Various New Methods for Computing Subresultant Polynomial Remainder Sequences (PRS's)

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Problem 1/2

Problem 1/2

- See https://en.wikipedia.org/wiki/Polynomial_greatest_common_divisor#Pseudo-remainder_sequences.

Problem 2/2

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- "Be aware that some computer algebra systems may normalize remainders from the Euclidean Algorithm which messes up the sign."

- The culprit is ...

- ... the function `prem(f, g, x)` for computing pseudo-remainders!!

Outline of the talk 1/2

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- We first present Sylvester's matrices of 1840 and 1853 and show how they can be used to compute, in $\mathbb{Z}[x]$, Euclidean subresultant prs's and Sturmian modified subresultant sequences (or modified subresultant prs), respectively.

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- We first present Sylvester's matrices of 1840 and 1853 and show how they can be used to compute, in $\mathbb{Z}[x]$, Euclidean subresultant prs's and Sturmian modified subresultant sequences (or modified subresultant prs), respectively.
- We next review the Collins-Brown-Traub method for computing subresultant polynomial remainder sequences (prs's) in $\mathbb{Z}[x]$ using pseudo-divisions. We will use the `sympy` pseudo-remainder function `prem(f, g, x)` and point out the drawback of this approach.

Outline of the talk 2/2

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- **2nd method**, which performs rational divisions, i.e. polynomial divisions are performed in $\mathbb{Q}[x]$ but the final result is in $\mathbb{Z}[x]$.
- **3nd method**, which triangularizes the Sylvester matrix of 1853 in $\mathbb{Z}[x]$, while using the Sylvester matrix of 1840 to compute the correct sign and value of the polynomial coefficients.

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In the first row of the first group (of m rows) are the coefficients of $f(x)$ with $m-1$ trailing zeros. The second row in this group differs from the first one in that its elements have been rotated to the right by one. A total of $m-1$ rotations are needed to construct the first group of rows.

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In the first row of the second group (of n rows) are the coefficients of $g(x)$ with $n-1$ trailing zeros. The second row in this group differs from the first one in that its elements have been rotated to the right by one. A total of $n-1$ rotations are needed to construct the second group of rows.

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The sylvester1 matrix of f, g is

$$\begin{pmatrix} 1 & 0 & -7 & 7 & 0 \\ 0 & 1 & 0 & -7 & 7 \\ 3 & 0 & -7 & 0 & 0 \\ 0 & 3 & 0 & -7 & 0 \\ 0 & 0 & 3 & 0 & -7 \end{pmatrix}.$$

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Both rows in the first pair have $2n - (n + 1)$ trailing zeros and both rows of the last pair have $2n - (n + 1)$ leading zeros. The second pair of rows differs from the first one in that the elements of both rows have been rotated to the right by one. A total of $2n - (n + 1)$ rotations are needed to construct `sylvester2`.

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For example, given the same polynomials $f = x^3 - 7x + 7$ and $g = 3x^2 - 7$, we have:

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If there are no gaps in the sequence of the degrees of the polynomials in the prs (i.e. no missing remainder), then the prs is called *complete*, else it is called *incomplete*.

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Definition

The *sign sequence* of a polynomial remainder sequence (prs) is the sequence of signs of the leading coefficients of its polynomials.

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Given $f(x), g(x) \in \mathbb{Z}[x]$ of degrees $\deg(f) = n$ and $\deg(g) = m$ with $n \geq m$ their (proper) *subresultant* prs is a sequence of polynomials similar to the *Euclidean* prs, the sequence obtained by applying in $\mathbb{Q}[x]$ Euclid's polynomial gcd algorithm on $f(x), g(x)$.

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The sign sequences of the Euclidean and subresultant prs's are identical for complete prs's. They may differ for incomplete prs's.

Modified subresultant prs

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Given $f(x), g(x) \in \mathbb{Z}[x]$ of degrees $\deg(f) = n$ and $\deg(g) = n - 1$, with $g = f'$, their *modified subresultant* prs is a sequence of polynomials similar to the sequence obtained by applying in $\mathbb{Q}[x]$ Sturm's algorithm on $f(x), g(x)$.

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▶ Sylvester had proved that for complete sequences, the proper or modified subresultant prs is identical to the Euclidean or Sturmian prs, respectively (computed in $\mathbb{Z}[x]$), where the polynomial remainder p_{i+2} in the latter sequences has been divided by the square of the leading coefficient of the polynomial p_i.

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▶ Moreover, Sylvester pointed out, that the coefficients of the polynomial remainders obtained as subresultants are the *smallest possible* without introducing rationals and without computing (integer) greatest common divisors.

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- For incomplete prs's, Pell and Gordon presented in 1917 a theorem, relating the polynomials in a Sturmian prs (computed with polynomial divisions in $\mathbb{Q}[x]$) with the corresponding ones of the modified subresultant prs — obtained in $\mathbb{Z}[x]$ by evaluating determinants of submatrices of `sylvester2`.

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- The corresponding result for incomplete Euclidean prs's and `sylvester1` was discovered 30 and 50 years latter by Habicht (1947) and Collins, Brown, Traub, ($\simeq$ 1970) respectively.

