Polynomial Real Root Isolation Using Vincent’s Theorem of 1836

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What is isolation?

Isolation of the real roots of a polynomial is the process of finding real open intervals such that each interval contains exactly one real root and every real root is contained in some interval.

To determine the values of the real roots, isolation is followed by approximation to any desired degree of accuracy.

One of — if not — the first to employ the isolation/approximation approach was Budan and we begin our talk with him.
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- One of — if not — the first to employ the isolation/approximation approach was Budan and we begin our talk with him.
Outline of the talk

Budan’s work of 1807
Vincent’s Theorem of 1836
Uspensky’s Extension of Vincent’s Theorem
Various Implementations of Vincent’s Theorem

Budan’s theorem and some other discoveries described in his book of 1807.

Vincent’s theorem of 1836, which is based on the one by Budan.

Uspensky’s extension of Vincent’s theorem, which appeared in his book published posthumously in 1948.

VAS, one of the three methods derived from Vincent’s theorem for the isolation of the real roots of polynomials.

Bounds on the values of the positive roots, which determine the efficiency of VAS.
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- VAS, one of the three methods derived from Vincent’s theorem for the isolation of the real roots of polynomials.
- Bounds on the values of the positive roots, which determine the efficiency of VAS.
Consider the polynomial $p(x) = a_n x^n + \cdots + a_1 x + a_0$, where $p(x) \in \mathbb{R}[x]$ and let $\var{p}$ represent the number of sign changes or variations (positive to negative and vice-versa) in the sequence of coefficients $a_n, a_{n-1}, \ldots, a_0$.

**Theorem**
The number $\var{p}$ of real roots — multiplicities counted — of the polynomial $p(x) \in \mathbb{R}[x]$ in the open interval $(0, \infty)$ is bounded above by $\var{p}$; that is, we have $\var{p} \geq \var{p}$. If $\var{p} > \var{p}$ then their difference is an even number.
Descartes’ rule of signs (1637) — saved from oblivion by Budan

Consider the polynomial

\[ p(x) = a_n x^n + \cdots + a_1 x + a_0, \]

where \( p(x) \in \mathbb{R}[x] \) and let \( \text{var}(p) \) represent the number of sign changes or variations (positive to negative and vice-versa) in the sequence of coefficients \( a_n, a_{n-1}, \ldots, a_0 \).
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**Theorem**

The number \( \varrho_+ (p) \) of real roots — multiplicities counted — of the polynomial \( p(x) \in \mathbb{R}[x] \) in the open interval \((0, \infty)\) is bounded above by \( \text{var}(p) \); that is, we have \( \text{var}(p) \geq \varrho_+ (p) \). If \( \text{var}(p) > \varrho_+ (p) \) then their difference is an even number.
Special cases of Descartes’ rule of signs

\[ \var(p) = 0 \iff \varrho + \varpi(p) = 0. \]

\[ \var(p) = 1 \implies \varrho + \varpi(p) = 1. \]

Obreschkoff in 1920-23 provided the conditions under which \( \var(p) = 1 \iff \varrho + \varpi(p) = 1. \)

These two special cases above will be used as termination criteria in the real root isolation method VAS.
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Special cases of Descartes’ rule of signs

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Historical Note on Budan (1761-1840)

From Wikipedia we see that Ferdinand Francois Desire Budan de Boislaurent is considered an amateur mathematician, who is best remembered for his discovery of a rule which gives the necessary condition for a polynomial equation to have no real roots within an open interval. Taken together with Descartes' Rule of signs, his theorem leads to an upper bound on the number of the real roots a polynomial has inside an open interval.
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Budan’s Book of 1807

Figure:
Budan’s theorem of 1807 — to be found in Budan’s book, Vincent’s paper of 1836 and our publications

\[
\begin{align*}
\text{If in an equation } p(x) = 0 \text{ we make two substitutions,} \\
x \leftarrow x + a \text{ and } x \leftarrow x + b, \text{ where } a < b, \\
\text{then:} \\
\var (p(x + a)) \geq \var (p(x + b)), \\
\text{the number } \varrho_{ab}(p) \text{ of real roots of } p(x) \text{ located between } a \text{ and } b, \\
\text{satisfies the inequality } \varrho_{ab}(p) \leq \var (p(x + a)) - \var (p(x + b)). \\
\text{if } \varrho_{ab}(p) < \var (p(x + a)) - \var (p(x + b)), \text{ then} \\
\var (p(x + a)) - \var (p(x + b)) - \varrho_{ab}(p) = 2k, \quad k \in \mathbb{N}.
\end{align*}
\]
Budan’s theorem of 1807 — to be found in Budan’s book, Vincent’s paper of 1836 and our publications

If in an equation \( p(x) = 0 \) we make two substitutions, \( x \leftarrow x + a \) and \( x \leftarrow x + b \), where \( a \) and \( b \) are real numbers such that \( a < b \), then:
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$\var(p(x + a)) \geq \var(p(x + b))$. 
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- $\text{var}(p(x + a)) \geq \text{var}(p(x + b))$.
- The number $\varnothing_{ab}(p)$ of real roots of $p(x)$ located between $a$ and $b$, satisfies the inequality $\varnothing_{ab}(p) \leq \text{var}(p(x + a)) - \text{var}(p(x + b))$. 

