DYNAMIC LINK ACTIVATION SCHEDULING IN MULTIHOP RADIO NETWORKS WITH FIXED OR CHANGING CONNECTIVITY

by
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Abstract

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Wireless communication networks exhibit fundamental differences from networks with dedicated point-to-point connections. Channel access and packet routing problems are closely related in radio networks, therefore should be considered jointly. Changing connectivity arises naturally in wireless systems in several cases (i.e. mobile radio nodes, meteor-burst channels) and should be taken into consideration by the channel access protocols. In this dissertation we consider queuing models that capture the above characteristics of radio networks and discuss throughput, delay performance and optimal dynamic control problems in these systems. Channel access and packet routing are studied jointly in a queuing system with interdependent servers that models a multihop radio network with scheduled link activation. Dynamic scheduling schemes are

investigated. The performance of a scheduling policy π is characterized by its stability region that is the set of arrival rates for which the network is stable under π. We obtain a scheduling policy which is optimal in the sense that its stability region dominates the stability region of every other policy and we characterize that region. Methods of stochastic stability theory are employed in this study. In addition to system stability, the issue of queuing delay is of particular importance in communication networks. For a tandem radio network we obtain link activation scheduling policies which are optimal with respect to delay in the stochastic ordering sense; the result in this case is obtained using sample path comparison arguments. Turning to the issue of changing connectivity, a single hop radio network is considered where the user connectivity is modeled by a stochastic process. The necessary and sufficient stabilizability condition as well as a stabilizing policy are obtained. In the case of a symmetric system with unlimited buffer capacity at each user, the channel allocation policy that minimizes the delay is obtained. When each user possesses a single buffer the channel allocation policy that minimizes both throughput and delay is specified. Finally stability issues are investigated in a general queueing network where there is routing and flow control at each queue. The implications of the stability results on deterministic flow networks are discussed.

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CHAPTER 1

Introduction

Wireless communication systems carry a large portion of traffic in today's communication networks. They appear in several forms, such as cellular networks, wireless indoor systems, satellite networks, meteor-burst or general mobile multihop networks. Their importance is expected to grow even more in the future since they provide the natural means for wireless access in tomorrows integrated communication systems. Nevertheless, the understanding we have for the operation of multihop radio networks is poor compared to dedicated link networks or single hop multiaccess channels ([BeG87]).

Multihop wired link packet switching networks have been studied heavily in the past and optimal distributed algorithms are available for the real time control of those systems. Similarly random access algorithms have been developed which achieve efficient utilization of the single hop multiaccess channel ([IIT85]). On the other hand, the research in multihop radio networks is still in a preliminary stage and in the existing systems ad-hoc methods are employed for resource allocation. There is a big research effort towards a better understanding of the principles of operation of multihop radio networks so that the next generation of wireless systems will be able to meet the current traffic demands which exceed by far the capabilities of the systems employed today.

Multihop radio network have certain characteristics that differentiate them from both dedicated link networks and single hop multiaccess channels. On one hand simultaneous transmissions of neighboring links in a radio network interfere; hence not all of the links may be used simultaneously. On the other hand, and unlike single hop multiaccess channels, more than one link may transmit without conflicts at the same time if they are appropriately spatially separated. Whether certain links may transmit simultaneously without conflicts or not is determined by the signaling forms used for transmission, the number of transceivers per node, and certain other specific features of the network. That is made clear in the next section where we discuss in detail the constraints for conflict free transmissions in certain networks. The number of different sets of links which may transmit simultaneously without conflicts is huge, usually of exponential order with respect to the size (number of links) of the network. Therefore an optimal link activation scheduling, that is, the selection of the transmitting links at each time slot is quite difficult to obtain.

If the destination node of a packet is not within the transmission range of the origin node, the packet needs forwarding to reach its destination; hence routing decisions should be made in addition to link activation scheduling. The traffic loading of the links is determined by the routing algorithms; the link activation scheduling is done such that the channel is allocated to neighboring links in a fair manner according to their loading. Clearly link activation and packet routing are strongly interrelated and should be treated jointly in order to have efficient network utilization.

Time varying connectivity is inherent in several types of radio networks including systems with mobile radio nodes, networks with meteor-burst communication channels, networks in environments with hard interference etc. In all the above cases the connectivity varies with time and the channel allocation algorithms should take it under consideration in a dynamic fashion in order to have efficient resource utilization.

In this dissertation we consider queueing models on which the above problems are addressed. Before we discuss in detail, in section 1.3, the problems we have considered, we provide a basis model of a radio network and review the previous work on the subject.

1.1 Multihop Radio Networks

A radio network consists of N nodes the radio connectivities of which are specified by the topology graph $G = (V, E)$. Each node of V corresponds to a radio node and a directed link (v, w) from node v to node w denotes that node w is within the transmission range of node v. A node v may communicate directly with node w if node w is within the transmission range of node v; that is a link of the topology graph corresponds to a radio link. A packet entering the system at some node i may have as its eventual destination any node of a set of nodes S_j so that as soon as the packet reaches any node at S_j it leaves the system. This assumption corresponds to the case where the actual destination of the packet is some node outside of the radio network which is connected through wired link connections with all nodes of S_j. Therefore, after a packet reaches a node of S_j it does not need the resources of the radio network any longer. If node i has no direct communication link to any node of S_j then it needs to forward the packet there and routing decisions must be made. The packet length is taken to be constant and the system is slotted with slot length equal to the packet length. The transmissions are synchronized to start in the beginning of a slot.

