & -3 & 4 & -5 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 5 & -4 & -3 & 2 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

```
> M2 := pivot( M1, 1, 2, -1 )
```

$$\begin{pmatrix} 0 & 6 & 5 & -4 & -3 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & -3 & 4 & -5 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 17 & 14 & -27 & 32 & -37 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The coefficients in the second and third row of $M2$ cannot be further reduced because the gcd of the elements in each row is 1. Moreover, since the signs of the diagonal elements are all positive nothing will change. Hence, the last two rows of $M2$ will replace the third and fourth rows of S.

```
> S[2] := M2[1]:;   S[3] := M2[2]:;   S
```

$$\begin{pmatrix} 1 & 1 & -1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 5 & -4 & -3 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & -3 & 4 & -5 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 17 & 14 & -27 & 32 & -37 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 5 & -4 & -3 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 5 & -4 & -3 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 5 & -4 & -3 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 & 5 & -4 & -3 & 2 & -1 \end{pmatrix}$$

The next matrix with three rows is formed by the two newly inserted rows in S, rotated appropriately when needed, and we repeat the pivoting procedure described above:

```
> M = row(S, 3), rotate( row(S, 2), -1), rotate( row(S, 3), -1 )]
```

$$\begin{pmatrix} 0 & 0 & 0 & 17 & 14 & -27 & 32 & -37 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & -3 & 4 & -5 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 17 & 14 & -27 & 32 & -37 & 0 & 0 & 0 \end{pmatrix}$$

```
> M1 := pivot( M, 0, 3 )
```

$$\begin{pmatrix} 0 & 0 & 0 & 17 & 14 & -27 & 32 & -37 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -48 & -24 & 36 & -48 & 102 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 17 & 14 & -27 & 32 & -37 & 0 & 0 & 0 \end{pmatrix}$$

```
> M2 := pivot( M1, 1, 4, -1 )
```

$$\begin{pmatrix} 0 & 0 & 0 & 17 & 14 & -27 & 32 & -37 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -48 & -24 & 36 & -48 & 102 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -264 & 684 & -720 & 42 & 0 & 0 & 0 \end{pmatrix}$$

First we remove the content from each one of the second and third rows of $M2$

```
> M2[1] := M2[1] / content( M2[1] )
```

$$\begin{pmatrix} 0 & 0 & 0 & 17 & 14 & -27 & 32 & -37 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -8 & -4 & 6 & -8 & 17 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -264 & 684 & -720 & 42 & 0 & 0 & 0 \end{pmatrix}$$

```
> M2[2] := M2[2] / content( M2[2] )
```

$$\begin{pmatrix} 0 & 0 & 0 & 17 & 14 & -27 & 32 & -37 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -8 & -4 & 6 & -8 & 17 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -44 & 114 & -120 & 7 & 0 & 0 & 0 \end{pmatrix}$$

Next we take care of the signs: The second row of $M2$ will replace the fifth row of S as is, since the diagonal element above -8 is positive; however, the third row of $M2$ will change sign since there is one negative element in the second row, on the diagonal.

```
> S[4] := M2[1]:;   S[5] := -M2[2]:;   S
```

$$\begin{pmatrix} 1 & 1 & -1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 5 & -4 & -3 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & -3 & 4 & -5 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 17 & 14 & -27 & 32 & -37 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -8 & -4 & 6 & -8 & 17 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 44 & -114 & 120 & -7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 5 & -4 & -3 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 5 & -4 & -3 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 & 5 & -4 & -3 & 2 & -1 \end{pmatrix}$$

Again, the next matrix with three rows is formed by the two newly inserted rows in S, rotated appropriately when needed:

```
> M := [row(S, 5), rotate( row(S, 4), -1 ), rotate( row(S, 5), -1 )]
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 44 & -114 & 120 & -7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -8 & -4 & 6 & -8 & 17 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 44 & -114 & 120 & -7 & 0 & 0 \end{pmatrix}$$

Continuing as indicated above we finally obtain the triangularized Sylvester matrix $T(S)$, shown below:

