# Polynomial Real Root Isolation Using Vincent's Theorem of 1836

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- To determine the values of the real roots, isolation is followed by approximation to any desired degree of accuracy.

- One of — if not — the first to employ the isolation / approximation approach was Budan and we begin our talk with him.

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- VAS, one of the three methods derived from Vincent's theorem for the isolation of the real roots of polynomials.
- Bounds on the values of the positive roots, which determine the efficiency of VAS.

# Descartes' rule of signs (1637) — saved from oblivion by Budan

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Consider the polynomial

$$p(x) = a_n x^n + \cdots + a_1 x + a_0,$$

where $p(x) \in \mathbb{R}[x]$ and let $var(p)$ represent the number of sign *changes* or *variations* (positive to negative and vice-versa) in the sequence of coefficients $a_n, a_{n-1}, \ldots, a_0$.

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**Theorem**

*The number $\varrho_+(p)$ of real roots — multiplicities counted — of the polynomial $p(x) \in \mathbb{R}[x]$ in the open interval $(0, \infty)$ is bounded above by $var(p)$; that is, we have $var(p) \geq \varrho_+(p)$. If $var(p) > \varrho_+(p)$ then their difference is an even number.*

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These two special cases above will be used as termination criteria in the real root isolation method VAS.

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# Historical Note on Budan (1761-1840)

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- Taken together with Descartes' Rule of signs, his theorem leads to an upper bound on the number of the real roots a polynomial has inside an open interval.

# Budan's Book of 1807

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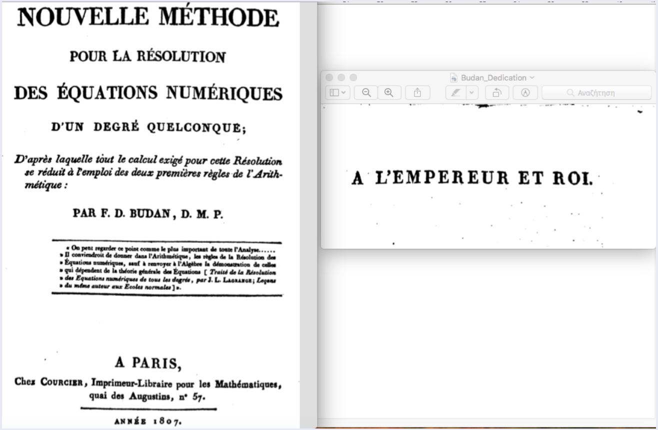

NOUVELLE MÉTHODE

POUR LA RÉSOLUTION

DES ÉQUATIONS NUMÉRIQUES

D'UN DEGRÉ QUELCONQUE;

*D'après laquelle tout le calcul exigé pour cette Résolution se réduit à l'emploi des deux premières règles de l'Arithmétique :*

PAR F. D. BUDAN, D. M. P.

« On peut regarder ce point comme le plus important de toute l'Analyse...... » Il conviendroit de donner dans l'Arithmétique, les règles de la Résolution des » Équations numériques, sauf à renvoyer à l'Algèbre la démonstration de celles » qui dépendent de la théorie générale des Équations [ *Traité de la Résolution » des Equations numériques de tous les degrés*, par J. L. LAGRANGE; *Leçons » du même auteur aux Ecoles normales* ] ».

A PARIS,

Chez COURCIER, Imprimeur-Libraire pour les Mathématiques, quai des Augustins, n° 57.

ANNÉE 1807.



Figure:

# Budan's theorem of 1807 — to be found in Budan's book, Vincent's paper of 1836 and our publications

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- $var(p(x + a)) \geq var(p(x + b))$.
- the number $\varrho_{ab}(p)$ of real roots of $p(x)$ located between $a$ and $b$, satisfies the inequality $\varrho_{ab}(p) \leq var(p(x + a)) - var(p(x + b))$.

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- the number $\varrho_{ab}(p)$ of real roots of $p(x)$ located between $a$ and $b$, satisfies the inequality $\varrho_{ab}(p) \leq var(p(x + a)) - var(p(x + b))$.
- if $\varrho_{ab}(p) < var(p(x + a)) - var(p(x + b))$, then $\{var(p(x + a)) - var(p(x + b))\} - \varrho_{ab}(p) = 2k, k \in \mathbb{N}$.