Neighboring transmissions are subject to radio interference; there are constraints in the simultaneous transmissions of neighboring links in order to avoid interference. As it has already been mentioned, those constraints depend on several different factors; the number of transceivers per node, the signaling forms used, the available frequency bands andother features of the network. The constraints vary in different networks. Two typical conflict constraints are the following,

1. If there is a single transceiver per node then at each time instant a given node may either transmit to exactly one other node or receive from exactly one other node without conflicts.

2. If there is a single frequency band then the transmission of node i is received without conflict by a node j within the transmission range of i, only if all the other nodes that have in their range node j are silent.

In a network with a single frequency band and one transceiver per node both constraints should be satisfied at each slot in order to have conflict free transmissions. We refer to those networks as networks with no secondary interference tolerance. If spread spectrum signaling is used then a node which is within the transmission range of several transmitting nodes may lock in the transmission of one of them and receives its transmission without interference from the others. In this case, the second constraint is not necessary for conflict free transmissions and we say that secondary interference is tolerated.

Clearly there is a channel sharing problem among neighboring nodes of the network. A multiaccess method is required to achieve efficient utilization of the locally common channel. Several random access schemes have been considered for the channel sharing in multihop networks which are generalizations of well known algorithms that have been developed for the multiaccess channel [HT85]. Their performance in the multihop case was not analogous to that of their counterparts in single hop networks. Another approach that has been taken to the channel allocation problem in radio networks is *scheduled link activation.* In scheduled link activation the transmitting links are selected such that conflicts are avoided. In this dissertation we concentrate on scheduled link activation.

Any set of links that can transmit simultaneously without conflicts is called a *transmission set.* A transmission set is represented by its corresponding *transmission vector*, a binary vector with one element for each link which is equal to one if the link belongs to the transmission set and to zero otherwise. In scheduled link activation, at each slot t the transmitting links constitute a transmission set; let $I(t)$ be the corresponding transmission vector. The collection $\{I(t)\}_{t=1}^{\infty}$ of the transmission vectors at all time slots constitute the schedule. The problem is to determine the schedule.

If there is more than one packet class in the network, where the packet classes are differentiated by the destinations of the packets, a decision should be taken at each slot about the class of the packet that is transmitted by each activated link. This decision is referred to as routing.

Scheduling schemes may be classified as either static or dynamic. In static schemes the schedule is determined in advance and the network state is not taken into consideration in a dynamic fashion. In dynamic schemes the transmission vector at slot t is determined based on some information about the network state in the previous slot. In this dissertation we concentrate on dynamic scheduling.

1.2 Previous work

Much of the previous work on the subject has focused on *fixed periodic* schedules which are as follows. A number T of transmission vectors $I_1, ..., I_T$ are selected and the schedule $\{I(t)\}_{t=1}^{\infty}$ is produced by the repetition of those T vectors, that is

$$I(t) = I_{t mod T+1}$$

In this case the problem of designing the schedule is reduced to the selection of the vectors $I_1, ..., I_T$ which is done such that performance is optimized according to some criteria.

Several approaches have been taken to that problem in [ChL87, EpT90, NeK85, PSK85, Sil82]. The optimization problems that have been posed were computationally intractable in most of the cases and heuristic algorithms have been proposed for their solution. Distributed algorithms for computation of the schedule have been proposed. In [RoS89] the problem of optimizing the long run average throughput and delay within the class of state independent (open loop) schedules is considered and it is shown, using dynamic programming techniques, that the optimal schedule is a fixed periodic schedule. The computation of the optimal schedule though is intractable. State dependent scheduling has been considered in [CiS89] where the transmission at each slot t is selected based on the number of packets at the network nodes at the beginning of the slot. The transmission set is computed heuristically based on the network's state. Distributed algorithms were proposed for the computation of the transmission set and their performance is evaluated by simulation.

Another problem that has been studied within the context of fixed periodic schedules is that of characterizing the set of vectors of link activation rates which are achievable by certain schedules. Given a fixed periodic schedule, the component of the vector $f = \frac{1}{T}\sum_{j=1}^{T} I_j$ that corresponds to a link e is the proportion of the slots at which link e is scheduled for activation. Given a vector f_0 we want to find a schedule with some period T, if there exists one, such that $f \leq \frac{1}{T}\sum_{j=1}^{T} I_j$. This problem can be shown to be equivalent to that of finding a representation of f as a convex combination of all transmission vectors. The latter depends heavily on the structure of the set of all transmission vectors. That problem has been studied for several different radio networks and in certain cases ([HaS88]) polynomial complexity algorithms have been found for its solution while in others ([Ar84]) the problem has been shown to be NP-hard. The problem of joint routing-scheduling has also been addressed in the above framework.

The issue of changing connectivity in communication networks has been addressed in the past in several different contexts. In radio networks that issue arises in a few cases. Networks with mobile radio nodes or with meteor burst channels are typical examples. Deterministic changing connectivity models have been considered in the past. In [Og88] a deterministic flow network with time varying link capacities is considered. The variation of the capacities of the links with time is assumed to be known. known. The problem is to determine a dynamic flow that maximizes the amount of commodity reaching the destination within some time τ. In [OrR90,91] the shortest path problem in a network where the edge weight changes with time is considered. Algorithms for finding the minimum weight path at all time instances are provided. Further related work is referenced in the above papers.