```
> S[6] := [0, 0, 0, 0, 0, 0, -16, 18, -6, 11, 0, 0]:;
> S[7] := [0, 0, 0, 0, 0, 0, 0, -86, 138, 31, 0, 0]:;
> S[8] := [0, 0, 0, 0, 0, 0, 0, 0, -30, -46, 43, 0]:;
> S[9] := [0, 0, 0, 0, 0, 0, 0, 0, 0, 506, -173, 0]:;
> S[10] := [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -331, 253]:;
> S[11] := [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1]:;
> TS := S
```

$$\begin{pmatrix} 1 & 1 & -1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 5 & -4 & -3 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & -3 & 4 & -5 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 17 & 14 & -27 & 32 & -37 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -8 & -4 & 6 & -8 & 17 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 44 & -114 & 120 & -7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -16 & 18 & -6 & 11 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -86 & 138 & 31 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -30 & -46 & 43 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 506 & -173 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -331 & 253 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

From the even rows of the triangularized matrix we extract the coefficients of the polynomials in the Sturm sequence, correcting the sign as indicated in Theorem 2.

Namely, the coefficients of the Sturm sequence are:

- from row 1: $[\ 1,\ \ 1,\ \ -1,\ \ -1,\ \ 1,\ \ -1,\ \ 1\ ]$
- from row 2: $[\ 6,\ \ 5,\ \ -4,\ \ -3,\ \ 2,\ \ -1\ ]$

- from row 4: $\left[\ 17,\quad 14,\quad -27,\quad 32,\quad -37\ \right]$
- from row 6: $\left[\ -44,\quad 114,\quad -120,\quad 7\ \right]$ ⇐`signs changed`=
- from row 8: $\left[\ -86,\quad 138,\quad 31\ \right]$
- from row 10: $\left[\ 506,\quad -173\ \right]$
- from row 12: -1 ⇐`sign changed`=

Indeed, using the function `sturm` of `Xcas` we see that, whereas the coefficients may differ in value (as expected, since Van Vleck removes the content from each polynomial), their signs are identical.