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October 2018, Swansea, Wales, UK
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- $\text{var}(p(x + a)) \geq \text{var}(p(x + b))$.
- The number $\varrho_{ab}(p)$ of real roots of $p(x)$ located between $a$ and $b$, satisfies the inequality $\varrho_{ab}(p) \leq \text{var}(p(x + a)) - \text{var}(p(x + b))$.
- If $\varrho_{ab}(p) < \text{var}(p(x + a)) - \text{var}(p(x + b))$, then $\{\text{var}(p(x + a)) - \text{var}(p(x + b))\} - \varrho_{ab}(p) = 2k, k \in \mathbb{N}$. 

Alkiviadis G. Akritas October 2018, Swansea, Wales, UK
Remarks on Budan’s theorem

From Budan’s theorem it follows that if the polynomials $p(x)$ and $p(x+1)$ have the same number of sign variations then $p(x)$ has no real roots in the interval $(0, 1)$.

On the other hand, if $p(x)$ has more sign variations than $p(x+1)$, Budan investigates the existence or absence of real roots in the interval $(0, 1)$ by mapping those roots in the interval $(0, \infty)$ so that he can use Descartes' rule of signs.
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Budan’s termination criterion for the interval \((0, 1)\)

To map the real roots of the interval \((0, 1)\) in the interval \((0, \infty)\)

Budan makes the pair of substitutions \(x \leftarrow 1\) and \(x \leftarrow 1 + x\) (which is equivalent to the substitution \(x \leftarrow \frac{1}{1 + x}\)). His termination criterion states that...

The number \(\varrho_{01}(p)\) of real roots in the open interval \((0, 1)\) — multiplicities counted — of the polynomial \(p(x) \in \mathbb{R}[x]\), is bounded above by the number of sign variations \(\text{var}_{01}(p)\), where

\[\text{var}_{01}(p) = \text{var}((x + 1)^{\deg(p)} p(1 + x)).\]

That is, we have \(\text{var}_{01}(p) \geq \varrho_{01}(p)\).
To map the real roots of the interval \((0, 1)\) in the interval \((0, \infty)\) Budan makes the pair of substitutions \(x \leftarrow \frac{1}{x}\) and \(x \leftarrow 1 + x\) (which is equivalent to the substitution \(x \leftarrow \frac{1}{1+x}\)). His **termination criterion** states that ...
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\[
\var{01}(p) = \var((x + 1)^{\text{deg}(p)} p\left(\frac{1}{x + 1}\right)).
\]

That is, we have \(\var{01}(p) \geq \varrho_{01}(p)\).
Budan’s Theorem overshadowed by Fourier’s Theorem — a

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Budan’s Theorem overshadowed by Fourier’s Theorem — a

Following a priority dispute, Budan’s theorem was overshadowed by an equivalent theorem by Fourier, which appears under the names Budan or Fourier or Fourier-Budan or Budan-Fourier.
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Figure: Fourier’s theorem in Serret’s Algebra, Vol. 1, 1877.
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Budan’s work of 1807

From Budan’s statement it is easier to deduce that
\( \varphi_0(p) = 0 \)

than it is from Fourier’s statement.

In his paper of 1836, Vincent presented

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CAVEAT: From Budan’s statement it is easier to deduce that
\[\text{var}(p(x)) - \text{var}(p(x + 1)) = 0 \implies \varrho_01(p) = 0,\]
than it is from Fourier’s statement.
CAVEAT: From Budan’s statement it is easier to deduce that
\[ \text{var}(p(x)) - \text{var}(p(x + 1)) = 0 \Rightarrow \varrho_{01}(p) = 0, \]
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In his paper of 1836, Vincent presented both the Budan and the Fourier statement of this crucial theorem.
Recapping Budan’s achievements — a

He had developed all the basic ingredients needed for the isolation of the real roots of polynomials and had a very modern point of view. However, he did not present a unifying theorem.

He revived Descartes' rule of signs — forgotten for about 160 years — and first isolates the positive roots. To isolate the negative roots he sets $x \leftarrow -x$ and treats them as positive.

To compute the coefficients of $p(x + 1)$ Budan developed in 1803 the special case, $a = 1$, of the Ruffini method to compute the coefficients of $p(x + a)$. Ruffini's method appeared in 1804 — and was independently rediscovered by Horner in 1819.
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Recapping Budan’s achievements — b

He uses his method to compute radicals, as in $x^3 - 1745$. If he knows the roots to be “far” away from 0 he can speed up his method by introducing substitutions of the form $x \leftarrow kx$, for $k = 10, 20, \text{etc}$. For example, with seven substitutions he can determine that $3\sqrt[3]{1745}$ is in the interval $(12, 13)$. However, in general, his method for real root isolation has exponential computing time.
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Historical Note on Vincent (1797-1868)

Until 1979 he was known as M. Vincent. (M. for Monsieur; see Lloyd, E. K.: “On the forgotten Mr. Vincent”; Historia Mathematica, 6, (1979), 448–450).