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The coefficients of $-42x + 63$ are determinants of submatrices of `sylvester1`, whereas for the last member of the sequence we have $-49 = \det(sylvester1(f, g, x))$

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$$x^3 - 7x + 7, \quad 3x^2 - 7, \quad 42x - 63, \quad 49.$$

The coefficients of $42x - 63$ are determinants of submatrices of `sylvester2`, whereas for the last member of the sequence we have $49 = \det(sylvester2(f, g, x))$

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- In `http://planetmath.org/sturmstheorem` it is stated:

- "Be aware that some computer algebra systems may normalize remainders from the Euclidean Algorithm which messes up the sign."

- Why is that?? Who is the culprit?

- The culprit is the function prem, that computes the pseudo-remainder. As explained below, we completely avoid it in our new methods.

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Euclidean PRS

Sturmian PRS

In summary ... 3/4

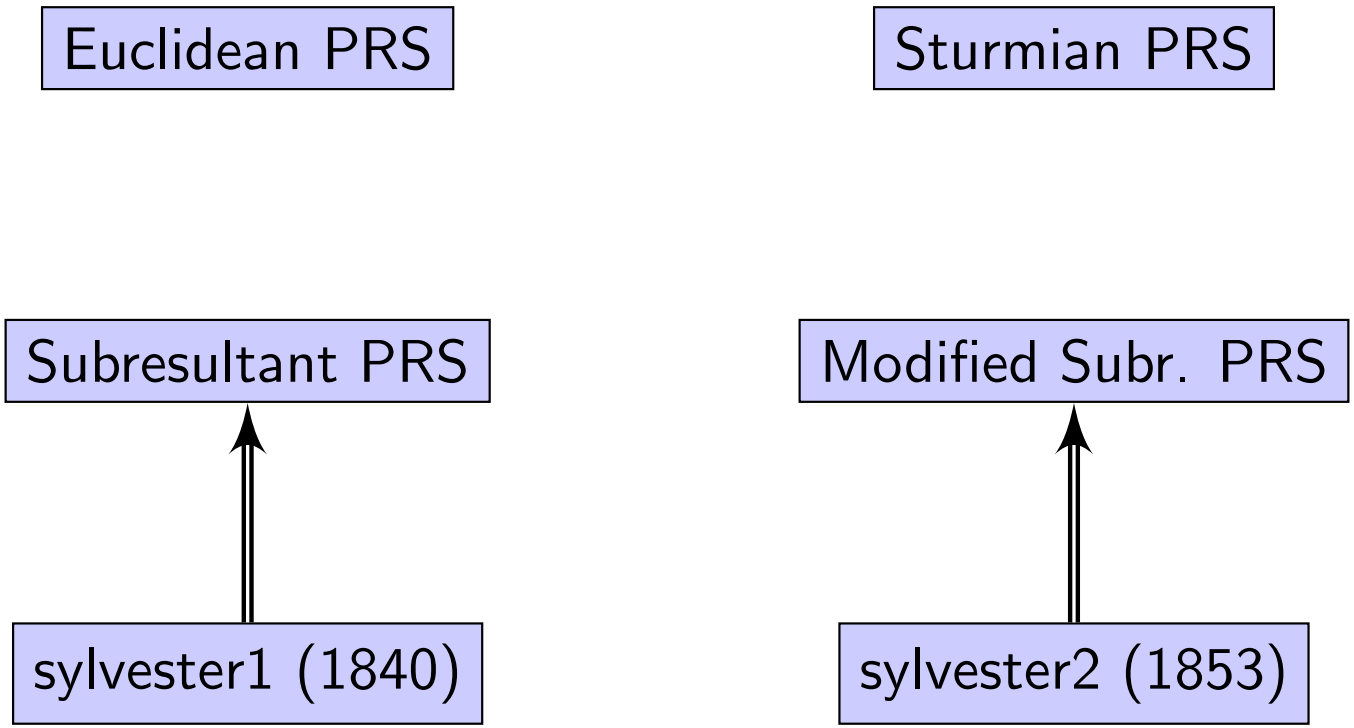
Euclidean PRS

Sturmian PRS

Subresultant PRS

Modified Subr. PRS

In summary ... 3/4



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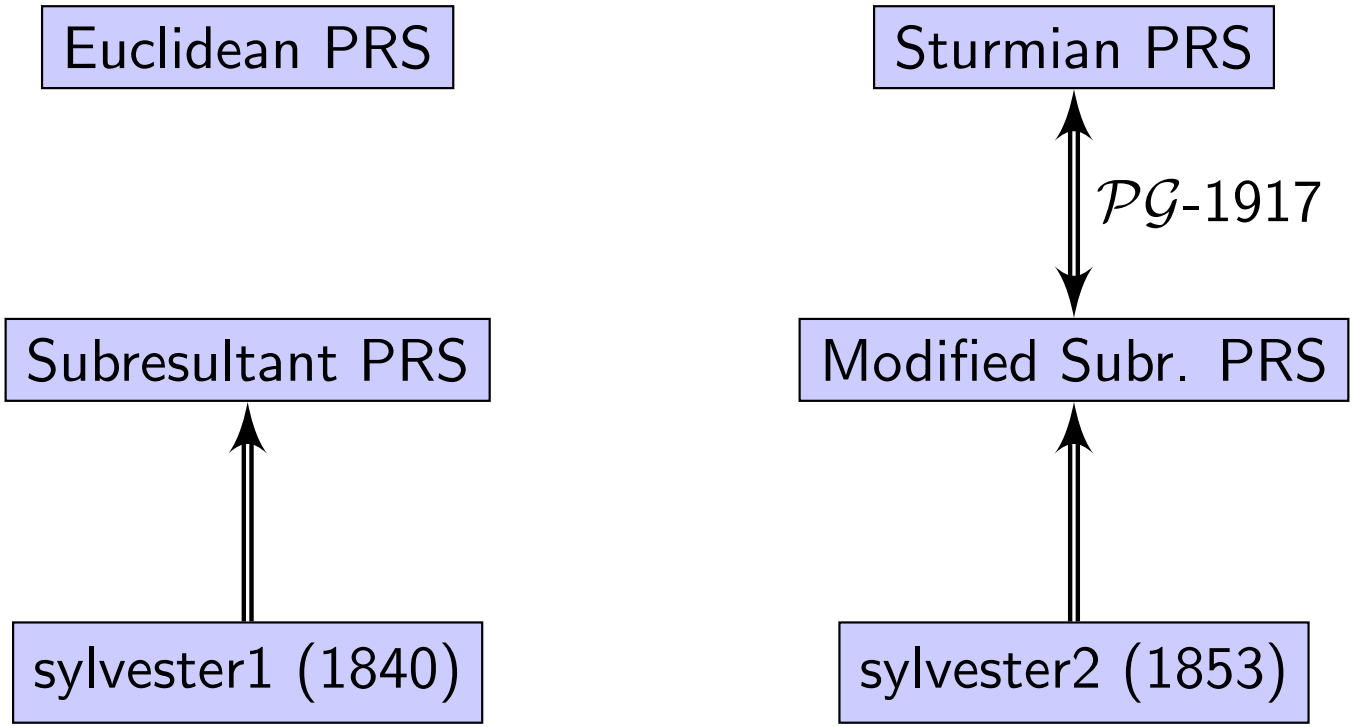

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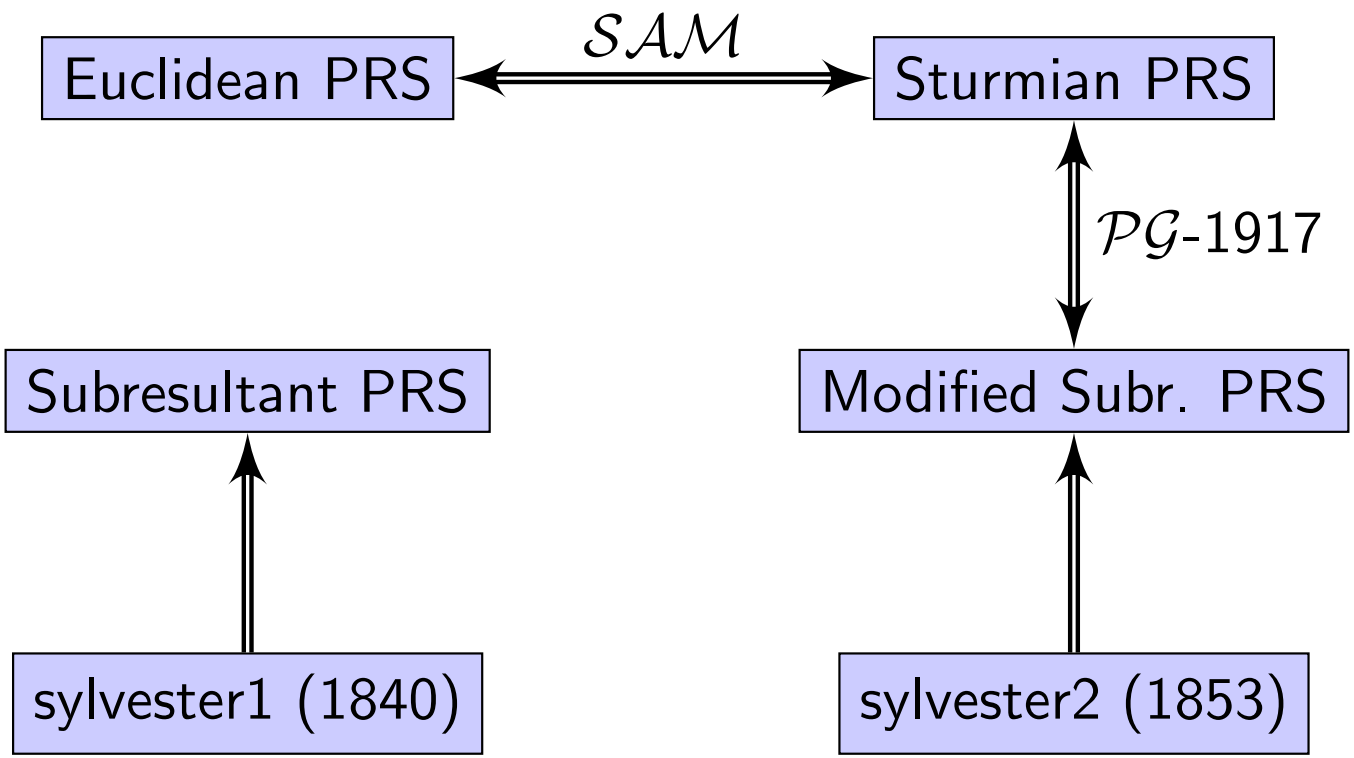