```
 >  sturm( x^6 + x^5 - x^4 - x^3 + x^2 - x + 1 )[1]
 [[1, 1, - 1, - 1, 1, - 1, 1], [6, 5, - 4, - 3, 2, - 1], [17, 14,
- 27, 32, - 37], [ - 44, 114, - 120, 7], [- 516, 828, 186], [9108,
- 3114], - 127359]
```

We have implemented Van Vleck's procedure in `Xcas` in the function `sturmSeqVanVleck`, which can be found at the link `http://inf-server.inf.uth.gr/~akritas/publications/VanVleck_Triang_CompleteSeq`.

The implentation of `sturmSeqVanVleck` is quite straightforward; it follows Example 6 and uses the following additional functions:

- `my_sqrfree2`, which converts a polynomial into a product of square free factors — a necessary condition for computing its Sturm sequence,
- `sylvester2`, which constructs the appropriate Sylvester matrix of two polynomials,
- `row2poly`, which converts a matrix row to a polynomial of a specified degree,
- `smallMatrixVanVleckRule`, which applies Van Vleck's sign rule to correctly compute the signs of the coefficients in the triangularized 3-row matrices, and
- `sturmSeqVanVleckRule`, which applies Van Vleck's sign rule to correctly compute the signs of the coefficients in the final triangularized matrix.

However, instead of removing the content of each polynomial, in `sturmSeqVanVleck` we follow Sylvester's practice for complete sequences and reduce the coefficients by dividing out the diagonal element "three" rows up [1]. This way the coefficients computed with Van Vleck's method are modified subresultants and they are the same as those obtained with the `sturm` function of `Xcas`.

3. The Generalized Triangularization Method for Computing in $\mathbb{Z}[x]$ Sturm Sequences of Any Kind. As we saw in Example 5 of Section 2.3, for incomplete Sturm sequences we cannot compute the exact signs of the polynomial coefficients using modified subresultants. Therefore, we cannot easily extend Van Vleck's triangularization method for complete Sturm sequences to *general* Sturm sequences, i.e., sequences that can be either complete or incomplete. The reason is that, in trying to compute general Sturm sequences by triangularizing `sylvester2` matrices, we faced the following major problems:

- In general, the coefficients computed by the matrix triangularization process are not modified subresultants and their signs may not be correct.
- We cannot use Theorem 2, Van Vleck's "sign rule", since it computes the correct signs of the polynomial coefficients of *only* complete Sturm sequences.

To wit, to correctly compute the signs of the polynomial coefficients of a general Sturm sequence, we clearly have to use another "sign rule" — one that is valid for both complete and incomplete Sturm sequences. This new rule is provided by the following theorem by Pell and Gordon [11], which also makes use of the same matrix `sylvester2`, used by Van Vleck:

Theorem 3 (Pell-Gordon, 1917)**.** *Let*

$$A = a_0x^n + a_1x^{n-1} + \cdots + a_n$$

and

$$B = b_0x^n + b_1x^{n-1} + \cdots + b_n$$

be two polynomials of the n th degree. Modify the process of finding the highest common factor of A and B by taking at each stage the negative of the remainder. Let the i th modified remainder be

$$R^{(i)} = r_0^{(i)}x^{m_i} + r_1^{(i)}x^{m_i-1} + \cdots + r_{m_i}^{(i)}$$

where (m_i+1) is the degree of the preceding remainder, and where the first (p_i-1) coefficients of $R^{(i)}$ are zero, and the p_i th coefficient $\varrho_i = r_{p_i-1}^{(i)}$ is different from *zero. Then for* $k = 0, 1, \ldots, m_i$ *the coefficients* $r_k^{(i)}$ *are given by*[13]

$$r_k^{(i)} = \frac{(-1)^{u_{i-1}}\,(-1)^{u_{i-2}} \cdots (-1)^{u_1}\,(-1)^{v_{i-1}}}{\varrho_{i-1}^{p_{i-1}+1}\,\varrho_{i-2}^{p_{i-2}+p_{i-1}} \cdots \varrho_1^{p_1+p_2}\,\varrho_0^{p_1}} \cdot \mathrm{Det}\,(i,k)\,, \tag{1}$$

where $u_{i-1} = 1 + 2 + \cdots + p_{i-1}$, $v_{i-1} = p_1 + p_2 + \cdots + p_{i-1}$ *and*

$$\mathrm{Det}\,(i,k) = \begin{vmatrix} a_0 & a_1 & a_2 & \cdots & \cdot & \cdot & \cdots & a_{2v_{i-1}} & a_{2v_{i-1}+1+k} \\ b_0 & b_1 & b_2 & \cdots & \cdot & \cdot & \cdots & b_{2v_{i-1}} & b_{2v_{i-1}+1+k} \\ 0 & a_0 & a_1 & \cdots & \cdot & \cdot & \cdots & a_{2v_{i-1}-1} & a_{2v_{i-1}+k} \\ 0 & b_0 & b_1 & \cdots & \cdot & \cdot & \cdots & b_{2v_{i-1}-1} & b_{2v_{i-1}+k} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & a_0 & a_1 & \cdots & a_{v_{i-1}} & a_{v_{i-1}+1+k} \\ 0 & 0 & 0 & \cdots & b_0 & b_1 & \cdots & b_{v_{i-1}} & b_{v_{i-1}+1+k} \end{vmatrix}.$$

P r o o f. The proof by induction of this theorem depends on two Lemmas and can be found in the original paper by Pell and Gordon.