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- On the other hand, if $p(x)$ has more sign variations than $p(x+1)$, Budan investigates the existence or absence of real roots in the interval $(0, 1)$ by mapping those roots in the interval $(0, \infty)$ so that he can use Descartes' rule of signs.

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- The number $\varrho_{01}(p)$ of real roots in the open interval $(0, 1)$ — multiplicities counted — of the polynomial $p(x) \in \mathbb{R}[x]$, is bounded above by the number of sign variations $var_{01}(p)$, where

$$var_{01}(p) = var((x+1)^{deg(p)} p(\frac{1}{x+1})).$$

That is, we have $var_{01}(p) \geq \varrho_{01}(p)$.

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121. Théorème de Budan. — *Étant donnée une équation quelconque* $f(x) = 0$ *de degré* $m$, *si dans les* $m+1$ *fonctions*

$$(1) \qquad f(x),\quad f'(x),\quad f''(x),\quad \ldots,\quad f^m(x)$$

*on substitue deux quantités réelles quelconques* $\alpha$ *et*

Figure: Fourier's theorem in Serret's Algebra, Vol. 1, 1877.

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- In his paper of 1836, Vincent presented *both* the Budan and the Fourier statement of this crucial theorem.

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- To compute the coefficients of $p(x+1)$ Budan developed in 1803 the special case, $a = 1$, of the Ruffini method to compute the coefficients of $p(x+a)$. Ruffini's method appeared in 1804 — and was independently rediscovered by Horner in 1819.

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- However, in general, his method for real root isolation has exponential computing time.

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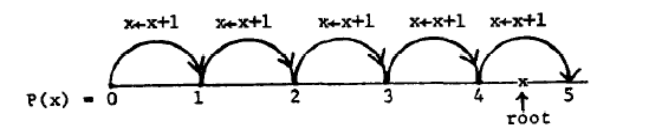
- In other words, searching for a real root Budan proceeds by taking *unit* steps of the form $x \leftarrow x + 1$.



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- He was a polymath. He wrote at least 30 papers on topics such as Mathematics, Archaeology, Philosophy, Ancient Greek Music etc.

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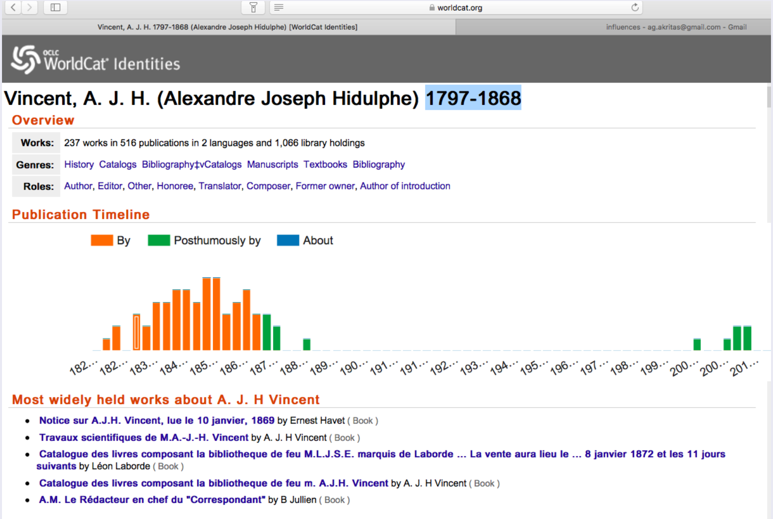

# Vincent's theorem of 1836

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If in a polynomial, $p(x)$, of degree $n$, with rational coefficients and simple roots we perform sequentially replacements of the form

$$x \leftarrow \alpha_1 + \frac{1}{x}, x \leftarrow \alpha_2 + \frac{1}{x}, x \leftarrow \alpha_3 + \frac{1}{x}, \ldots$$

where $\alpha_1 \geq 0$ is an arbitrary non negative integer and $\alpha_2, \alpha_3, \ldots$ are arbitrary positive integers, $\alpha_i > 0$, $i > 1$, then the resulting polynomial either has *no* sign variations or it has *one* sign variation. In the first case there are *no* positive roots whereas in the last case the equation has exactly one positive root, represented by the continued fraction

$$\alpha_1 + \cfrac{1}{\alpha_2 + \cfrac{1}{\alpha_3 + \cfrac{1}{\ddots}}}$$

# Remarks on Vincent's Theorem — a

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- The requirement of the theorem that the roots of the polynomial be simple, does not restrict its generality, because we can always apply square free factorization and obtain polynomials with simple roots. That is, employing polynomial gcd computations, we can always obtain the factorization

$$p(x) = p_1(x)p_2(x)^2 \cdots p_k(x)^k,$$

where the roots of each $p_i(x), i = 1, \ldots, k$ are simple.