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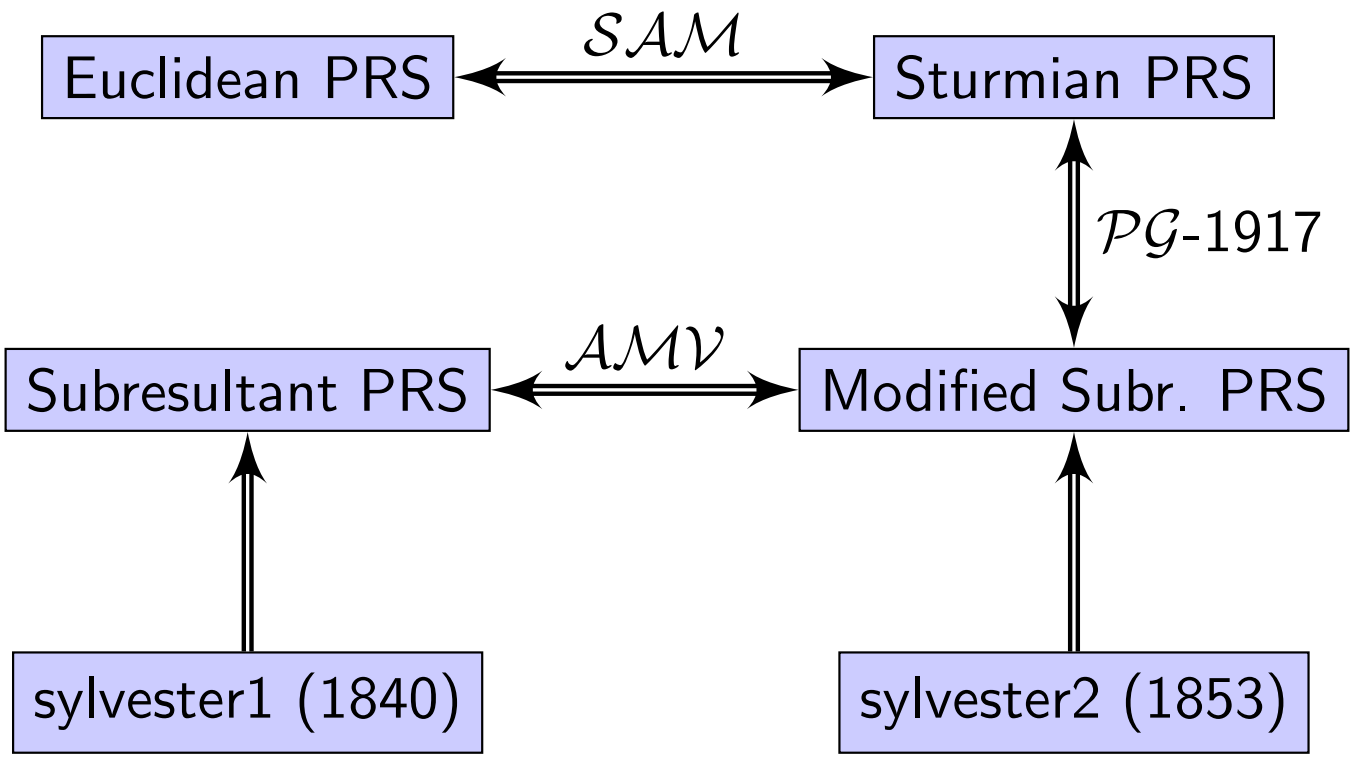