As indicated elsewhere [7], we use a modification of formula (1) to compute the coefficients of the Sturm sequence. In our case $p_0 = \deg(A) - \deg(B) = 1$, since B is the derivative of A and, hence, the modified formula is shown below with the changes appearing in bold:

$$r_k^{(i)} = \frac{(-1)^{u_{i-1}}\,(-1)^{u_{i-2}} \cdots (-1)^{u_1}\,(-1)^{u_0}\,(-1)^{v_{i-1}}}{\varrho_{i-1}^{\boldsymbol{p_{i-1}+p_i-\mathbf{degDiffer}}}\,\varrho_{i-2}^{p_{i-2}+p_{i-1}} \cdots \varrho_1^{p_1+p_2}\,\varrho_0^{\boldsymbol{p_0}+p_1}} \cdot \frac{\mathrm{Det}\,(i,k)}{\boldsymbol{\varrho_{-1}}}, \tag{2}$$

where $\varrho_{-1} = a_0$, the leading coefficient of A and `degDiffer` is the difference between the expected degree m_i and the actual degree of the remainder.

It should be noted that in the general case $p_0 = \deg(A) - \deg(B)$ and that the division $\dfrac{\mathrm{Det}\,(i,k)}{\varrho_{-1}}$ is possible if the leading coefficient of A is the only element in the first column of `sylvester2`. Moreover, if the leading coefficient of A is negative we work with the polynomial negated and at the end we reverse the signs of all polynomials in the sequence. □

To see how equation (2) of Theorem 3 is used in the general triangularization process, suppose that we have computed with the latter the i-th polynomial remainder

$$s^{(i)} = s_0^{(i)} x^{m_i} + s_1^{(i)} x^{m_i-1} + \cdots + s_{m_i}^{(i)}, \tag{3}$$

[13] It is understood in (1) that $\varrho_0 = b_0$, $p_0 = 0$, and that $a_i = b_i = 0$ for $i > n$.

where $s_k^{(i)} \in \mathbb{Z}, 0 \leq k \leq m_i$; in general, the coefficients $s_k^{(i)}$ are not modified subresultants and we are not sure about the correctness of their signs.

To compute the correct sign of $s^{(i)}$ we evaluate equation (2) *only* for the leading coefficient $r_j^{(i)}$, that is, for $k = j$ where j is the smallest integer for which $\frac{\text{Det}\,(i,j)}{\varrho_{-1}} \neq 0$. Then if $\text{sgn}(s_j^{(i)}) \neq \text{sgn}(r_j^{(i)})$ we set $s^{(i)} = -s^{(i)}$.

Having computed the correct sign of $s^{(i)}$ we can force its coefficients to equal the corresponding modified subresultants by multiplying $s^{(i)}$ times $\left|\frac{\text{Det}\,(i,j)}{s_j^{(i)} \cdot \varrho_{-1}}\right|$. In other words, by computing just *one* determinant, Det (i,j), and forming the product

$$\left|\frac{\text{Det}\,(i,j)}{s_j^{(i)} \cdot \varrho_{-1}}\right| \cdot s^{(i)} \tag{4}$$

we obtain the i th Sturmian polynomial remainder, whose k-th coefficient is the modified subresultant shown below:[14]

$$\frac{(-1)^{u_{i-1}}(-1)^{u_{i-2}}\cdots(-1)^{u_1}(-1)^{u_0}(-1)^{v_{i-1}}}{\text{sgn}(\varrho_{i-1})^{p_{i-1}+\boldsymbol{p_i}-\textbf{degDiffer}}\,\text{sgn}(\varrho_{i-2})^{p_{i-2}+p_{i-1}}\cdots\text{sgn}(\varrho_1)^{p_1+p_2}\,\text{sgn}(\varrho_0)^{\boldsymbol{p_0}+p_1}} \cdot \frac{\text{Det}\,(i,k)}{\boldsymbol{\varrho_{-1}}}. \tag{5}$$

This is so, because of the existing ratio equality

$$\left|\frac{\text{Det}\,(i,j)}{s_j^{(i)} \cdot \varrho_{-1}}\right| = \left|\frac{\text{Det}\,(i,k)}{s_k^{(i)} \cdot \varrho_{-1}}\right| \tag{6}$$

that holds for $j < k \leq m_i$; recall that $s_j^{(i)}$ is the leading coefficient of the polynomial remainder $s^{(i)}$, in equation (3), computed by the generalized matrix triangularization method.

From the above it is obvious that for our purposes we only need expression (5). The overhead in this generalized method is that we have to keep track of all the variables in Theorem 3 and compute one determinant for each polynomial remainder. An example will make everything clear.

Example 7. Using the generalized matrix triangularization method we will compute the incomplete Sturm sequence of the polynomial $p(x) = 2x^5 - 3x^4 -$

[14] Positive leading coefficients will not affect the sign of expression (5) and, hence, they can be ignored alltogether.

3, the same one used in Example 5 of Section 2.3, where we failed to successfully complete the same task using modified subresultants because the sign of the last term was wrong. We choose this example to save space and energy, since we have already seen the `sylvester2` matrix and have computed all the required determinants.

We begin by constructing the `sylvester2` matrix S, which will remain unchanged so that we can compute the various modified subresultants; we also make a copy of it S', which will be triangularized.

As in Van Vleck' procedure we form the 3-row matrices $M, M1, M2$ and from the last one we obtain a candidate for the first remainder, namely $72x^3+300$; moreover, the second and third rows of $M2$ will replace, respectively, the 3 rd and 4 th row of S'.