1.3 Outline of the dissertation

In this dissertation we focus on dynamic radio network models. We address first the issue of joint routing and scheduling in a general multihop, multidestination radio network. A queueing system with interdependent servers is considered as a model for the network. The dependency among the servers is described by the definition of those subsets of them that can be activated simultaneously and reflects the conflict constraints in the radio network. We study the problem of scheduling the server activation under the constraints imposed by the dependency among them. The performance criterion of a scheduling policy π is its throughput. That is characterized by its stability region C_π that is the set of vectors of arrival and service rates for which the system is stable. A policy π_0 is obtained which is optimal in the sense that its stability region C_{π_0} is a superset of the stability region of every other scheduling policy. The stability region C_{π_0} is characterized. The optimal policy π_0 is difficult to implement, however we obtain an easily implementable adaptive version of π_0 that has the same stability properties. We study also the behavior of the network for arrival rates that lie outside the stability region. Methods of stochastic stability theory are employed in our study. We rely on the Markov property of the queue length process to obtain the stability results.

The issue of queueing delay is of particular importance in addition to stability in a communication network. That is studied in a tandem radio network. The constraint for nonconflicting transmissions is that no two links incident to the same node should be activated simultaneously. At each time t the set of activated links is selected based on the lengths of the queues at time $t-1$. The activated links should satisfy the constraint for nonconflicting transmissions. Each radio node receives exogeneous traffic of constant length packets. Two assumptions are made about the traffic. The first, is that all packets have a common destination that is one end-node of the tandem; the second, is that the destination of each packet is an immediate neighbor of the node at which the packet enters the network. Under the first traffic assumption the system corresponds to a tandem queueing system with interdependent servers and a scheduling policy is obtained which is samplepath wise optimal. Under the second traffic assumption the system corresponds to a set of parallel queues with interdependent servers; it is shown that the optimal policy activates at each slot the maximum possible number of servers. The optimality results are obtained using sample path comparison arguments.

The issue of time varying connectivity is addressed in chapter 4. We consider a dynamic model of changing connectivity. A queueing model of a single hop radio network is considered consisting of N parallel queues (radio nodes) competing for the attention of a single server (central station). At each time slot each queue may or may not be *connected* to the server. The server is allocated to one of the connected queues at each slot; the allocation decision is based on the lengths of the connected queues only. At the end of each slot, service may be completed with a given fixed probability. In the case of infinite buffers, necessary and sufficient conditions are obtained for stability of the system in terms of the different system parameters. The allocation policy that minimizes the delay for the special case of symmetric queues (i.e. queues with equal arrival, service and connectivity statistics) is provided. In a system with a single buffer per queue an allocation policy is obtained that maximizes the throughput and minimizes the delay. The delay optimality results hold in a stochastic ordering sense and obtained using sample path comparison arguments.

In the last chapter we consider a general queueing network with routing and flow control at each queue. Necessary and sufficient stabilizability conditions on the arrival and service rates of the system are obtained. An alternative proof of the maxflow-mincut theorem is given based on the stability properties of the queueing network.

Finally, before we proceed, a few words about notation. The random quantities are denoted by upper case letters; for the nonrandom quantities we reserve the lower case letters. Vectors are denoted by boldface characters. A random process, that is a sequence of random variables indexed by time, is denoted by the same symbol as the corresponding random variable and a time index. We number the equations independently in each chapter; hence equations in different chapters may have the same numbers. No confusion is caused though since whenever we refer to an equation of another chapter we mention the chapter as well. Also the notation in different chapters overlaps in a few cases. Whenever we refer though to an entity of a different chapter we explicitly mention that so there should be no confusion.

CHAPTER 2

Joint routing and scheduling for maximum throughput

2.1 Introduction

In this chapter we address the issue of joint routing and scheduling in a *constraint* queueing model of a mutihop radio network. The queueing network has arbitrary topology and multiple servers. The servers are interdependent in that they can not provide service simultaneously. The dependency among them is reflected on the constraints which specify exactly which subsets of servers may be active simultaneously. The servers correspond to the links and the constraints disallow simultaneous transmisions for neighboring links. We consider slotted time. At each time slot, routing decisions are taken for the served customers and eligible sets of servers are selected for activation. We assume that these decisions are made in a centralized fashion and are based on global knowledge of the queue lengths in the entire network. We assume that buffering at each queue is infinite. We consider the system to be *stable* if the queues do not tend to increase without bound. We wish to find control policies under which the system is stable for given arrival and service rates. Indeed, we characterize the region of arrival and service rate vectors for which there exists some stabilizing policy, and do find a policy which in fact stabilizes the system for all arrival and service rate vectors in that region. Such a policy is in a sense optimal as far as throughput is concerned.

In addition to multihop radio networks, the constrained queueing model is appropriate for other resource allocation problems as well. A model of a database with concurrency control and locking has been considered in [Ke85, MiW84, Mi85]; the constrained queueing system that we study in this chapter captures that database model where the constraints reflect the locking constraints of the database and the policy that we propose provides a concurrency control algorithm that achieves maximum throughput. In [BaW90] a generalized multiserver queue is proposed as a model of certain parallel processing systems; that multiserver queue can also be modeled by an appropriate constrained queueing system.

This chapter is organized as follows. In section 2.2 we describe the constrained queueing model and we discuss how it corresponds to a multihop radio network. In section 2.3 we state the stability performance criteria and we present the optimality results. In section 2.4 the behavior of the system in the instability region is investigated. In section 2.5 we give another maximum throughput policy which is easily implementable. In section 2.6 we demonstrate how the constrained queueing system is appropriate for other resource allocation problems.