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- It employs Descartes' termination test, which is very efficiently executed.

- The theorem does not provide a bound on the number of substitutions $x \leftarrow \alpha_1 + \frac{1}{x}, x \leftarrow \alpha_2 + \frac{1}{x}, x \leftarrow \alpha_3 + \frac{1}{x}, \ldots$ that need to be performed in order to obtain a polynomial with at most one sign variation.

# Vincent's search for a root

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Like Budan, Vincent searches for roots — that is, he computes each partial quotient $\alpha_i$ — by performing substitutions of the form $x \leftarrow x + 1$ — which correspond to $\alpha_i \leftarrow \alpha_i + 1$ — until the number of sign variations changes. Then he needs to investigate the existence or absence of real roots in $(0, 1)$ using Budan's termination criterion.

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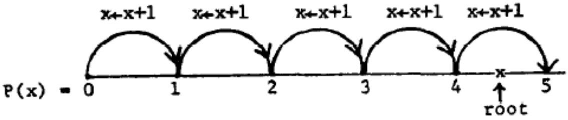

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- In the 19-th century the theorem appeared with its proof but without examples only in Serret's Algebra — at least in the fourth edition of 1877 — and in its Russian translation.

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- He was fully aware of Budan's work and used almost all the tools developed by Budan in 1807.

- What can be considered a step backward, is that he did not use Budan's method for computing the coefficients of $p(x+1)$. Instead, he computes them by employing Pascal's triangle.

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- Finally, as in Budan's case, his real root isolation method has exponential computing time.

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- He joined the faculty of Stanford University in 1929-30 and 1930-31 as acting professor of mathematics. He was professor of mathematics at Stanford from 1931 until his death.

# Extension of Vincent's theorem by Uspensky

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If $\Delta$ is the smallest distance between any two roots of $p(x)$ having simple roots and degree $n$ and $F_i$ is the $i$-th Fibonacci number (seed numbers 1, 1) we need to perform at most $m$ substitutions

$$x \leftarrow \alpha_1 + \frac{1}{x}, x \leftarrow \alpha_2 + \frac{1}{x}, x \leftarrow \alpha_3 + \frac{1}{x}, \ldots, x \leftarrow \alpha_m + \frac{1}{\xi}$$

to obtain a polynomial with at most 1 sign variation. The index $m$ is defined by

$$F_{m-1}\Delta > \frac{1}{2}, \qquad \Delta F_m F_{m-1} > 1 + \frac{1}{\epsilon}$$

where

$$\epsilon = (1 + \frac{1}{n})^{\frac{1}{n-1}} - 1.$$

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- As we will see, the circle at -1 with radius $\epsilon$ greatly underestimates the sector into which all other roots have to move, so that $var(p) = 1 \Leftarrow \varrho_+(p) = 1$.

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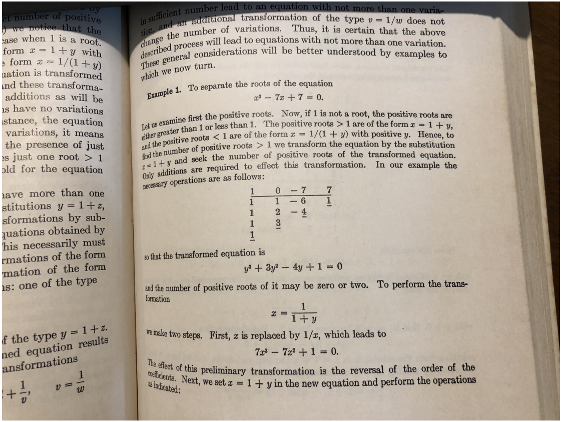

in sufficient number lead to an equation with not more than one variation, and an additional transformation of the type $v = 1/w$ does not change the number of variations. Thus, it is certain that the above described process will lead to equations with not more than one variation. These general considerations will be better understood by examples to which we now turn.