```
> M := subMat( S, 1, 0, 3, 9 )
```

$$\begin{pmatrix} 0 & 10 & -12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -3 & 0 & 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 10 & -12 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

```
> M1 := pivot( M, 0, 1 )
```

$$\begin{pmatrix} 0 & 10 & -12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -6 & 0 & 0 & 0 & -30 & 0 & 0 & 0 \\ 0 & 0 & 10 & -12 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

```
> M2 := pivot( M1, 1, 2, -1 )
```

$$\begin{pmatrix} 0 & 10 & -12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -6 & 0 & 0 & 0 & -30 & 0 & 0 & 0 \\ 0 & 0 & 0 & 72 & 0 & 0 & 300 & 0 & 0 & 0 \end{pmatrix}$$

```
> S'
```

$$\begin{pmatrix}
2 & -3 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 \\
0 & 10 & -12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -6 & 0 & 0 & 0 & -30 & 0 & 0 & 0 \\
0 & 0 & 0 & 72 & 0 & 0 & 300 & 0 & 0 & 0 \\
0 & 0 & 2 & -3 & 0 & 0 & 0 & -3 & 0 & 0 \\
0 & 0 & 0 & 10 & -12 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & -3 & 0 & 0 & 0 & -3 & 0 \\
0 & 0 & 0 & 0 & 10 & -12 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & -3 & 0 & 0 & 0 & -3 \\
0 & 0 & 0 & 0 & 0 & 10 & -12 & 0 & 0 & 0
\end{pmatrix}$$

To see if we have to correct the sign of the polynomial or to change its coefficients to modified subresultants we have to evaluate expression (5) which now becomes

$$\frac{(-1)^{u_0}\,(-1)^{v_0}}{\operatorname{sgn}(\varrho_0)^{p_0+p_1-\mathrm{degDiffer}_1}} \cdot \frac{\mathrm{Det}\,(i,k)}{\varrho_{-1}}. \tag{7}$$

The variables are as follows:

- $i = 1$, for the first remainder,
- $\varrho_{-1} = 2$, for the leading coefficient of $p(x)$,
- $\dfrac{\mathrm{Det}\,(1,0)}{\varrho_{-1}} = 72$, as computed in Example 5; this implies that the degree of the first remainder is 3, that is $d_{r_1} = 3$,
- $\mathrm{degDiffer}_1 = 0$, where in the terminology of Theorem 3, $\mathrm{degDiffer}_1 = m_1 - d_{r_1} = 3 - 3 = 0$,
- $p_0 = 1$, the number of leading zeros in the derivative of $p(x)$ if the former is also considered of degree 5; this is equivalent to $p_0 = \deg(p) - \deg(p') = 1$, the difference in degrees between $p(x)$ and its derivative $p'(x) = 10x^4 - 12x^3$,
- $p_1 = 1$, since $p_1 = \deg(p') - d_{r_1} = 1$,[15]
- p_List $= [p_0] = [1]$; we need this list so that we can compute variable v,
- $u_0 = 1$, since $u_0 = 1 + 2 + \cdots + p_0 = 1$ as stated in Theorem 3,

[15] *Caveat:* Even though we need p_1 in order to compute expression (7), it should be appended to p_List in the next round, when $i = 2$.

- u_List $= [u_0] = [1]$; each member of the list will be used as an exponent to -1 as shown in expression 5,
- $v_0 = 1$, since $v_0 = p_0 + p_1 + \cdots + p_0 = 1$ as stated in Theorem 3, and
- $\mathrm{sgn}(\varrho_0) = \mathrm{sgn}(10) = 1$, for the sign of the leading coefficient of $p'(x)$.[16]

Replacing the variables in expression (7) with their values we obtain 72, which exactly matches — both in sign and value — the leading coefficient of the remainder we computed by matrix triangularization. Hence, the first Sturmian remainder is $r^{(1)} = 72x^3 + 300$.

To compute the second Sturmian remainder we use rows 3 and 4 of S' to form the 3-row matrices $M, M1, M2$ and from the last one we obtain a candidate for the second remainder, namely $1800x - 2160$. Obviously, the second row of $M2$ will replace the 5 th row of S' but the 3 rd row of $M2$ will now replace the 7 th row of S'![17] Row 6 in S' is the reduntant row — equivalent to the extra polynomial of degree 1 we obtained in Example 5 — and so can be replaced again by the second row of $M2$ rotated by one.