2.2 The constrained queueing model

We consider a network consisting of L nodes and N links. The connectivity of the system is represented by the directed graph $G = (V, E)$, where V is the set of nodes and E is the set of links (fig. 1). Each link coresponds to a server that serves customers residing at the origin node of the link; after service the customers are directed to the destination node of the link. The origin and destination nodes of link i are denoted by $q(i)$ and $h(i)$ respectively. The terms servers and links are used interchangably in the following. A customer may enter the network at any node. Its destination is a subset of the network nodes in the sense that as long as the customer reach any of these nodes it leaves the system. Each customer reaches its destination by appropriate routing through the network. There are J customer classes which are distinguished by the destinations of the customers. The set of destination nodes for class j is V_j. At each node l customers of all classes are queued, except of those classes j for which node l is a destination, that is $l \in V_j$ (any customer of the latter classes leaves the system as long as it reaches node l). We consider slotted time . At each slot t certain links originating from node l provide service; those are the active links at slot t. Notice that the customers are not committed to specific outgoing links of a node l by the time they reach l but at the beginning of each slot a decision is taken which customers (of which classes) are allocated at which links. This decision corresponds to routing.

There are constraints in the simultaneous activation of the serves in the sense that certain servers can not provide service at the same time. An *activation set* is a set of servers which can be activated in the same slot. An activation set is represented by its *activation vector*, that is a binary vector with N elements; the ith element corresponds to server i, and is equal to 1 if server i belongs to the activation set and to 0 otherwise. The terms activation set and activation vector will be used interchangably in the rest of the chapter. The *constraint set* S consists of all activation vectors of the system; this set completely specifies the activation constraints. We make the following assumption about the structure of the constraint set which is natural in the systems we consider.

A.1 Every subset of an activation set is an activation set itself.

At the beginning of each slot an activation set of links is selected that provide service during the slot. This is reffered as scheduling in the following.

2.2.1 Queue length dynamics

The servers are synchronized to start service at the beginning of a slot. We control the system through the selection, at each time slot, of the activation set and of the class of the customer assigned to each activated server for service. The binary variable $E_{ij}(t)$ indicates whether server i is activated in slot t or not and which customer class it serves; if $E_{ij}(t) = 1$ server i is activated and serves a customer of class j otherwise it is not. A customer served by server i in slot t completes service with some probability m_i. More specifically we consider a binary variable $M_i(t)$ and a customer served by server i during slot t completes service and moves from queue $q(i)$ to queue $h(i)$ if $M_i(t) = 1$; otherwise it remains at queue $q(i)$. The vector $\mathbf{E}(t) = (E_{ij}(t) : \ i = 1,..,N, \ j = 1,..,J)$, indicates which class each server serves at slot t. A binary vector $\mathbf{e} = (e_{ij} : \ i = 1,..,N, \ j = 1,..,J)$ is a *multiclass activation vector* if the corresponding vectors $\mathbf{e}^j = (e_{ij} : \ i = 1,..,N)$, $j = 1,..,J$ are such that $\sum_{j=1}^{J} \mathbf{e}^j \in S$. Let $\mathcal{E}$ by the collection of all multiclass activation vectors. At each slot t the vector $\mathbf{E}(t)$ is selected from the set $\mathcal{E}$. The decisions are based on the number of customers of each class in each queue; the queues have unlimited capacity (infinite buffers). This information is represented as follows. Let $X_{lj}(t)$ be the number of customers of class j at queue i by the end of slot t (or the beginning of slot $t + 1$). The vector $\mathbf{X}(t) = (X_{lj}(t) : \ i = 1,..,L, \ j = 1,..,J)$ consists of the lengths of the queues of all customer classes and is called the multiclass queue length vector at slot t. We denote by $\mathcal{X}$ the space where the vector $\mathbf{X}(t)$ lies.

Consider a function $g : \mathcal{X} \to \mathcal{E}$; if $g(\mathbf{x}) = \mathbf{e} = (e_{ij} : i = 1,..,N, \ j = 1,...,J)$ then denote the vector $\mathbf{e}^j$ by $g^j(\mathbf{x})$. An *activation rule* is a function $g : \mathcal{X} \to \mathcal{E}$ with the property that no servers are considered activated for non-existing customers, that is to say, the number of servers of queue l activated by the