Vincent is best known for his Cours de Géométrie Élémentaire, 1826, which reached a sixth edition and was published in German as well.

He was a polymath. He wrote at least 30 papers on topics such as Mathematics, Archaeology, Philosophy, Ancient Greek Music etc.
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Vincent’s Publications Timeline
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Overview
- Works: 237 works in 516 publications in 2 languages and 1,066 library holdings
- Genres: History, Catalogs, Bibliography, Catalogs, Manuscripts, Textbooks, Bibliography
- Roles: Author, Editor, Other, Honoree, Translator, Composer, Former owner, Author of introduction

Publication Timeline

Most widely held works about A. J. H Vincent
- Notice sur A.J.H. Vincent, lue le 10 janvier, 1869 by Ernest Havet (Book)
- Travaux scientifiques de M.A.-J.-H. Vincent by A. J. H Vincent (Book)
- Catalogue des livres composant la bibliothèque de feu M.L.J.S.E. marquis de Laborde ... La vente aura lieu le ... 8 janvier 1872 et les 11 jours suivants by Léon Laborde (Book)
- Catalogue des livres composant la bibliothèque de feu m. A.J.H. Vincent by A. J. H Vincent (Book)
- A.M. Le Rédacteur en chef du "Correspondant" by B Julian (Book)
Vincent’s theorem of 1836

If in a polynomial, \( p(x) \), of degree \( n \), with rational coefficients and simple roots we perform sequentially replacements of the form

\[
x \leftarrow \alpha_1 x + 1,
\]

\[
x \leftarrow \alpha_2 x + 1,
\]

\[
x \leftarrow \alpha_3 x + 1,
\]

... where \( \alpha_1 \geq 0 \) is an arbitrary non-negative integer and \( \alpha_2, \alpha_3, ... \) are arbitrary positive integers, \( \alpha_i > 0, i > 1 \), then the resulting polynomial either has no sign variations or it has one sign variation. In the first case there are no positive roots whereas in the last case the equation has exactly one positive root, represented by the continued fraction

\[
\alpha_1 + 1 \alpha_2 + 1 \alpha_3 + 1 ...
\]
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x \leftarrow \alpha_1 + \frac{1}{x}, x \leftarrow \alpha_2 + \frac{1}{x}, x \leftarrow \alpha_3 + \frac{1}{x}, \ldots
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where \( \alpha_1 \geq 0 \) is an arbitrary non-negative integer and \( \alpha_2, \alpha_3, \ldots \) are arbitrary positive integers, \( \alpha_i > 0, i > 1 \), then the resulting polynomial either has no sign variations or it has one sign variation. In the first case there are no positive roots whereas in the last case the equation has exactly one positive root, represented by the continued fraction

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\]
Remarks on Vincent’s Theorem — a

The requirement of the theorem that the roots of the polynomial be simple, does not restrict its generality, because we can always apply square free factorization and obtain polynomials with simple roots. That is, employing polynomial gcd computations, we can always obtain the factorization

\[ p(x) = p_1(x)p_2(x) \cdots p_k(x), \]

where the roots of each \( p_i(x) \), \( i = 1, \ldots, k \) are simple.
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\[ p(x) = p_1(x)p_2(x)^2 \cdots p_k(x)^k, \]

where the roots of each \( p_i(x), i = 1, \ldots, k \) are simple.
Remarks on Vincent’s Theorem — b

The substitutions of the form $x \leftarrow \alpha_1 + \alpha_2 x, \ldots$ can be compactly written in the form of a Möbius substitution $M(x) = ax + b, cx + d$.

It employs Descartes' termination test, which is very efficiently executed.

The theorem does not provide a bound on the number of substitutions $x \leftarrow \alpha_1 + \alpha_2 x, x \leftarrow \alpha_3 + \alpha_4 x, \ldots$ that need to be performed in order to obtain a polynomial with at most one sign variation.

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The substitutions of the form \( x \leftarrow \alpha_1 + \frac{1}{x}, \ldots \) can be compactly written in the form of a Möbius substitution

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- The theorem does not provide a bound on the number of substitutions \( x \leftarrow \alpha_1 + \frac{1}{x}, x \leftarrow \alpha_2 + \frac{1}{x}, x \leftarrow \alpha_3 + \frac{1}{x}, \ldots \) that need to be performed in order to obtain a polynomial with at most one sign variation.
Vincent’s search for a root

Like Budan, Vincent searches for roots — that is, he computes each partial quotient $\alpha_i$ by performing substitutions of the form $x \leftarrow x + 1$ — which correspond to $\alpha_i \leftarrow \alpha_i + 1$ — until the number of sign variations changes. Then he needs to investigate the existence or absence of real roots in $(0, 1)$ using Budan’s termination criterion.
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... where it was rediscovered by me in 1975 and formed the subject of my Ph.D. Thesis (1978).
Recapping Vincent’s achievements — a
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He presented and proved a theorem that unified the basic ingredients needed for the isolation of the real roots of polynomials. His theorem lacked a certain feature, but nonetheless was a significant step forward.
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- He presented and proved a theorem that unified the basic ingredients needed for the isolation of the real roots of polynomials. His theorem lacked a certain feature, but nonetheless was a significant step forward.