```
> M := [[0,0,0,72,0,0,300,0,0,0],[0,0,0,-6,0,0,0,-30,0,0],
[0,0,0,0,72,0,0,300,0,0]]
```

$$\begin{pmatrix} 0 & 0 & 0 & 72 & 0 & 0 & 300 & 0 & 0 & 0 \\ 0 & 0 & 0 & -6 & 0 & 0 & 0 & -30 & 0 & 0 \\ 0 & 0 & 0 & 0 & 72 & 0 & 0 & 300 & 0 & 0 \end{pmatrix}$$

```
>  M1 :=  pivot( M, 0, 2 )
```

$$\begin{pmatrix} 0 & 0 & 0 & 72 & 0 & 0 & 300 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1800 & -2160 & 0 & 0 \\ 0 & 0 & 0 & 0 & 72 & 0 & 0 & 300 & 0 & 0 \end{pmatrix}$$

```
 >  M2 := rowSwap( M1, 1, 2 )
```

$$\begin{pmatrix} 0 & 0 & 0 & 72 & 0 & 0 & 300 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 72 & 0 & 0 & 300 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1800 & -2160 & 0 & 0 \end{pmatrix}$$

[16] If the leading coefficient is positive, its sign will not play any role in the evaluation of the sign of expression (7); hence, it can be left out altogether.

[17] If we start enumeration with 0, the number of the row in S' that will be replaced is indicated by the number of leading zeros in the 3 rd row of $M2$.

```
> S'
```

$$\begin{pmatrix} 2 & -3 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 10 & -12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -6 & 0 & 0 & 0 & -30 & 0 & 0 & 0 \\ 0 & 0 & 0 & 72 & 0 & 0 & 300 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 72 & 0 & 0 & 300 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 72 & 0 & 0 & 300 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1800 & -2160 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10 & -12 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -3 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 10 & -12 & 0 & 0 & 0 \end{pmatrix}$$

To see if we have to correct the sign of the polynomial or to change its coefficients to modified subresultants we have to evaluate again expression (5) which now becomes

$$\frac{(-1)^{u_1}\,(-1)^{u_0}\,(-1)^{v_1}}{\operatorname{sgn}(\varrho_1)^{p_1+p_2-\mathrm{degDiffer}_2}\,\operatorname{sgn}(\varrho_0)^{p_0+p_1}} \cdot \frac{\mathrm{Det}\,(i,k)}{\varrho_{-1}}. \tag{8}$$

Below are the new variables and the ones that *changed*:

- $i = 2$, for the second remainder,
- $\dfrac{\mathrm{Det}\,(2,1)}{\varrho_{-1}} = 2160$, as computed in Example 5; this implies that the degree of the second remainder is 1, that is $d_{r_2} = 1$,
- $\mathrm{degDiffer}_2 = 1$, where in the terminology of Theorem 3, $\mathrm{degDiffer}_2 = m_2 - d_{r_2} = 2 - 1 = 1$,
- $p_2 = 2$, since $p_2 = d_{r_1} - d_{r_2} = 3 - 1 = 2$,
- p_List $= [p_0, p_1] = [1, 1]$; we need this list so that we can compute variable v,
- $u_1 = 1$, since $u_1 = 1 + 2 + \cdots + p_1 = 1$ as stated in Theorem 3,
- u_List $= [u_0, u_1] = [1, 1]$; each member of the list will be used as an exponent to -1 as shown in expression 5,

- $v_1 = 2$, since $v_1 = p_0 + p_1 + \cdots + p_1 = 1 + 1 = 2$ as stated in Theorem 3, and
- $\mathrm{sgn}(\varrho_1) = \mathrm{sgn}(72) = 1$, for the sign of the leading coefficient of the first remainder $r^{(1)}(x)$.

Replacing the variables in expression (8) with their values we obtain 2160, which means that the sign of the polynomial $1800x - 2160$ was correctly computed by matrix triangularization. However, the values of the coefficients are not modified subresultants. This is easily rectified by multiplying $1800x - 2160$ times $2160/1800$. Indeed, we have