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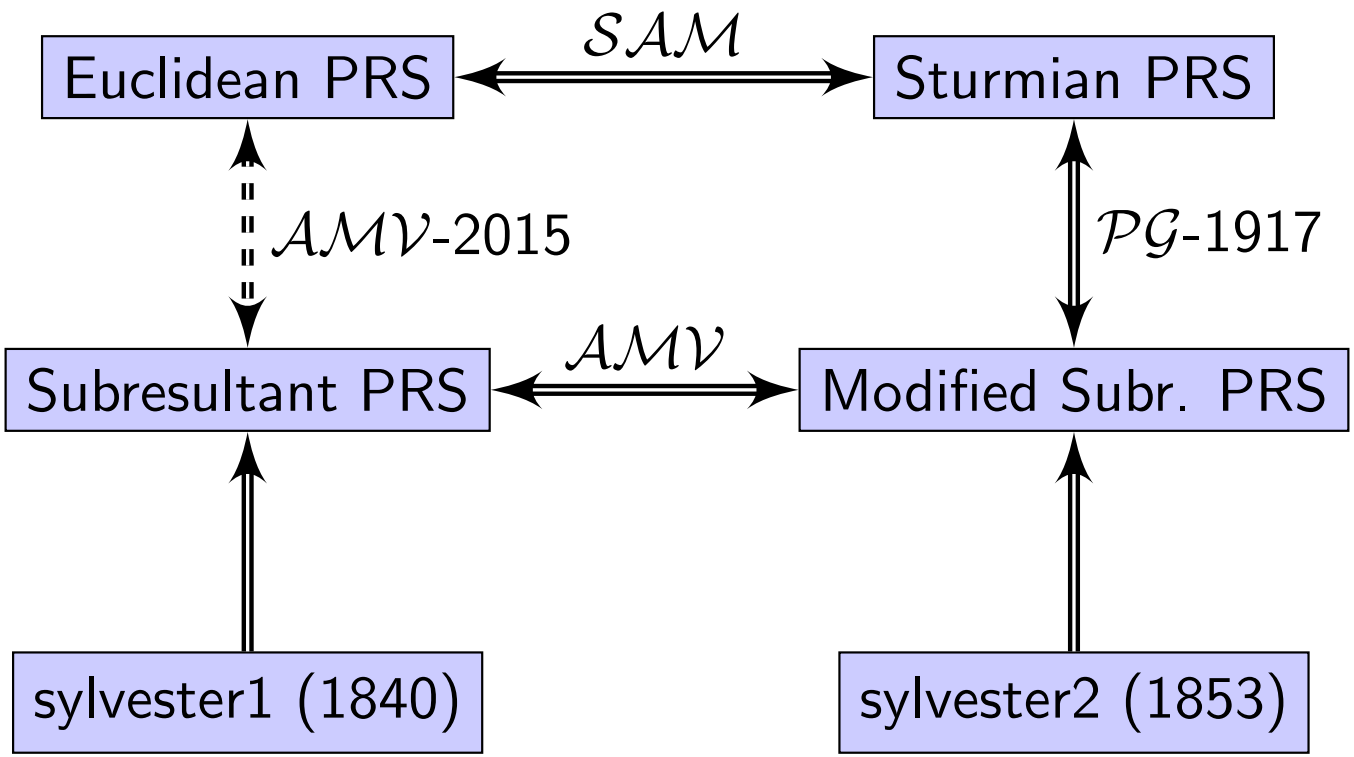

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- The problem with `prem(f, g, x)` is that whereas ...
- ... it correctly computes the subresultant prs,
- ... it computes the wrong signs of the Euclidean prs.
- Hence, `prem(f, g, x)` should be avoided.

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- If $LC(R_{i-1})$ is the leading coefficient of the divisor R_{i-1}, and $\delta = degree(R_{i-2}) - degree(R_{i-1}) + 1$, then (to exactly divide by the leading coefficient of the divisor) `prem` premultiplies the dividend times the leading coefficient of the divisor according to the formula

$$LC(R_{i-1})^{\delta} \cdot R_{i-2} = q_{i-2} \cdot R_{i-1} + R_i.$$

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- The expression β_i is given by the recurrence relations

$$\psi_1 = -1, \quad \beta_1 = (-1)^{\delta_1},$$
$$\psi_i = \frac{(-LC(R_{i-2}))^{\delta_{i-1}-1}}{\psi_{i-1}^{\delta_{i-1}-2}}, \qquad i > 1,$$
$$\beta_i = -LC(R_{i-2}) \cdot \psi_i^{\delta_i - 1}, \qquad i > 1,$$

where ...

Collins-Brown-Traub (CBT) subresultant prs algorithm 2/2

- This pre-multiplication, however, leads to an increase in the size of the coefficients, and a certain factor, β_i, is divided out to reduce the coefficients to subresultants.

- The expression β_i is given by the recurrence relations

$$\psi_1 = -1, \quad \beta_1 = (-1)^{\delta_1},$$
$$\psi_i = \frac{(-LC(R_{i-2}))^{\delta_{i-1}-1}}{\psi_{i-1}^{\delta_{i-1}-2}}, \qquad i > 1,$$
$$\beta_i = -LC(R_{i-2}) \cdot \psi_i^{\delta_i - 1}, \qquad i > 1,$$

where ...

- $$\delta_i = degree(R_{i-2}) - degree(R_{i-1}) + 1, \qquad i \geqslant 1.$$

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- If the a prs is complete, then the sign sequences are identical; that is, equation $LC(R_{i-1})^{\delta} \cdot R_{i-2} = q_{i-2} \cdot R_{i-1} + R_i$ can be safely used without any problem.

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- However, if the prs is incomplete, then the sign sequences are not necessarily identical and this can be really confusing.

- After all, `prem(f, g, x)` is used to facilitate divisions in $\mathbb{Z}[x]$ and its purpose is not to distort sign sequences.

Example

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- Consider the polynomials $f = x^8 + x^6 - 3x^4 - 3x^3 + 8x^2 + 2x - 5$ and $g = 3x^6 + 5x^4 - 4x^2 - 9x + 21$ which result in an incomplete Euclidean prs of degrees 8, 6, 4, 2, 1, 0.

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- On the other hand, performing polynomial divisions in $\mathbb{Z}[x]$ with the help of `prem(f, g, x)` the sign sequence of the prs is

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Question

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- Would you use the function `prem(f, g, x)` on the polynomials of the previous example to compute their (generalized) Sturm sequence??

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- Of course you would not ...