- He was fully aware of Budan’s work and used almost all the tools developed by Budan in 1807.

- What can be considered a step backward, is that he did not use Budan’s method for computing the coefficients of \( p(x + 1) \). Instead, he computes them by employing Pascal’s triangle.
Recapping Vincent’s achievements — b
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The nature of the partial quotients $\alpha_1, \alpha_2, \alpha_3\ldots$ is not clear.
Recapping Vincent’s achievements — b

- The nature of the partial quotients $\alpha_1, \alpha_2, \alpha_3 \ldots$ is not clear.

- Unclear is also the effect of the substitutions $x \leftarrow \alpha_1 + \frac{1}{x}, x \leftarrow \alpha_2 + \frac{1}{x}, x \leftarrow \alpha_3 + \frac{1}{x}, \ldots$ on the roots with positive real part.
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  $x \leftarrow \alpha_1 + \frac{1}{x}, x \leftarrow \alpha_2 + \frac{1}{x}, x \leftarrow \alpha_3 + \frac{1}{x}, \ldots$ on the roots with positive real part.

- Finally, as in Budan’s case, his real root isolation method has exponential computing time.
1 Budan’s work of 1807

2 Vincent’s Theorem of 1836

3 Uspensky’s Extension of Vincent’s Theorem
   - Uspensky’s Bound on the Number of Substitutions
   - An Example
   - Recapping

4 Various Implementations of Vincent’s Theorem
Historical Note on Uspensky (1883-1947)

Uspensky was born in Mongolia, the son of a Russian diplomat. He graduated from the University of St. Petersburg in 1906 and received his doctorate from the University of St. Petersburg in 1910. He was a member of the Russian Academy of Sciences from 1921. He joined the faculty of Stanford University in 1929-30 and 1930-31 as acting professor of mathematics. He was professor of mathematics at Stanford from 1931 until his death.

Alkiviadis G. Akritas

October 2018, Swansea, Wales, UK
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- He joined the faculty of Stanford University in 1929-30 and 1930-31 as acting professor of mathematics. He was professor of mathematics at Stanford from 1931 until his death.
If $\Delta$ is the smallest distance between any two roots of $p(x)$ having simple roots and degree $n$ and $F_i$ is the $i$-th Fibonacci number (seed numbers 1, 1) we need to perform at most $m$ substitutions

$x \leftarrow \alpha_1 + 1 \cdot x,$

$x \leftarrow \alpha_2 + 1 \cdot x,$

$x \leftarrow \alpha_3 + 1 \cdot x,$

..., $x \leftarrow \alpha_m + 1 \cdot \xi$

to obtain a polynomial with at most 1 sign variation. The index $m$ is defined by

$$F_m - 1 \cdot \Delta > \frac{1}{2},$$

$$\Delta \cdot F_m > 1 + \epsilon,$$

where $\epsilon = \left(1 + \frac{1}{n}\right)^{\frac{1}{n} - 1}.$
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where

$$\epsilon = (1 + \frac{1}{n})^{\frac{1}{n-1}} - 1.$$
Remarks on Uspensky’s Theorem

Uspensky’s proof is unnecessarily complicated and the bound $m$ on the number of substitutions is way too high.

From his theorem it follows that if a polynomial $p(x)$ has one positive root and all other roots with positive real part have been moved — through a suitable Möbius substitution — inside a circle with center at $-1$ and radius $\epsilon$, then $\varphi(p) = 1$.

As we will see, the circle at $-1$ with radius $\epsilon$ greatly underestimates the sector into which all other roots have to move, so that $\varphi(p) = 1 \Leftrightarrow \varphi(p) = 1$. 
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As we will see, the circle at -1 with radius $\epsilon$ greatly underestimates the sector into which all other roots have to move, so that $\text{var}(p) = 1 \iff \varphi_+(p) = 1$. 
Uspensky Uses the Same Example as Vincent — a
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Figure: Uspensky uses Budan’s method, by then a special case of the established Ruffini-Horner method.
Uspensky Uses the Same Example as Vincent — b
Figure: At the terminal nodes we have $M_L(x) = \frac{2x+3}{x+2}$ and $M_R(x) = \frac{x+3}{x+2}$. 
Uspensky’s search for a root — a

Uspensky was not able to deduce from Fourier’s statement that $\varrho_0 - \varrho_1 = 0$ implies $\varrho_0 = 0$. So the fact that there is no sign variation loss after the substitution $x \leftarrow x + 1$ means nothing to him.