```
> simplify( ( 2160 / 1800 ) * ( 1800x - 2160 ) )
```

$$2160 \cdot x - 2592$$

and checking back with Example 5 we see that -2592 is also a modified subresultant. Therefore, the second Sturmian remainder is $r^{(2)} = 2160x - 2592$.

To compute the third, and final, remainder in the Sturm sequence we form matrix M, which now has 4 rows![18] This is one of the new features in this extended method that we describe. The rows of M can be formed either from rows of S' or from the correctly computed remainders. We follow the first approach and form matrices $M, M1, M2, M3$. From $M3$, the triangularized form of matrix M, we obtain as candidate remainder the constant 2475194112000, which replaces the last row of S'. The 2 nd and 3 rd rows of $M3$ replace, respectively, the 8 th and 9 th row of S'.

```
> m1:=[0,0,0,0,0,0,1800,-2160,0,0]:;
> m2:=[0,0,0,0,0,0,0,1800,-2160,0]:;
> m3 := [0,0,0,0,0,0,0,0,1800,-2160]:;
> m4 := [0,0,0,0,0,0,72,0,0,300]:;
> M := [m1, m2, m3, m4]
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1800 & -2160 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1800 & -2160 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1800 & -2160 \\ 0 & 0 & 0 & 0 & 0 & 0 & 72 & 0 & 0 & 300 \end{pmatrix}$$

```
> M1 := pivot( M, 0, 6 )
```

[18] That is, M becomes a matrix with $(3 + \mathrm{degDiffer})$ rows.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1800 & -2160 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1800 & -2160 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1800 & -2160 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 155520 & 0 & 540000 \end{pmatrix}$$

```
> M2 := pivot( M1, 1, 7, -1 )
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1800 & -2160 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1800 & -2160 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1800 & -2160 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 335923200 & 972000000 \end{pmatrix}$$

```
> M3 := pivot( M2, 2, 8, -2 )
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1800 & -2160 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1800 & -2160 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1800 & -2160 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2475194112000 \end{pmatrix}$$

```
> S'
```

$$\begin{pmatrix} 2 & -3 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 10 & -12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -6 & 0 & 0 & 0 & -30 & 0 & 0 & 0 \\ 0 & 0 & 0 & 72 & 0 & 0 & 300 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 72 & 0 & 0 & 300 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 72 & 0 & 0 & 300 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1800 & -2160 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1800 & -2160 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1800 & -2160 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2475194112000 \end{pmatrix}$$

To see if we have to correct the sign of the constant polynomial 2475194112000 or to change it to a modified subresultant we have to evaluate again expression (5) which now becomes

$$\frac{(-1)^{u_2}(-1)^{u_1}(-1)^{u_0}(-1)^{v_2}}{\operatorname{sgn}(\varrho_2)^{p_2+p_3-\text{degDiffer}_3}\operatorname{sgn}(\varrho_1)^{p_1+p_2}\operatorname{sgn}(\varrho_0)^{p_0+p_1}} \cdot \frac{\operatorname{Det}(i,k)}{\varrho_{-1}}. \tag{9}$$

Below are the new variables and the ones that *changed*:

- $i = 3$, for the second remainder,
- $\dfrac{\mathrm{Det}\,(3,0)}{\varrho_{-1}} = 11459232$, as computed in Example 5; this implies that the degree of the third remainder is 0, that is $d_{r_3} = 0$,
- $\mathrm{degDiffer}_3 = 0$, where in the terminology of Theorem 3, $\mathrm{degDiffer}_3 = m_3 - d_{r_3} = 0 - 0 = 0$,
- $p_3 = 1$, since $p_3 = d_{r_2} - d_{r_3} = 1 - 0 = 1$,
- p_List $= [p_0, p_1, p_2] = [1, 1, 2]$; we need this list so that we can compute variable v,
- $u_{i-1} = u_2 = 3$, since $u_2 = 1 + 2 + \cdots + p_2 = 3$ as stated in Theorem 3,
- u_List $= [u_0, u_1, u_2] = [1, 1, 3]$; each member of the list will be used as an exponent to -1 as shown in expression 5,
- $v_{i-1} = v_2 = 4$, since $v_2 = p_0 + p_1 + \cdots + p_2 = 1 + 1 + 2 = 4$ as stated in Theorem 3, and
- $\mathrm{sgn}(\varrho_2) = \mathrm{sgn}(2160) = 1$, for the sign of the leading coefficient of the first remainder $r^{(2)}(x)$.

Replacing the variables in expression (9) with their values we obtain -11459232, which means that our constant with correct sign is -2475194112000. Obviously, this value — computed by matrix triangularization — is not a modified subresultant. This is easily rectified by multiplying -2475194112000 times $11459232/2475194112000$. Indeed, we have

$$\frac{11459232}{2475194112000} \cdot (-2475194112000) = -11459232$$

and the third member of the Sturm sequence is $r^{(3)} = -11459232$.

Therefore the Sturm sequence of $p(x) = 2x^5 - 3x^4 - 3$ is

$$\left[2x^5 - 3x^4 - 3,\ \ 10x^4 - 12x^3,\ \ 72x^3 + 300,\ \ 2160x - 2592,\ \ -11459232\right],$$

which agrees with the result obtained with the function `sturm` of `Xcas`.[19]

```
>  sturm( 2x^5 - 3x^4 - 3 )[1]
```

[19] The last term of the Sturm sequence in `Xcas` is *not* a modified subresultant! However, this is just a minor detail.

$$[[2, -3, 0, 0, 0, -3], [10, -12, 0, 0, 0], [72, 0, 0, 300], [2160, -2592], -1782139760640]$$

We have implemented version 1 of the generalized matrix triangularization (VanVleck-Pell-Gordon) procedure in `Xcas` in the function `sturmSeqVanVleckPellGordon`. This function also uses `sylvester2`, `my_sqrfree2` and `row2poly` as does `sturmSeqVanVleck`. However, instead of Van Vleck's "sign rule", it uses the functions:

- `gaps`, which is activated when a pivot takes place in the process of triangularizing the 3-row matrix M. It uses the theorem by Pell and Gordon to compute the correct sign of the remainder and also to force its coefficients to become modified subresultants,
- `compute_correct_sign`, used by `gaps` only when a pivot took place in a complete sequences. It also uses the Pell-Gordon theorem to determine the correct sign of the remainder.

The whole program is at the link `http://inf-server.inf.uth.gr/~akritas/publications/VanVleck_Pell_Gordon`.

4. Conclusions. We have presented two matrix triangularization methods for computing, in $\mathbb{Z}[x]$, the Sturm sequence of a polynomial $p(x)$.

The first method is due to Van Vleck and applies only to complete Sturm sequences. In this method Van Vleck's own "sign rule" is used to compute the correct signs of the polynomials in the sequence. To reduce the size of the coefficients — instead of removing the content as was done by Van Vleck — we use an old theorem by Sylvester and force the coefficients to become modified subresultants. This method is extremely fast and — based on the assumption that the Sturm sequence will be complete — it should be used when there are no missing terms in the polynomial whose Sturm sequence we want to compute.

The second method — called VanVleck-Pell-Gordon — was developed by us and applies to both complete and incomplete Sturm sequences. We use Theorem 3 of 1917, by Pell and Gordon, not only to compute the correct sign of the polynomial remainders, but also to force their coefficients to become modified subresultants. The extra cost in this method is that we have to compute one determinant (of increasing dimensions) for each polynomial remainder. However, this cost turns out to be negligible given the probabilistic algorithm of `Xcas` for computing large determinants.

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