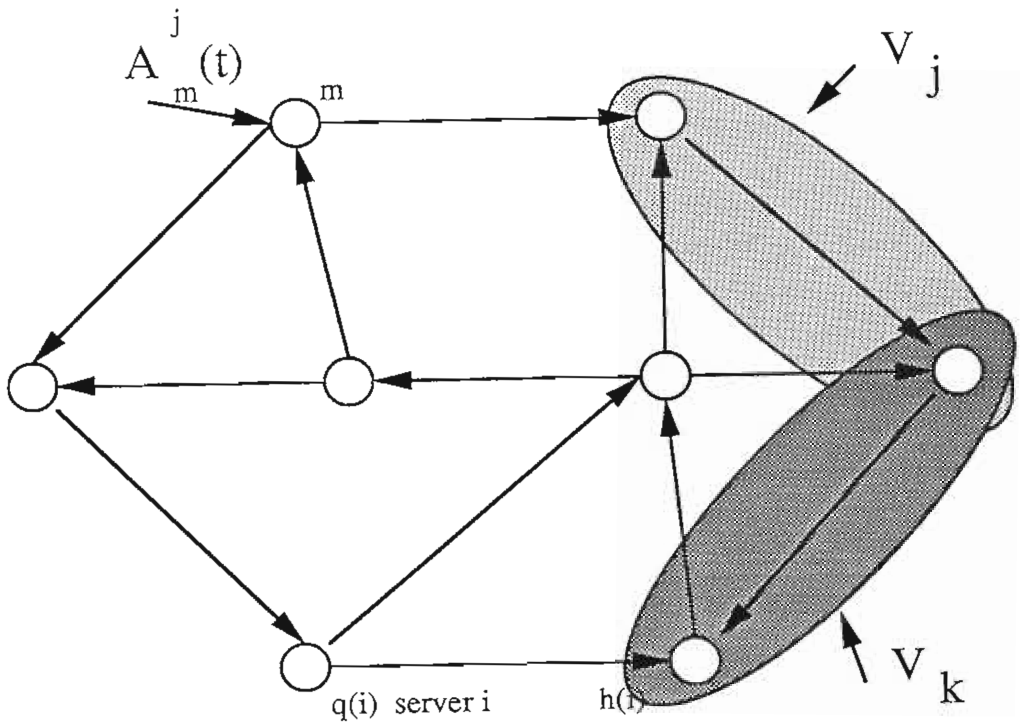

Figure 1. The topology graph of a constrained queueing network

activation vector $g^j(\mathbf{x})$ are less than or equal to x_{lj}; where servers of queue l are those servers i for which $q(i) = l$. An *activation policy* is a collection of activation rules g_t, $t = 1, 2, ..$; at slot t we have $\mathbf{E}(t) = g_t(\mathbf{X}(t-1))$. Until section 5 we consider stationary policies that is policies which use the same activation rule at each slot. In section 5 it will become apparent that we do not gain anything with respect to stability if we consider nonstationary policies in addition to stationary. The class of all stationary activation policies is denoted by G. When the network is operated by policy π with activation rule g, at slot $t+1$ we have $\mathbf{E}^j(t+1) = g^j(\mathbf{X}(t))$ where $\mathbf{E}^j(t) = (E_{ij}(t) : \ i = 1, .., N)$ is the activation vector of class j at slot t. The state of the system evolves according to the equations:

$$\mathbf{X}^j(t+1) = \mathbf{X}^j(t) + R^j\mathbf{M}(t+1)\mathbf{E}^j(t+1) + \mathbf{A}^j(t+1), \quad t = 0, 1, \ldots, \ j = 1, .., J \tag{2.1}$$

where $\mathbf{M}(t)$ is a diagonal matrix, the ith diagonal element of which is equal to $M_i(t)$, $\mathbf{X}^j(t) = (X_{lj}(t) : \ l = 1, .., L)$ is the vector of the queue lengths of class j by the end of slot t, $\mathbf{A}^j(t) = (A_{lj}(t) : \ l = 1, .., L)$ is a vector with its lth element $A_{lj}(t)$ being equal to the number of customers of class j arriving at queue l during slot t and R^j is an $L \times N$ matrix that reflects the connectivities of the queues among themselves and with the destination node of class j. Matrix R^j is called the routing matrix of class j. The element of R^j in its lth row and

ith column is

$$r^j_{li} = \begin{cases} 1, & \text{if } h(i) = l \text{ and queue } l \text{ is not connected} \\ & \text{with the destination node of class } l \\ -1, & \text{if } q(i) = l \\ 0, & \text{otherwise.} \end{cases}$$

We assume that $\{A_{lj}(t)\}_{t=1}^{\infty}$, $\{M_i(t)\}_{i=1}^{\infty}$ are i.i.d. sequences of random variables for all $l = 1,..,L$, $j = 1,..,J$, $i = 1,..,N$. Furthermore we assume that the above processes are independent among themselves and the second moments of the arrival processes $E[A^2_{lj}(t)]$ are finite. Under those statistical assumptions and for any policy in G the queue length process $\{\mathbf{X}(t)\}_{t=1}^{\infty}$ is a Markov chain. Finally we make the following assumption concerning the topology of the network.

A.2 If a customer of class j_0 may reach some queue l_0 then this customer may be forwarded from queue l_0 to some destination node of class j_0 if an appropriate route is selected. More specifically, if there is a sequence of servers $i_1,..,i_n$ such that $a_{q(i_1)j_0} > 0$, $h(i_m) = q(i_{m+1})$, $m = 1,..,n-1$ then there exists a sequence of servers $i'_1,..,i'_{n'}$ such that $h(i_n) = q(i'_1)$, $h(i'_m) = q(i'_{m+1})$, $m = 1,..,n'-1$ and there exists a link in E_d from $h(i'_{n'})$ to the destination node of class j_0.

The above queueing model corresponds to nultihop radio networks as follows.

2.2.2 The correspondence with multihop radio networks

Consider the radio network model introduced in section 1.2. A packet entering the system at some node i may have as eventual destination any node of a set of nodes S_j in the sense that whichever node of S_j the packet reaches it leaves the system. This assumption correspond the case where the actual destination of the packet is some node outside of the radio network which is connected through wired link connections with all nodes of S_j. Therefore after a packet reaches a node of S_j it does not need the resources of the radio network any more. We consider a multidestination system with J sets of eventual destinations $S_1, .., S_J$. Notice that two destination sets S_j and S_m may overlap. We distinguish the packets in different classes according to their eventual destinations. The packet length is constant and the system is slotted with slot length equal to the packet length. The transmissions are synchronized to start in the beginning of a slot.