**Example 1.** To separate the roots of the equation

$$x^3 - 7x + 7 = 0.$$

Let us examine first the positive roots. Now, if 1 is not a root, the positive roots are either greater than 1 or less than 1. The positive roots $> 1$ are of the form $x = 1 + y$, and the positive roots $< 1$ are of the form $x = 1/(1 + y)$ with positive $y$. Hence, to find the number of positive roots $> 1$ we transform the equation by the substitution $x = 1 + y$ and seek the number of positive roots of the transformed equation. Only additions are required to effect this transformation. In our example the necessary operations are as follows:

| | | | |
|---|---|---|---|
| 1 | 0 | −7 | 7 |
| 1 | 1 | −6 | 1 |
| 1 | 2 | −4 | |
| 1 | 3 | | |
| 1 | | | |

so that the transformed equation is

$$y^3 + 3y^2 - 4y + 1 = 0$$

and the number of positive roots of it may be zero or two. To perform the transformation

$$x = \frac{1}{1+y}$$

we make two steps. First, $x$ is replaced by $1/x$, which leads to

$$7x^3 - 7x^2 + 1 = 0.$$

The effect of this preliminary transformation is the reversal of the order of the coefficients. Next, we set $x = 1 + y$ in the new equation and perform the operations as indicated:

Figure: Uspensky uses Budan's method, by then a special case of the established Ruffini-Horner method.

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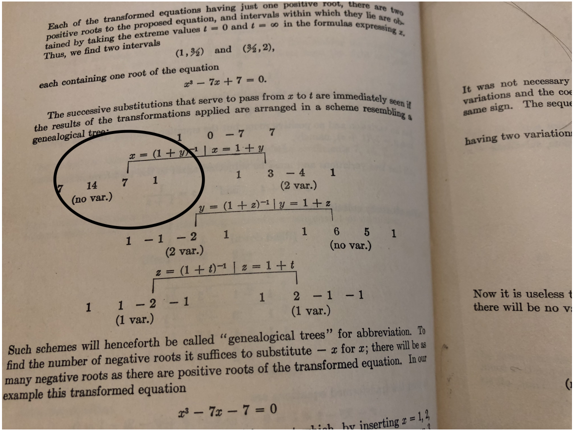
# Uspensky Uses the Same Example as Vincent — b



Each of the transformed equations having just one positive root, there are two positive roots to the proposed equation, and intervals within which they lie are obtained by taking the extreme values $t = 0$ and $t = \infty$ in the formulas expressing $x$. Thus, we find two intervals

$$(1, 3/2) \quad \text{and} \quad (3/2, 2),$$

each containing one root of the equation

$$x^3 - 7x + 7 = 0.$$

The successive substitutions that serve to pass from $x$ to $t$ are immediately seen if the results of the transformations applied are arranged in a scheme resembling a genealogical tree:

1 0 −7 7

$x = (1+y)^{-1} \mid x = 1 + y$

7 14 7 1 (no var.) | 1 3 −4 1 (2 var.)

$y = (1+z)^{-1} \mid y = 1 + z$

1 −1 −2 1 (2 var.) | 1 6 5 1 (no var.)

$z = (1+t)^{-1} \mid z = 1 + t$

1 1 −2 −1 (1 var.) | 1 2 −1 −1 (1 var.)

Such schemes will henceforth be called "genealogical trees" for abbreviation. To find the number of negative roots it suffices to substitute $-x$ for $x$; there will be as many negative roots as there are positive roots of the transformed equation. In our example this transformed equation

$$x^3 - 7x - 7 = 0$$

... which, by inserting $x = 1, 2,$ ...

It was not necessary ...
variations and the coe...
same sign. The seque...
having two variation...
Now it is useless t...
there will be no v...

Figure: At the terminal nodes we have $M_L(x) = \frac{2x+3}{x+2}$ and $M_R(x) = \frac{x+3}{x+2}$.