To make sure there is no root in $(0, 1)$ Uspensky “reinvented” Budan’s termination test and after each substitution of the form $x \leftarrow x + 1$, he also performs the reduntant substitution $x \leftarrow (x + 1)^{\deg(p)}$. 

Alkiviadis G. Akritas  
October 2018, Swansea, Wales, UK
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Recapping Uspensky’s achievements — a

He definitely kept Vincent’s theorem alive, and extended it by including the missing feature. He proved that the purpose of the substitutions
\[
x \leftarrow \alpha_1 + x, \\
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x \leftarrow \alpha_3 + x, ...
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is to force the roots with positive real part inside a circle with center at -1 and radius \( \epsilon \).

He presented the real root isolation process in tree form and reintroduced Budan's method for computing the coefficients of \( p(x+1) \).
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As we saw, he just doubled the computing time of Vincent’s method.

Therefore, as in Budan’s and Vincent’s cases, the presented real root isolation method has exponential computing time.
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Budan's work of 1807

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4 Various Implementations of Vincent’s Theorem
   - Vincent’s theorem by Alesina and Galuzzi (2000)
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Historical note on Alesina and Galuzzi

Alesina and Galuzzi understood Vincent’s theorem so thoroughly that they gave an equivalent version of it — the bisections version — and provided a generalization of Budan’s termination test for the interval (0, 1).

Moreover, they were the ones who discovered Obreschkoff’s Sector (or Cone) and Circles theorem in his book of 1963 and used it to prove Vincent’s theorem.
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- Moreover, they were the ones who discovered Obreschkoff’s Sector (or Cone) and Circles theorem in his book of 1963 and used it to prove Vincent’s theorem.
Let $f(z)$, be a real polynomial of degree $n$, which has only simple roots. It is possible to determine a positive quantity $\delta$ so that for every pair of positive real numbers $a, b$ with $|b - a| < \delta$, every transformed polynomial of the form

$$\phi(z) = (1 + z)^n f\left(\frac{a + bz}{1 + z}\right)$$

has exactly 0 or 1 variations. The second case is possible if and only if $f(z)$ has a simple root within the open interval $(a, b)$. 
Sketch of the proof of Vincent’s theorem

If a real polynomial has one positive simple root $x_0$ and all the other — possibly multiple — roots lie in the sector $S_{\sqrt{3}} = \{ x = -\alpha + i\beta | \alpha > 0 \text{ and } \beta^2 \leq 3\alpha^2 \}$ then the sequence of its coefficients has exactly one sign variation.
Sketch of the proof of Vincent’s theorem

► Obreschkoff’s theorem of 1920-23, gives a much superior bound (to Uspensky’s) on the number of interval bisections (or equivalently substitutions) that need to be performed in order to obtain a polynomial with one sign variation. It states that ...

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then the sequence of its coefficients has exactly one sign variation.
Real root isolation using Vincent’s theorem

To isolate the positive roots of a polynomial $p(x)$, all we have to do is compute — for each root — the variables $a, b, c, d$ of the corresponding Möbius substitution

$$M(x) = \frac{ax + b}{cx + d}$$

that leads to a transformed polynomial

$$f(x) = (cx + d)^n p\left(\frac{ax + b}{cx + d}\right)$$

with one sign variation.
Two different ways to isolate the real roots:
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Crucial observation:

The variables $a, b, c, d$ of a Möbius substitution $M(x) = \frac{ax+b}{cx+d}$ (in Vincent’s theorem) leading to a transformed polynomial with one sign variation can be computed:
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- or, by bisections, leading to the methods developed by:
  (a) Vincent, Collins and Akritas (1976), the VCA bisection method, and
  (b) Vincent, Alesina and Galuzzi (2000), the VAG bisection method.
As we pointed out, Vincent’s method is exponential because each partial quotient $\alpha_i$ is computed by a series of unit increments $\alpha_i \leftarrow \alpha_i + 1$, equivalent to substitutions of the form $x \leftarrow x + 1$.

In 1978 I completed my Ph.D. thesis where I computed each partial quotient $\ell_b$ as the lower bound on the values of the positive roots of a polynomial. This made Vincent’s method polynomial.

In my thesis I made two plausible assumptions: (a) that $\ell_b$ computes the integer part of the smallest positive root, and (b) that its value is bounded by the size of the polynomial coefficients.

That is, we now set $\alpha_i \leftarrow \ell_b$ or, equivalently, we perform the substitution $x \leftarrow x + \ell_b$, which takes about the same time as the substitution $x \leftarrow x + 1$. 

The second method derived from Vincent’s Theorem
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Alkiviadis G. Akritas

October 2018, Swansea, Wales, UK
The *ideal* step
The *ideal* step

Figure: This way the theoretical computing time of Vincent’s method became polynomial.
Note that in general the ideal lower bound is bigger than the computed bound, i.e. $\ell_b > \ell_{\text{computed}}$.

The efficiency of the VAS algorithm depends on the algorithm used to evaluate $\ell_{\text{computed}}$.