The radio network is modeled by a constrained queueing system with $|V|$ queues and $|E|$ servers. Each queue corresponds to a network node and each server to a radio link. There are J customer classes; each class contains packets with a specific destination. The service process $\{M_i(t)\}_{t=1}^{\infty}$ of a link i has the following interpretation. If link i transmits at slot t the packet is correctly received if $M_i(t) = 1$, otherwise it is lost and it has to be retransmitted. Note that since we select the transmiting links at each slot such that conflicts are avoided the possible packet losses which are modeled by the service process are due to channel inefficiencies. A set of servers constitute an activation set if the corresponding set of links of the radio network is a transmission set. The topology graph $G' = (V', E')$ of the constrained queueing system is very similar to G. The set of nodes V' is the union of V'_q and V'_d where V'_q is identical to V and V'_d contains one node for each packet class. The set of links is $E' = E'_s \bigcup E'_d$ where E'_s is identical to E and E'_d contains a link (v, w) from node $v \in V'_q$ to node $w \in V'_d$ if node v of the radio network belongs to the destination set S_j of the class of packets that correspond to node $w \in V'_d$. When secondary interference is tolerated the constrained set S contains all matchings of G where the weights are updated at each sklot.

2.3 Stability considerations

The system is stable if the queue length process reaches a steady state and does not blow to infinity. When the Markov chain $\mathbf{X}$ is irreducible, stability of the system is equivalent to ergodicity of $\mathbf{X}$. Under the general assumptions we made about the constraint set and the topology of the queueing system we can not guarante irreducibility of the queue length process. In the general case the state space is partitioned in transient and recurrent states. We consider the system to be stable if all recurrent states are positive recurrent and the queue length process hits the recurrent states with probability one; that is $\mathbf{X}$ does not remain in the set of transient states for ever. In the following we state our definition of stability after we recall some basic facts from Markov chain theory ([KSK76]).

A state $\mathbf{x}$ is *reachable* by some state $\mathbf{y}$ if $P(\mathbf{X}(t+n) = \mathbf{x}|\mathbf{X}(t) = \mathbf{y}) > 0$ for some $n \geq 1$. The states $\mathbf{x}$ and $\mathbf{y}$ *communicate* if they are reachable by each other. A set of states R is *closed* if $P(\mathbf{X}(t+1) = \mathbf{x}|\mathbf{X}(t) = \mathbf{y}) = 0$ for all $\mathbf{y} \in R$, $\mathbf{x} \notin R$. The state space of the chain is partitioned in the sets T, R_1, $R_2, ..$ where R_j, $j = 1, 2, ..$ are closed sets of communicating states and T contains all states which do not belong to any closed set of communicating states and therefore are transient. For any $\mathbf{x} \in T$ assume that $\mathbf{X}(0) = \mathbf{x}$ and consider the time

$$\tau_{\mathbf{x}} = \begin{cases} \infty, & \text{if } \mathbf{X}(t) \in T, \ \forall t > 0 \\ \min\{t > 0 : \ \mathbf{X}(t) \notin T\}, & \text{otherwise} \end{cases} \tag{3.1}$$

at which the chain hits some of the sets R^j for first time when it starts at $t = 0$ from state $\mathbf{x}$ at $t = 0$. If $\cup_{j=1}^{\infty} R_j = \emptyset$ then apparently $\tau_{\mathbf{x}} = \infty$. We can define now stability as follows.

Definition 3.1: The system is stable if for the queue length process $\mathbf{X}$ we have

$$P(\tau_{\mathbf{y}} < \infty) = 1 \ \ \forall \mathbf{y} \in T \tag{3.1a}$$

and all states $\mathbf{x} \in \cup_{j=1}^{\infty} R_j$ are positive recurrent.

Next theorem states sufficient conditions for stability of the system according to definition 3.1. Those conditions involve the drift of a test (Liapunov) function on the state space of the chain. In the case of irreducible chains similar conditions are sufficient for ergodicity of the chain and have been studied extensively ([Fo53]).

Theorem 3.1: Consider a Markov chain $\mathbf{X}(t)$ with state space $\mathcal{X}$. If there exists a lower bounded real function $V : \mathcal{X} \to \mathbf{R}$, an $\epsilon > 0$ and a finite subset $\mathcal{X}_0$ of $\mathcal{X}$ such that

$$E[V(\mathbf{X}(t+1)) - V(\mathbf{X}(t))|\mathbf{X}(t) = \mathbf{y}] \leq -\epsilon \qquad \text{if} \quad \mathbf{y} \notin \mathcal{X}_0, \tag{3.2}$$

$$E[V(\mathbf{X}(t+1))|\mathbf{X}(t) = \mathbf{y}] < \infty \qquad \text{if} \quad \mathbf{y} \in \mathcal{X}_0 \tag{3.3}$$

then for the time $\tau_{\mathbf{x}}$ as defined in (3.1) we have

$$P(\tau_{\mathbf{x}} < \infty) = 1 \ \ \forall \mathbf{x} \in T$$

and all states $\mathbf{x} \in \cup_{j=1}^{\infty} R_j$ are positive recurrent.