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To make sure there is no root in $(0, 1)$ Uspensky "reinvented" Budan's termination test and after each substitution of the form $x \leftarrow x+1$, he also performs the reduntant substitution

$$x \leftarrow (x+1)^{\deg(p)} p(\frac{1}{x+1}).$$

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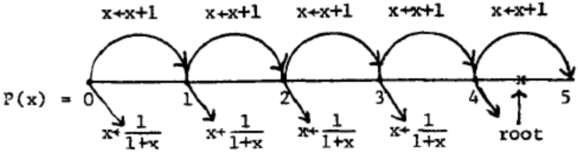

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- He proved that the purpose of the substitutions $x \leftarrow \alpha_1 + \frac{1}{x}, x \leftarrow \alpha_2 + \frac{1}{x}, x \leftarrow \alpha_3 + \frac{1}{x}, \ldots$ is to force the roots with positive real part inside a circle with center at -1 and radius $\epsilon$.

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- He presented the real root isolation process in tree form and reintroduced Budan's method for computing the coefficients of $p(x+1)$.

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- Therefore, as in Budan's and Vincent's cases, the presented real root isolation method has exponential computing time.

# Table of contents

# Historical note on Alesina and Galuzzi

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- Alesina and Galuzzi understood Vincent's theorem so thoroughly that they gave an equivalent version of it — the bisections version — and provided a generalization of Budan's termination test for the interval $(0, 1)$.

- Moreover, they were the ones who discovered Obreschkoff's Sector (or Cone) and Circles theorem in his book of 1963 and used it to prove Vincent's theorem.

# Vincent's Bisections theorem — by Alesina and Galuzzi, 2000

Let $f(z)$, be a real polynomial of degree $n$, which has only simple roots. It is possible to determine a positive quantity $\delta$ so that for every pair of positive real numbers $a, b$ with $|b-a| < \delta$, every transformed polynomial of the form

$$\phi(z) = (1+z)^n f(\frac{a+bz}{1+z})$$

has exactly 0 or 1 variations. The second case is possible if and only if $f(z)$ has a simple root within the open interval $(a, b)$.

# Sketch of the proof of Vincent's theorem

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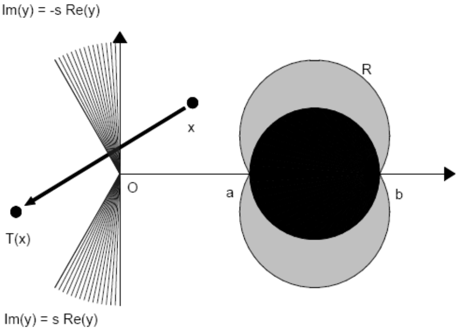
If a real polynomial has one positive simple root $x_0$ and all the other — possibly multiple — roots lie in the sector

$$S_{\sqrt{3}} = \{x = -\alpha + \imath\beta \mid \alpha > 0 \quad \textit{and} \quad \beta^2 \leq 3\alpha^2\}$$

then the sequence of its coefficients has exactly one sign variation.

# View of Obreschkoff's Cone and Circles. Diagram by Alesina and Galuzzi, 2000.



# Real root isolation using Vincent's theorem

To isolate the positive roots of a polynomial $p(x)$, all we have to do is compute — for *each* root — the variables $a, b, c, d$ of the corresponding Möbius substitution

$$M(x) = \frac{ax + b}{cx + d}$$

that leads to a transformed polynomial

$$f(x) = (cx + d)^n p(\frac{ax + b}{cx + d})$$

with one sign variation.

# Two different ways to isolate the real roots:

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**Crucial observation:**

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- either by continued fractions, leading to the continued fractions method developed by Vincent, Akritas and Strzeboński, (1978 / 1993 / 2008) the `VAS` *continued fractions* method,
- or, by bisections, leading to the methods developed by:

(a) Vincent, Collins and Akritas (1976), the `VCA` *bisection* method, and

(b) Vincent, Alesina and Galuzzi (2000), the `VAG` *bisection* method.

# The second method derived from Vincent's Theorem

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- In my thesis I made 2 plausible assumptions: (a) that $\ell b$ computes the integer part of the smallest positive root, and (b) that its value is bounded by the size of the polynomial coefficients.

- That is, we now set $\alpha_i \leftarrow \ell b$ or, equivalently, we perform the substitution $x \leftarrow x + \ell b$, which takes about the same time as the substitution $x \leftarrow x + 1$.