In the next section we will present two algorithms for evaluating $\ell_{\text{computed}}$.  

Ideal vs computed lower bound
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In the next section we will present two algorithms for evaluating \( \ell b_{\text{computed}} \).
The VAS algorithm — Input / Output
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**VAS, 1978:**

**Input:** The square-free polynomial $p(x) \in \mathbb{Z}[x], p(0) \neq 0$, and the Möbius transformation $M(x) = \frac{ax+b}{cx+d} = x, \ a, b, c, d \in \mathbb{Z}$

**Output:** A list of isolating intervals of the **positive** roots of $p(x)$

**Figure:** The fastest implementation of Vincent’s theorem.
The VAS algorithm
The VAS algorithm

1. \( \text{var} \leftarrow \text{the number of sign changes of } p(x) \);
2. if \( \text{var} = 0 \) then RETURN \( \emptyset \);
3. if \( \text{var} = 1 \) then RETURN \( \{a, b]\} \quad /\quad a = \min(M(0), M(\infty)), \quad b = \max(M(0), M(\infty)) \);
4. \( \ell b \leftarrow \text{a lower bound on the positive roots of } p(x) \);
5. if \( \ell b > 1 \) then \( \{p \leftarrow p(x + \ell b), M \leftarrow M(x + \ell b)\} \);
6. \( p_{01} \leftarrow (x + 1)^{\deg(p)} p\left(\frac{1}{x+1}\right), M_{01} \leftarrow M\left(\frac{1}{x+1}\right) \quad /\quad \text{Look for real roots in } ]0, 1[ \);
7. \( m \leftarrow M(1) \quad /\quad \text{Is } 1 \text{ a root?} \);
8. \( p_{1\infty} \leftarrow p(x + 1), M_{1\infty} \leftarrow M(x + 1) \quad /\quad \text{Look for real roots in } ]1, +\infty[ \);
9. if \( p(1) \neq 0 \) then
10. \qquad RETURN \( \text{VAS}(p_{01}, M_{01}) \cup \text{VAS}(p_{1\infty}, M_{1\infty}) \)
11. else
12. \quad RETURN \( \text{VAS}(p_{01}, M_{01}) \cup \{[m, m]\} \cup \text{VAS}(p_{1\infty}, M_{1\infty}) \)
13. end

Figure: The fastest implementation of Vincent’s theorem.
Budan’s work of 1807
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Various Implementations of Vincent’s Theorem

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Bounds on the values of the positive roots of polynomials

Computing time analysis of VAS
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With the help of the Alesina-Galuzzi papers and without any assumptions, Sharma proved that VAS has polynomial computing time.
It was Adam Strzeboński of Wolfram Research, who in 1993 implemented “VAS” in Mathematica and at the same time introduced the substitution $x \leftarrow \ell \cdot b$ computed $x$, whenever $\ell \cdot b$ computed $> 16$. The value 16 was determined experimentally.

The Strzeboński substitution improved VAS even further.
Strzeboński’s contribution to Vincent’s method

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 Bounds on the values of the positive roots of polynomials
To compute the lower bound $\ell b$ of $p(x)$ we replace $x \leftarrow \frac{1}{x}$, compute the upper bound $ub$ of $p(\frac{1}{x})$ and set $\ell b = \frac{1}{ub}$. 
Bounds on the values of the positive roots

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Snag in 1978: Even though Cauchy and Lagrange had presented upper bounds on the values of the positive roots of a real polynomial, the only suitable bounds available in the English mathematical literature before my Ph.D. thesis in 1978 were on the absolute values of the roots.
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 Bounds on the absolute values of the roots work fine for the bisection methods, where they are computed only once at the start of the process.
To compute the lower bound $\ell b$ of $p(x)$ we replace $x \leftarrow \frac{1}{x}$, compute the upper bound $ub$ of $p\left(\frac{1}{x}\right)$ and set $\ell b = \frac{1}{ub}$.

Snag in 1978: Even though Cauchy and Lagrange had presented upper bounds on the values of the positive roots of a real polynomial, the only suitable bounds available in the English mathematical literature before my Ph.D. thesis in 1978 were on the absolute values of the roots.

Bounds on the absolute values of the roots work fine for the bisection methods, where they are computed only once at the start of the process.