Proof: For any state $\mathbf{y} \in \mathcal{X}$ consider the Markov chain starting from $\mathbf{y}$ ($\mathbf{X}(0) = \mathbf{y}$) and the time

$$\sigma_{\mathbf{y}} = \begin{cases} \infty, & \text{if } \mathbf{X}(t) \notin \mathcal{X}_0 \ \forall t > 0 \\ \min\{t > 0 : \ \mathbf{X}(t) \in \mathcal{X}_0\}, & \text{otherwise} \end{cases}$$

We show that

$$E[\sigma_{\mathbf{y}}] < \infty \tag{3.4}$$

Assume first that $\mathbf{y} \notin \mathcal{X}_0$. Consider the process $V(t) = V(\mathbf{X}(t))1\{\sigma_{\mathbf{y}} > t\}$ and let $\mathcal{F}_t = \sigma(\mathbf{X}(1), .., \mathbf{X}(t))$. We have

$$E[V(t+1)|\mathcal{F}_t] = E[V(t+1); \ \sigma_{\mathbf{y}} > t|\mathcal{F}_t]P(\sigma_{\mathbf{y}} > t)$$

$$+E[V(t+1);\ \sigma_{\mathbf{y}} \leq t|\mathcal{F}_t]P(\sigma_{\mathbf{y}} \leq t)$$

If $\sigma_{\mathbf{y}} \leq t$, then $V(t+1) = 0$ therefore

$$E[V(t+1)|\mathcal{F}_t] = E[V(t+1);\ \sigma_{\mathbf{y}} > t|\mathcal{F}_t]P(\sigma_{\mathbf{y}} > t)$$

$$\leq E[V(\mathbf{X}(t+1));\ \sigma_{\mathbf{y}} > t|\mathcal{F}_t] = 1\{\sigma_{\mathbf{y}} > t\}E[V(\mathbf{X}(t+1))|\mathcal{F}_t] \leq V(t) - \epsilon 1\{\sigma_{\mathbf{y}} > t\}$$

By taking expectations above we get

$$0 \leq E[V(t+1)] \leq E[V(t)] - \epsilon P(\sigma_{\mathbf{y}} > t) \tag{3.5}$$

and by replacing recursivelly from (3.5) we get

$$0 \leq E[V(t+1)] \leq V(\mathbf{y}) - \epsilon \sum_{k=0}^{t} P(\sigma_{\mathbf{y}} > k) \tag{3.6}$$

If we let $t \to \infty$ in (3.6) we get

$$0 \leq V(\mathbf{y}) - \epsilon E[\sigma_{\mathbf{y}}] \Rightarrow E[\sigma_{\mathbf{y}}] \leq \epsilon^{-1} V(\mathbf{y}) \tag{3.7}$$

If $\mathbf{y} \in \mathcal{X}_0$ then

$$E[\sigma_{\mathbf{y}}] = \sum_{\mathbf{x} \in \mathcal{X}_0} p_{\mathbf{yx}} + \sum_{\mathbf{x} \notin \mathcal{X}_0} p_{\mathbf{yx}} E[\sigma_{\mathbf{y}} + 1] \leq 1 + \epsilon^{-1} \sum_{\mathbf{x} \notin \mathcal{X}_0} p_{\mathbf{yx}} V(\mathbf{x})$$

where $p_{\mathbf{yx}} = P(\mathbf{X}(t+1) = \mathbf{x}|\mathbf{X}(t) = \mathbf{y})$. From (3.7) we have $E[\sigma_{\mathbf{y}}] < \infty$ in this case also. Hence (3.4) hold for all $\mathbf{y}$ and it implies that the chain visits set $\mathcal{X}_0$ infinitely often from whichever state it starts. Since $\mathcal{X}_0$ is finite one of its elements, let say $\mathbf{z}$, is visited infinitely often. Because of that state $\mathbf{z}$ is recurrent and can not belong to T since T contains only transient states. Therefore $\mathbf{z} \in \cup_{i=1}^{\infty} R_i$ and (3.1) holds for all $\mathbf{x} \in T$.

If the chain is restricted in any closed communicating class R_i then it is irreducible. Using the well known Foster's criterion ([As87]), conditions (3.2) and (3.3) imply that all states in R_i are positive recurrent. ◇

Corollary 3.1: When conditions (3.2), (3.3) hold then there is a finite number of closed sets of communicating states.

Proof: Consider a closed set of communicating states R_i. If the system starts from a state $\mathbf{y} \in R_i$ then it will remain in R_i for ever. If R_i has no common element with $\mathcal{X}_0$ then the chain will never hit $\mathcal{X}_0$ therefore $\sigma_{\mathbf{y}} = \infty$ a.s. which a contradiction with 4.2. Hence each set R_i should have a common element with $\mathcal{X}_0$ and since $\mathcal{X}_0$ is finite and the sets R_i are disjoint there can be only a finite number of them. ◇

A function V such that in theorem 3.1 is usually called stochastic Liapunov function.

2.3.1 Scheduling for maximum throughput

We would like the system to be stable for a wide range of arrival and service rates. The arrival rate of class j to queue l, $E[A_{lj}(t)]$ is denoted by a_{lj}. The multiclass arrival rate vector $\mathbf{a} = (a_{lj} : \ l = 1, .., L, \ j = 1, .., J)$ consists of the arrival rates of all classes at all queues. The service rate $E[M_i(t)]$ of server i is denoted by m_i; the service rate vector is $\mathbf{m} = (m_i : i = 1, .., N)$. We quantify the performance of an activation policy by its stability region.

Definition 3.2: Stability region C_π of policy π is the set of multiclass arrival rate vectors $\mathbf{a}$ for which the system is stable under π.

We wish a policy π to have a large stability region. The largest the stability region the better the policy is.

Definition 3.3: A policy π_1 *dominates* another policy π_2 if $C_{\pi_2} \subset C_{\pi_1}$.