# The *ideal* step

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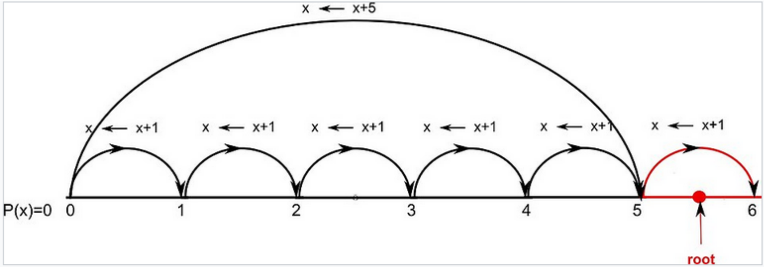

VAS searching for a root: The *ideal* lower bound is 5, hence $x \leftarrow x + 5$.

Figure: This way the theoretical computing time of Vincent's method became polynomial.

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- Note that in general the ideal lower bound is bigger than the computed bound, i.e.

$$\ell b > \ell b_{computed}.$$

- The efficiency of the VAS algorithm depends on the algorithm used to evaluate $\ell b_{computed}$.

- In the next section we will present two algorithms for evaluating $\ell b_{computed}$.

# The VAS algorithm — Input / Output

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## VAS, 1978:

**Input:** The square-free polynomial $p(x) \in \mathbb{Z}[x], p(0) \neq 0$, and the Möbius transformation $M(x) = \frac{ax+b}{cx+d} = x,\ a, b, c, d \in \mathbb{Z}$

**Output:** A list of isolating intervals of the positive roots of $p(x)$

Figure: The fastest implementation of Vincent's theorem.

# The VAS algorithm

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$var \longleftarrow$ the number of sign changes of $p(x)$;
**if** $var = 0$ **then RETURN** $\emptyset$;
**if** $var = 1$ **then RETURN** $\{]a, b[\}$ `// a = min(M(0),M(∞)), b = max(M(0),M(∞));`
$\ell b \longleftarrow$ a lower bound on the positive roots of $p(x)$;
**if** $\ell b > 1$ **then** $\{p \longleftarrow p(x + \ell b), M \longleftarrow M(x + \ell b)\}$;
$p_{01} \longleftarrow (x+1)^{deg(p)} p(\frac{1}{x+1}), M_{01} \longleftarrow M(\frac{1}{x+1})$ `// Look for real roots in` $]0, 1[$ ;
$m \longleftarrow M(1)$ `// Is 1 a root?` ;
$p_{1\infty} \longleftarrow p(x+1), M_{1\infty} \longleftarrow M(x+1)$ `// Look for real roots in` $]1, +\infty[$ ;
**if** $p(1) \neq 0$ **then**
    **RETURN** $\mathtt{VAS}(p_{01}, M_{01}) \bigcup \mathtt{VAS}(p_{1\infty}, M_{1\infty})$
**else**
    **RETURN** $\mathtt{VAS}(p_{01}, M_{01}) \bigcup \{[m, m]\} \bigcup \mathtt{VAS}(p_{1\infty}, M_{1\infty})$
**end**

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- Because of the assumptions made in my thesis, VAS was considered exponential until Sharma's Ph.D. Thesis came out in 2007.

- With the help of the Alesina-Galuzzi papers and without any assumptions, Sharma proved that VAS has polynomial computing time.

# Strzeboński's contribution to Vincent's method

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- It was Adam Strzeboński of Wolfram Research, who in 1993 implemented "VAS" in *Mathematica* and at the same time introduced the substitution $x \leftarrow \ell b_{computed} \cdot x$, whenever $\ell b_{computed} > 16$. The value 16 was determined experimentally.

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- The Strzeboński substitution improved VAS even further.

# Bounds on the values of the positive roots

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- Bounds on the absolute values of the roots work fine for the bisection methods, where they are computed only once at the start of the process.

- By contrast, at each step of the process, the VAS continued fractions method relies heavily on the repeated estimation of lower bounds on the values of the positive roots of polynomials.