By contrast, at each step of the process, the VAS continued fractions method relies heavily on the repeated estimation of lower bounds on the values of the positive roots of polynomials.
I came across Cauchy's theorem in N. Obreschkoff's book Verteilung und Berechnung der Nullstellen reeller Polynome, (East) Berlin, 1963. It states the following:

Let \( p(x) = \alpha_n x^n + \alpha_{n-1} x^{n-1} + \ldots + \alpha_0, (\alpha_n > 0) \) be a polynomial of degree \( n > 0 \), with \( \alpha_{n-k} < 0 \) for at least one \( k, 1 \leq k \leq n \). If \( \lambda \) is the number of negative coefficients, then an upper bound on the values of the positive roots of \( p(x) \) is given by

\[
ub_{Cauchy} = \max \{ 1 \leq k \leq n : \alpha_n - k \alpha_n \} \sqrt{-\lambda \alpha_n - k \alpha_n}.
\]
Cauchy’s bound

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Let \( p(x) = \alpha_n x^n + \alpha_{n-1} x^{n-1} + \ldots + \alpha_0 \), \((\alpha_n > 0)\) be a polynomial of degree \( n > 0 \), with \( \alpha_{n-k} < 0 \) for at least one \( k \), \(1 \leq k \leq n\). If \( \lambda \) is the number of negative coefficients, then an upper bound on the values of the positive roots of \( p(x) \) is given by

\[
ub_C = \max_{\{1 \leq k \leq n : \alpha_{n-k} < 0\}} \sqrt[k]{-\frac{\lambda \alpha_{n-k}}{\alpha_n}}.
\]
Let $p(x) \in \mathbb{R}[x]$ be such that the number of variations of signs of its coefficients is even. If $p(x) = c_1x^{d_1} - b_1x^{m_1} + c_2x^{d_2} - b_2x^{m_2} + \ldots + c_kx^{d_k} - b_kx^{m_k} + g(x)$, with $g(x) \in \mathbb{R^+}[x]$, $c_i > 0$, $b_i > 0$, $d_i > m_i > d_i + 1$ for all $i$, the number $u_S = \max\left\{ \frac{b_1c_1}{d_1-m_1}, \ldots, \frac{b_kc_k}{d_k-m_k} \right\}$ is an upper bound for the positive roots of the polynomial $p$ for any choice of $c_1, \ldots, c_k$. 

Ștefănescu's theorem for pairing terms
(Ștefănescu’s theorem, 2005) Let \( p(x) \in R[x] \) be such that the number of variations of signs of its coefficients is even. If

\[
p(x) = c_1x^{d_1} - b_1x^{m_1} + c_2x^{d_2} - b_2x^{m_2} + \ldots + c_kx^{d_k} - b_kx^{m_k} + g(x),
\]

with \( g(x) \in R_+[x], c_i > 0, b_i > 0, d_i > m_i > d_{i+1} \) for all \( i \), the number

\[
ub_s = \max \left\{ \left( \frac{b_1}{c_1} \right)^{1/(d_1-m_1)}, \ldots, \left( \frac{b_k}{c_k} \right)^{1/(d_k-m_k)} \right\}
\]

is an upper bound for the positive roots of the polynomial \( p \) for any choice of \( c_1, \ldots, c_k \).
Our splitting and pairing of terms in Cauchy’s bound

Alkiviadis G. Akritas

October 2018, Swansea, Wales, UK
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For Cauchy’s bound, the splitting and pairing of terms can be seen if we rewrite the formula as

\[ ub_C = \max_{1 \leq k \leq n: \alpha_n - k < 0} \left\{ \sqrt{\frac{\alpha}{\lambda}} \right\} \]
Bounds with quadratic complexity

Cauchy’s upper bound has linear time complexity; that is, each negative coefficient is paired with just one positive coefficient.

Main idea of quadratic bounds:
Each negative coefficient of the polynomial is paired with all the preceding positive coefficients and the minimum of the computed values is associated with this coefficient. The maximum of all those minimums is taken as the estimate of the bound.
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Bounds with quadratic complexity

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Main idea of quadratic bounds:
- Each negative coefficient of the polynomial is paired with all the preceding positive coefficients and the **minimum** of the computed values is associated with this coefficient. The **maximum** of all those minimums is taken as the estimate of the bound.
For the polynomial $p(x) \in \mathbb{R}[x]$

$$p(x) = \alpha_n x^n + \alpha_{n-1} x^{n-1} + \ldots + \alpha_0,$$

$(\alpha_n > 0)$

each negative coefficient $a_i < 0$ is “paired” with each one of the preceding positive coefficients $a_j$ divided by $2t_j$ — where $t_j$ is initially set to 1 and is incremented each time the positive coefficient $a_j$ is used — and the minimum is taken over all $j$; subsequently, the maximum is taken over all $i$.

That is, we have:

$$\text{ub}_{\text{LMQ}} = \max \{ a_i < 0 \} \quad \min \{ a_j > 0 : j > i \} \cdot \sqrt{\frac{-a_i}{a_j}}^{2t_j}.$$
Local Max Quadratic, (LMQ)

For the polynomial $p(x) \in \mathbb{R}[x]$

$$p(x) = \alpha_n x^n + \alpha_{n-1} x^{n-1} + \ldots + \alpha_0, \quad (\alpha_n > 0),$$

each negative coefficient $a_i < 0$ is “paired” with each one of the preceding positive coefficients $a_j$ divided by $2^{t_j}$ — where $t_j$ is initially set to 1 and is incremented each time the positive coefficient $a_j$ is used — and the minimum is taken over all $j$; subsequently, the maximum is taken over all $i$.