- ... even though it turns out that it does not disturb the sign sequence of this particular example.

Example where `prem(f, g, x)` fails in Sturm sequence

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- On the other hand, performing polynomial divisions in $\mathbb{Z}[x]$ with the help of `prem(f, g, x)` the wrong sign sequence of the prs is

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- The recently discovered theorem by Pell-Gordon (1917) allows us to correctly compute the signs of any Euclidean subresultant prs and renders `prem(f, g, x)` useless.

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- The recently discovered theorem by Pell-Gordon (1917) allows us to correctly compute the signs of any Euclidean subresultant prs and renders `prem(f, g, x)` useless.

- We have written the `sympy` function `euclid_PG(f, g, x, method=0)` which performs polynomial divisions in $\mathbb{Q}[x]$ — and hence computes the correct signs; subsequently, the rationals are converted to integers using the Pell-Gordon theorem.

A more efficient solution using `remZ(f, g, x)`

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- Premultiply times the absolute value of the leading coefficient of the divisor; that is, use the function `remZ(f, g, x)`, which implements the equation

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- `remZ(f, g, x)` is equivalent to `euclid_PG(f, g, x, method=1)` and, hence, the sign sequences of Euclidean or Sturmian prs's in $\mathbb{Z}[x]$ are identical with the corresponding ones computed in $\mathbb{Q}[x]$.

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- Collateral damage: Using `remZ(f, g, x)` the CBT subresultant prs algorithm does not work properly. Hence, new methods need to be developed!

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Preliminaries

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- ... that Pell and Gordon perform the polynomial divisions in $\mathbb{Q}[x]$ and that ....

- ... they evaluate determinants of submatrices of `sylvester2`.

- It is worth noting that the `sylvester2` matrix is equivalent to the `bezout` matrix.

The Pell-Gordon theorem 1/3

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Theorem

(Pell-Gordon, 1917) *Let* $A = a_0x^n + a_1x^{n-1} + \cdots + a_n$ *and* $B = b_0x^n + b_1x^{n-1} + \cdots + b_n$ *be two polynomials of the n-th degree. Modify the process of finding the highest common factor of A and B by taking at each stage the negative of the remainder. Let the i-th modified remainder be*

$$R^{(i)} = r_0^{(i)}x^{m_i} + r_1^{(i)}x^{m_i-1} + \cdots + r_{m_i}^{(i)}$$

where $(m_i + 1)$ *is the degree of the preceding remainder, and where the first* $(p_i - 1)$ *coefficients of* $R^{(i)}$ *are zero, and the* p_i*-th coefficient* $\varrho_i = r_{p_i-1}^{(i)}$ *is different from zero. Then ...*

The Pell-Gordon theorem 2/3

The Pell-Gordon theorem 2/3

Theorem

... *for* $k = 0, 1, \ldots, m_i$ *the coefficients* $r_k^{(i)}$ *are given by*

$$r_k^{(i)} = \frac{(-1)^{u_{i-1}}(-1)^{u_{i-2}} \cdots (-1)^{u_1}(-1)^{v_{i-1}}}{\varrho_{i-1}^{p_{i-1}+1}\varrho_{i-2}^{p_{i-2}+p_{i-1}} \cdots \varrho_1^{p_1+p_2}\varrho_0^{p_1}} \cdot \det(i, k),$$

where $u_{i-1} = 1 + 2 + \cdots + p_{i-1}$, $\quad v_{i-1} = p_1 + p_2 + \cdots + p_{i-1}$ *and ...*

The Pell-Gordon theorem 3/3

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Theorem

...

$$\det(i,k) = \begin{vmatrix} a_0 & a_1 & a_2 & \cdots & \cdot & \cdot & \cdots & a_{2v_{i-1}} & a_{2v_{i-1}+1+k} \\ b_0 & b_1 & b_2 & \cdots & \cdot & \cdot & \cdots & b_{2v_{i-1}} & b_{2v_{i-1}+1+k} \\ 0 & a_0 & a_1 & \cdots & \cdot & \cdot & \cdots & a_{2v_{i-1}-1} & a_{2v_{i-1}+k} \\ 0 & b_0 & b_1 & \cdots & \cdot & \cdot & \cdots & b_{2v_{i-1}-1} & b_{2v_{i-1}+k} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & a_0 & a_1 & \cdots & a_{v_{i-1}} & a_{v_{i-1}+1+k} \\ 0 & 0 & 0 & \cdots & b_0 & b_1 & \cdots & b_{v_{i-1}} & b_{v_{i-1}+1+k} \end{vmatrix}.$$

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- We modified the formula as shown below with the changes appearing in red

$$r_k^{(i)} = \frac{(-1)^{u_{i-1}} (-1)^{u_{i-2}} \cdots (-1)^{u_1} (-1)^{u_0} (-1)^{v_{i-1}}}{\varrho_{i-1}^{p_{i-1}+\mathbf{p_i}-\mathbf{degDiffer}} \varrho_{i-2}^{p_{i-2}+p_{i-1}} \cdots \varrho_1^{p_1+p_2} \varrho_0^{\mathbf{p_0}+p_1}} \cdot \frac{\det(i,k)}{\varrho_{\mathbf{-1}}},$$

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where ...