# Cauchy's bound

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- I came across Cauchy's theorem in N. Obreschkoff's book Verteilung und Berechnung der Nullstellen reeller Polynome, (East) Berlin, 1963. It states the following:

Let $p(x) = \alpha_n x^n + \alpha_{n-1} x^{n-1} + \ldots + \alpha_0, \quad (\alpha_n > 0)$ be a polynomial of degree $n > 0$, with $\alpha_{n-k} < 0$ for at least one $k$, $1 \leq k \leq n$. If $\lambda$ is the number of negative coefficients, then an upper bound on the values of the positive roots of $p(x)$ is given by

$$ub_C = \max_{\{1 \leq k \leq n : \alpha_{n-k} < 0\}} \sqrt[k]{-\frac{\lambda \alpha_{n-k}}{\alpha_n}}.$$

# Ştefănescu's theorem for pairing terms

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- *(Ştefănescu's theorem, 2005)* Let $p(x) \in R[x]$ be such that the number of variations of signs of its coefficients is **even**. If

$$p(x) = c_1 x^{d_1} - b_1 x^{m_1} + c_2 x^{d_2} - b_2 x^{m_2} + \ldots + c_k x^{d_k} - b_k x^{m_k} + g(x),$$

with $g(x) \in R_+[x], c_i > 0, b_i > 0, d_i > m_i > d_{i+1}$ for all $i$, the number

$$ub_S = \max\left\{ \left(\frac{b_1}{c_1}\right)^{1/(d_1 - m_1)}, \ldots, \left(\frac{b_k}{c_k}\right)^{1/(d_k - m_k)} \right\}$$

is an upper bound for the positive roots of the polynomial $p$ for any **choice** of $c_1, \ldots, c_k$.

## Our splitting and pairing of terms in Cauchy's bound

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- For Cauchy's bound, the splitting and pairing of terms can be seen if we rewrite the formula as

$$ub_C = \max_{\{1 \leq k \leq n : \alpha_{n-k} < 0\}} \sqrt[k]{-\frac{\alpha_{n-k}}{\frac{\alpha_n}{\lambda}}}$$

# Bounds with quadratic complexity

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**Main idea of quadratic bounds:**

- Each negative coefficient of the polynomial is paired with all the preceding positive coefficients and the minimum of the computed values is associated with this coefficient. The maximum of all those minimums is taken as the estimate of the bound.

# Local Max Quadratic, (LMQ)

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- For the polynomial $p(x) \in \mathbb{R}[x]$

$$p(x) = \alpha_n x^n + \alpha_{n-1} x^{n-1} + \ldots + \alpha_0, \quad (\alpha_n > 0),$$

each negative coefficient $a_i < 0$ is "paired" with each one of the preceding positive coefficients $a_j$ divided by $2^{t_j}$ — where $t_j$ is initially set to 1 and is incremented each time the positive coefficient $a_j$ is used — and the minimum is taken over all $j$; subsequently, the maximum is taken over all $i$.

That is, we have:

$$ub_{LMQ} = \max_{\{a_i<0\}} \min_{\{a_j>0 : j>i\}} \sqrt[j-i]{-\frac{a_i}{\frac{a_j}{2^{t_j}}}}.$$

# Example

Consider the polynomial

$$x^3 + 10^{100}x^2 - 10^{100}x - 1,$$

which has one sign variation and, hence, one positive root equal to 1

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**With Cauchy's linear bound, we pair the terms:**

- $\{\frac{x^3}{2}, -10^{100}x\}$ and $\{\frac{x^3}{2}, -1\}$,

and taking the maximum of the radicals we obtain a bound estimate of $1.41421 * 10^{50}$.

# Example

Consider the polynomial

$$x^3 + 10^{100}x^2 - 10^{100}x - 1,$$

which has one sign variation and, hence, one positive root equal to 1

**With LMQ, the "Local Max" quadratic bound, we compute:**

- the minimum of the two radicals obtained from the pairs of terms $\{\frac{x^3}{2}, -10^{100}x\}$ and $\{\frac{10^{100}x^2}{2}, -10^{100}x\}$ which is 2, and
- the minimum of the two radicals obtained from the pairs of terms $\{\frac{x^3}{2^2}, -1\}$ and $\{\frac{10^{100}x^2}{2^2}, -1\}$ which is $\frac{2}{10^{50}}$.
- Therefore, the obtained estimate of the bound is $max\{2, \frac{2}{10^{50}}\} = 2$.

# Good old quadratic complexity bounds

# Good old quadratic complexity bounds

- Using *LMQ*, the performance of the `VAS` real root isolation method was speeded up by an average overall factor of 40%.