That is, we have:

$$ub_{LMQ} = \max \{a_i < 0\} \min \{a_j > 0 : j > i\} \left( j - i \sqrt{-\frac{a_i}{a_j}} \right) \frac{a_i}{2^{t_j}}.$$
Example

Consider the polynomial

\[ x^3 + 10^{100}x^2 - 10^{100}x - 1, \]

which has one sign variation and, hence, one positive root equal to 1.
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which has one sign variation and, hence, one positive root equal to 1.

With Cauchy’s linear bound, we pair the terms:

\[ \{ \frac{x^3}{2}, -10^{100}x \} \text{ and } \{ \frac{x^3}{2}, -1 \}, \]

and taking the maximum of the radicals we obtain a bound estimate of 1.41421 * 10^{50}. 
Example

Consider the polynomial

\[ x^3 + 10^{100}x^2 - 10^{100}x - 1, \]

which has one sign variation and, hence, one positive root equal to 1.

With LMQ, the “Local Max” quadratic bound, we compute:

- the minimum of the two radicals obtained from the pairs of terms \( \{ \frac{x^3}{2}, -10^{100}x \} \) and \( \{ \frac{10^{100}x^2}{2}, -10^{100}x \} \) which is 2, and
- the minimum of the two radicals obtained from the pairs of terms \( \{ \frac{x^3}{2^2}, -1 \} \) and \( \{ \frac{10^{100}x^2}{2^2}, -1 \} \) which is \( \frac{2}{10^{50}} \).
- Therefore, the obtained estimate of the bound is \( \max \{ 2, \frac{2}{10^{50}} \} = 2 \).
Good old quadratic complexity bounds
Using **LMQ**, the performance of the VAS real root isolation method was speeded up by an average overall factor of **40%**.
VAS vs VCA on Mignotte polynomials

The Mignotte polynomials are of the form:

\[ x^n - 2(c \cdot x - 1)^2, \text{ for } c, n \geq 3, \]

have only 4 real roots and as the degree increases, 2 of the 3 positive roots get closer and closer together.

We test our methods on the Mignotte polynomial:

\[ x^{300} - 2(5x - 1)^2. \]
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VAS has been implemented in *Mathematica* — version 7 shown below
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Figure: Isolating and approximating real roots with Mma 7
VCA has been implemented in maple — version 11 shown below
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— and it takes 170 seconds to just isolate the roots of Mignotte’s polynomial of degree 300.
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— and it takes 170 seconds to just isolate the roots of Mignotte’s polynomial of degree 300.

```
> with(RootFinding):
> f := x^{300} - 2(5x - 1)^2;
> st := time( ) : Isolate(f, digits = 250) : time( ) - st,
    170.431
```

**Figure:** To isolate Mignotte’s poly of degree 300
Therefore, ...
Therefore, ...

VAS can be many thousand times faster than the fastest implementation of VCA.
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VAS can be many thousand times faster than the fastest implementation of VCA.

Moreover, as the following frames indicate, VAS can be many times faster than numeric methods, which cannot compute just the positive roots! They compute all the roots (real and complex).
Consider the polynomial
\[ f(x) = 10^{999}(x - 1)^{50} - 1 \]
with the 2 positive roots \( \neq 1 \).

The numeric method \texttt{NRoots} used in Mma 7 takes 12.933 seconds to find the two positive roots with 30 digits of accuracy.
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Using Mma 7 (1/3 frames)

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Budan’s work of 1807
Vincent’s Theorem of 1836
Uspensky’s Extension of Vincent’s Theorem
Various Implementations of Vincent’s Theorem

Using Mma 7 (2/3 frames)

On the other hand, the function RootIntervals, i.e. the VAS continued fractions method, isolates the two positive roots in 5 * 10^{-16} seconds ...

Figure: Using the function RootIntervals in Mma 7

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```
ints = RootIntervals[f][[1]] // Timing
{5.60316 \times 10^{-16}, {{0, 1}, {1, 2}}}
```

**Figure:** Using the function `RootIntervals` in Mma 7
Using Mma 7 (3/3 frames)
...and approximates them to 30 digits of accuracy in practically no time at all!
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```
ints = Last[ints];
FindRoot[f, {x, #[[1]], #[[2]]}, Method -> Brent,
         WorkingPrecision -> 30, MaxIterations -> 200] &/@ ints //
Timing
{0., {{x -> 0.999999999999999999989528714519},
      {x -> 1.00000000000000000001047128548}}}
```

**Figure:** Using the function `FindRoot` in Mma 7
Concluding remarks

The theoretical results by Alesina-Galuzzi and Sharma improved our understanding of Vincent's theorem. Additionally, Ştef˘ anescu's theorem of 2005 and our discovery and use of LMQ, the quadratic complexity bound on the values of the positive roots, made VAS the fastest real root isolation method.

However, when we try to isolate the roots of a sparse polynomial of very large degree, say 100000, most CASs run out of memory. To solve the problem, the VAS continued fractions method has been implemented using interval arithmetic.
Concluding remarks

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References


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