- ... $\varrho_{-1} = a_0$, is the leading coefficient of A and `degDiffer` is the difference between the expected degree m_i and the actual degree of the remainder.

Remarks 2/2

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- Clearly, using the modified formula mentioned in the previous frame and multiplying times the absolute value of the denominator we uniquely determine the complete or incomplete Sturmian polynomial remainders in $\mathbb{Z}[x]$.

The main application of the Pell-Gordon theorem

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- The failed attempts to solve the "signs problem" lasted close to 30 years, until P.S. Vigklas discovered the Pell-Gordon theorem.

- Pell and Gordon showed the relationship that exists between the polynomials of an incomplete prs computed by divisions in $\mathbb{Q}[x]$ and the ones whose coefficients are computed as modified subresultants (determinants of submatrices) of the `sylvester2` matrix.

The essence of the Pell-Gordon theorem 1/2

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▶ The Pell-Gordon theorem proved that if we want to compute an incomplete Sturm sequence by Van Vleck's matrix triangularization method, the only way to obtain the correct signs is to evaluate — for each polynomial remainder — one modified subresultant (determinant of a submatrix) of the `sylvester2` matrix.

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▶ In other words, the essence — or the main idea — of the Pell-Gordon theorem is that we can now use determinant evaluation as a new method for computing modified subresultant prs's.

The essence of the Pell-Gordon theorem prs's 2/2

The essence of the Pell-Gordon theorem prs's 2/2

- We used the above idea not only to compute the correct signs but also to force all coefficients of the polynomials in the Sturm sequence to become modified subresultants of the `sylvester2` matrix. This way, Van Vleck's triangularization method works also for incomplete prs's and, we are thus able to compute modified subresultant prs's of any kind.

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- Variations of this main idea have been used to develop new methods for computing (proper) subresultant prs's of any kind. These methods are discussed below.

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The main idea

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- ... where now — for each polynomial remainder — we evaluate one subresultant (determinant of a submatrix) of the `sylvester1` matrix.

- This technique is applied in all three new methods for computing subresultant prs's. Moreover, instead of `sylvester1` we can also use the `bezout` matrix, which has smaller dimensions.

Details of the main idea 1/2

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- Suppose that we have computed the i-th polynomial remainder of *expected* degree (expDeg) m_i

$$s^{(i)} = s_0^{(i)} x^{m_i} + s_1^{(i)} x^{m_i - 1} + \cdots + s_{m_i}^{(i)},$$

where all the coefficients $s_k^{(i)}, 0 \leq k \leq m_i$ are either integers or rationals, depending on the method used.

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where all the coefficients $s_k^{(i)}, 0 \leq k \leq m_i$ are either integers or rationals, depending on the method used.

- To compute the correct sign of the polynomial $s^{(i)}$ and to make its coefficients subresultants we evaluate the determinant $\det(i,j) \neq 0$ of the sub-matrix of `sylvester1` corresponding to the *actual* leading coefficient $s_j^{(i)}$ of $s^{(i)}$.

Details of the main idea 2/2

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$$\frac{s^{(i)}}{s_j^{(i)}} \cdot \det(i, j)$$

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that holds for $j < k \leq m_i$.

- Recall that $s_j^{(i)}$ is the *actual* leading coefficient of the polynomial remainder $s^{(i)}$.

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▶ This new method performs operations in $\mathbb{Z}[x]$ and computes the subresultant prs by:

- premultipling the dividend times the *absolute* value of the leading coefficient of the divisor (`remZ`),

- evaluating the subresultant of `sylvester1` corresponding to the leading coefficient of the remainder in order to obtain its correct sign and to force its coefficients to become subresultants *without* the quantities β_i of the CBT method.

In other words ...

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- If $s^{(i)}$ is the i-th remainder computed with `remZ`, $s_j^{(i)}$, is its leading coefficient and $\det(i, j)$ is the subresultant of `sylvester1` corresponding to $s_j^{(i)}$, ...

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- Clearly, in the above formula, the sign and the value of the leading coefficient of the resulting polynomial are those of the determinant.

Advantage of the method

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- The strong and most important point of this new method is that the function `remZ` does *not* distort the sign sequence of the Euclidean prs computed either in $\mathbb{Q}[x]$ or in $\mathbb{Z}[x]$ with `remZ`.

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- The strong and most important point of this new method is that the function `remZ` does *not* distort the sign sequence of the Euclidean prs computed either in $\mathbb{Q}[x]$ or in $\mathbb{Z}[x]$ with `remZ`.

- We have implemented this method in sympy as the function `subresultants_remZ(f, g, x)`, and have made it available for use.

Outline of the second method — original version

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- With this method we compute each remainder by performing polynomial divisions in $\mathbb{Q}[x]$.