# VAS vs VCA on Mignotte polynomials

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- The Mignotte polynomials are of the form $x^n - 2(c \cdot x - 1)^2$, for $c, n \geq 3$, have only 4 real roots and as the degree increases, 2 of the 3 positive roots get closer and closer together.

# VAS vs VCA on Mignotte polynomials

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- We test our methods on the Mignotte polynomial

$$x^{300} - 2(5x - 1)^2$$

.

# VAS has been implemented in *Mathematica* — version 7 shown below

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- — and it takes 0.046 seconds to isolate and approximate the roots of Mignotte's polynomial of degree 300.

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▶ — and it takes 0.046 seconds to isolate and approximate the roots of Mignotte's polynomial of degree 300.

```
In[1]:= f := x^300 - 2 (5 x - 1)^2;

In[9]:= ints = RootIntervals[f][[1]] // Timing

Out[9]= {0.031, {{-2, 0}, {0, 1/5}, {1/5, 1/4}, {1, 3}}}

In[10]:= ints = Last[ints];
         FindRoot[f, {x, #[[1]], #[[2]]}, Method → Brent, WorkingPrecision → 150, MaxIterations → 200] & /@ ints //
          Timing

Out[10]= {0.015, {{x →
       -1.0144385320669281488172557391616077462987206190052230818125849684107751217279585133642868348\
         441369799525549844971713898995570158375762282142622484198}, {x →
       0.19999999999999999999999999999999999999999999999999999999999999999999999999999999999999999999\
        999999999999999999999999999999999999999999958861793771948}, {x →
       0.20000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000\
        000000000000000000000000000000000000000000041138206229655}, {x →
       1.0117175091291073215531547258788715881455524805339758641772534125171800615654039763046715053353\
        40148223065465258510769575765146865990995434417439919722}}}
```

Figure: Isolating and approximating real roots with Mma 7

# VCA has been implemented in maple — version 11 shown below

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— and it takes 170 seconds to just isolate the roots of Mignotte's polynomial of degree 300.

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```
> with(RootFinding) :
> f := x^300 - 2(5 x - 1)^2;
                              f := x^300 - 2 (5 x - 1)^2
> st := time( ) : Isolate(f, digits = 250) : time( ) - st;
                                          170.431
>
```

Figure: To isolate Mignotte's poly of degree 300

# Therefore, ...

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VAS can be many thousand times faster than the fastest implementation of VCA.

# Therefore, ...

VAS can be many thousand times faster than the fastest implementation of VCA.

Moreover, as the following frames indicate, VAS can be many times faster than numeric methods, which cannot compute just the positive roots! They compute all the roots (real and complex).

# Using Mma 7 (1/3 frames)

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Consider the polynomial

$$f = 10^{999}(x-1)^{50} - 1$$

with the 2 positive roots $\neq 1$.

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with the 2 positive roots $\neq 1$.

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# Using Mma 7 (1/3 frames)

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▶ The numeric method `NRoots` used in Mma 7 takes 12.933 seconds to find the two positive roots with 30 digits of accuracy.

```
f := 10^999 (x - 1)^50 - 1

Select[NRoots[f == 0, x, 30], Im[#[[2]]] == 0 &] //
 Timing

{12.933, x == 0.9999999999999999999989528714519 ||
  x == 1.0000000000000000000001047128548}
```

# Using Mma 7 (2/3 frames)

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```
ints = RootIntervals[f][[1]] // Timing
{5.60316×10^-16, {{0, 1}, {1, 2}}}
```

Figure: Using the function RootIntervals in Mma 7

# Using Mma 7 (3/3 frames)

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- . . . and approximates them to 30 digits of accuracy in practically no time at all!

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```
ints = Last[ints];
FindRoot[f, {x, #[[1]], #[[2]]}, Method → Brent,
    WorkingPrecision → 30, MaxIterations → 200] & /@ ints //
 Timing

{0., {{x → 0.9999999999999999999989528714519},
  {x → 1.00000000000000000000001047128548}}}
```

Figure: Using the function FindRoot in Mma 7

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- However, when we try to isolate the roots of a sparse polynomial of very large degree, say 100000, most CASs run out of memory.

- To solve the problem the *VAS* continued fractions method has been implemented using interval arithmetic.

# References

# References


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