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- To force the coefficients of each remainder into $\mathbb{Z}[x]$ we multiply the remainder times the absolute value of the denominator in the (modified) formula of the Pell-Gordon Theorem (note that $\det(i,k)$ is a modified subresultant of `sylvester2`)

$$r_k^{(i)} = \frac{(-1)^{u_{i-1}}(-1)^{u_{i-2}}\cdots(-1)^{u_1}(-1)^{u_0}(-1)^{v_{i-1}}}{\varrho_{i-1}^{p_{i-1}+\mathbf{p_i}-\mathbf{degDiffer}}\varrho_{i-2}^{p_{i-2}+p_{i-1}}\cdots\varrho_1^{p_1+p_2}\varrho_0^{\mathbf{p_0}+p_1}}\cdot\frac{\det(i,k)}{\varrho_{-\mathbf{1}}}.$$

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- Then, the correct sign of each remainder is obtained by evaluating the sign of the subresultant (of `sylvester1` or `bezout`) corresponding to its leading coefficient.

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- If $s^{(i)}$ is the i-th polynomial remainder computed with polynomial division in $\mathbb{Q}[x]$, $s_j^{(i)}$ is its leading coefficient, and $\det(i, j)$ is the subresultant of `sylvester1` corresponding to $s_j^{(i)}$, ...

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- We have implemented this method in sympy as the function `subresultants_PG2(f, g, x)`, and have made it available for use.

It is more efficient because ...

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- We now use only `sylvester1` to both convert the rational coefficients of each remainder to absolute values of subresultants and simultaneously compute the correct sign of its leading coefficient.

It is more efficient because ...

- We now use only `sylvester1` to both convert the rational coefficients of each remainder to absolute values of subresultants and simultaneously compute the correct sign of its leading coefficient.

- In other words, we do not have to keep track of the absolute value of the denominator in the modified formula.

Remarks on the third method

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- It is worth noting that with this method, and for the first time in the literature, we have an explicit interplay between the two Sylvester matrices, `sylvester1` and `sylvester2`.

Outline of the third method — original version 1/2

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- With this method we do not perform polynomial divisions; instead, the remainders are computed in $\mathbb{Z}[x]$ by matrix triangularization.

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- We triangularize Sylvester's matrix `sylvester2` to compute each polynomial - candidate for the subresultant prs and, then, we evaluate the subresultant (determinant of a submatrix) of `sylvester1`, corresponding to the leading coefficient of the candidate under consideration, to make it a member of the subresultant prs.

Outline of the third method — original version 2/2

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- In other words, if $s^{(i)}$ is the i-th polynomial remainder computed in $\mathbb{Z}[x]$ by triangularizing `sylvester2`, $s_j^{(i)}$ is its leading coefficient, and $\det(i,j)$ is the subresultant of `sylvester1` corresponding to $s_j^{(i)}$, ...

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- ... then

$$\frac{s^{(i)}}{s_j^{(i)}} \cdot \det(i,j)$$

is the i-th polynomial of the subresultant prs.

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- ... then

$$\frac{s^{(i)}}{s_j^{(i)}} \cdot \det(i,j)$$

is the i-th polynomial of the subresultant prs.

- We have implemented this method in `sympy` as the function `subresultants_triang(f, g, x, method=0)`. Set `method=1` to print out the triangularized matrix.

Outline of the third method — a more efficient version 1/2

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- In the more efficient method we take advantage of the fact that polynomial division is equivalent to the triangularization of a "small" matrix. Hence, we do not triangularize the `sylvester2` matrix, but a number of smaller matrices from each one of which we obtain one polynomial - candidate for the subresultant prs.

Outline of the third method — a more efficient version 1/2

- In the more efficient method we take advantage of the fact that polynomial division is equivalent to the triangularization of a "small" matrix. Hence, we do not triangularize the `sylvester2` matrix, but a number of smaller matrices from each one of which we obtain one polynomial - candidate for the subresultant prs.

- We then evaluate the subresultant (determinant of the submatrix) of `sylvester1` corresponding to the leading coefficient of each polynomial - candidate, and turn it into a member of the subresultant prs by adjusting its sign and coefficients.

Outline of the second method — a more efficient version 2/2

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- In other words, if $s^{(i)}$ is the i-th polynomial remainder computed in $\mathbb{Z}[x]$ by triangularizing a "smaller" matrix, $s_j^{(i)}$ is its leading coefficient, and $\det(i,j)$ is the subresultant of `sylvester1` corresponding to $s_j^{(i)}$, ...

Outline of the second method — a more efficient version 2/2

- In other words, if $s^{(i)}$ is the i-th polynomial remainder computed in $\mathbb{Z}[x]$ by triangularizing a "smaller" matrix, $s_j^{(i)}$ is its leading coefficient, and $\det(i,j)$ is the subresultant of `sylvester1` corresponding to $s_j^{(i)}$, ...

- ... then

$$\frac{s^{(i)}}{s_j^{(i)}} \cdot \det(i,j)$$

is the i-th polynomial of the subresultant prs.

Outline of the second method — a more efficient version 2/2

- In other words, if $s^{(i)}$ is the i-th polynomial remainder computed in $\mathbb{Z}[x]$ by triangularizing a "smaller" matrix, $s_j^{(i)}$ is its leading coefficient, and $\det(i,j)$ is the subresultant of `sylvester1` corresponding to $s_j^{(i)}$, ...

- ... then

$$\frac{s^{(i)}}{s_j^{(i)}} \cdot \det(i,j)$$

is the i-th polynomial of the subresultant prs.

- We have implemented this method in sympy as the function `subresultants_triang2(f, g, x)`, and have made it available for use.

Table of contents

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