If policy π_1 dominates policy π_2 the system is stable under π_1 whenever it is stable under π_2 (fig. 2). Two policies are not always comparable since it may no one dominates the other. This is the case for policies π_3 and π_1 in fig. 2.

Definition 3.4: The stability region of the system is

$$C = \bigcup_{\pi \in G} C_\pi$$

The set C contains all arrival rate vectors for which there exists a policy in G that stabilize the system. An *optimal policy*, that is one which dominates any other policy in G, should have stability region that is a superset of the stability region of any other policy in G; therefore it should have stability region equal to C. Such a policy is called maximum throughput policy in the rest of the chapter. Notice that since not any two policies have comparable stability regions, it is not clear at all whether a maximum throughput policy exists or not. One of our main results is that such an optimal policy exists indeed.

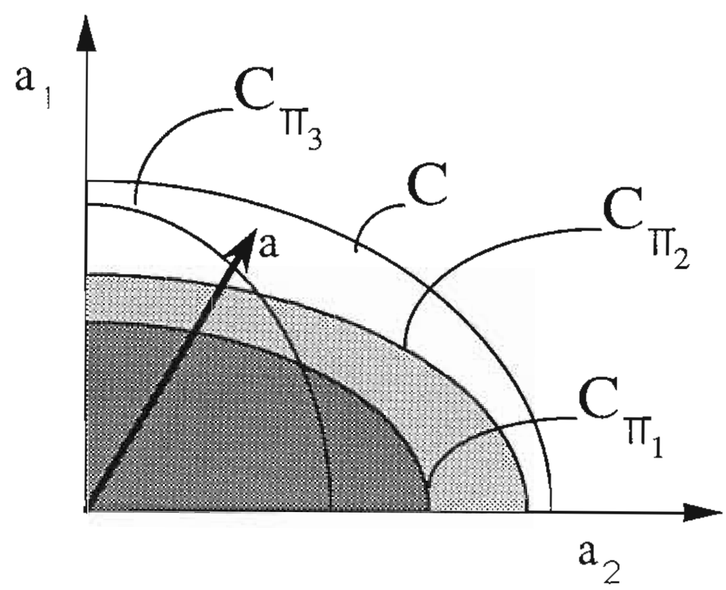

Figure 2. Stability regions

2.3.2 Maximum Throughput Policy

Policy π_0 that we specify next achieves maximum throughput. The activation rule for π_0 is denoted by $g_0(\cdot)$; the vector $\mathbf{E}(t) = g_0(\mathbf{X}(t-1))$ is selected in three stages.

Stage 1. For each server i a weight $D_i(t)$ is selected as follows. For each class j and server i consider the quantity

$$D_{ij}(t) = \begin{cases} (X_{q(i)j}(t-1) - X_{h(i)j}(t-1))m_i, & \text{if } h(i) \notin V_j \\ X_{q(i)j}(t-1)m_i, & \text{if } h(i) \in V_j \end{cases}$$

Let $D_i(t) = \max_{j=1,..,J}\{D_{ij}(t)\}$ be the weight of server i and $\mathbf{D}(t) = (D_i(t) : i = 1, .., N)$ the weight vector at slot t.

Stage 2. A maximum weighted activation vector $\hat{\mathbf{c}}$ is selected from S

$$\hat{\mathbf{c}} = \arg\max_{\mathbf{c}\in S}\{\mathbf{D}^T(t)\mathbf{c}\}$$

If more than one vector $\mathbf{c}$ achieves the maximum, $\hat{\mathbf{c}}$ is selected arbitrarily among them.

Stage 3. Let $\hat{j}_i$ be the class for which $D_i(t) = D_{i\hat{j}_i}(t)$ for each server i; if more than one classes satisfy the above inequality then $\hat{j}_i$ can be any of these classes. The multiclass activation vector $\mathbf{E}(t)$ is as follows

$$E_{ij}(t) = \begin{cases} 1, & \text{if } \hat{c}_i = 1,\ j = \hat{j}_i \text{ and } X_{q(i)j}(t-1) \text{ is greater than} \\ & \text{the number of servers that serve queue } q(i) \\ 0, & \text{otherwise} \end{cases}$$

Remarks

1. If $D_{ij}(t)$ is greater than zero and server i serves a customer of class j during slot t then the quantity $D_{ij}(t)$ tends to be reduced. That is the difference between $X_{h(i)j}(t)$ and $X_{g(i)j}(t)$ is diminished. Policy π_0 selects $\mathbf{E}(t)$ such that the servers i and the corresponding classes j for which $D_{ij}(t)$ is larger are activated. In other words π_0 tends for each class to equalize the queue lengths of the same class in different network nodes, giving priority to the servers and classes for which this difference is larger.

2. The implementation of policy π_0 requires at each time slot t the solution of the optimization problem

$$\max_{\mathbf{c}\in S}\{\mathbf{D}^T(t)\mathbf{c}\} \tag{3.8}$$

The number of possible activation vectors (the cardinality of S) is usually large compared to the number of servers; in fact it is of exponential order with respect to the number of servers most of the times. Therefore solution of the above optimization problem by exhaustive search of all activation vector is out of the question. In certain cases the constraint set S has a specific structure that can be utilized for the solution of (3.8). In section 5 the constraint sets are illustrated for several communication and computer systems. Finding efficient algorithms for the solution of (3.8) given the constraint set S in each particular application is important for the implementation of π_0.

2.3.3 Characterization of the Stability Region

In this section we characterize the system stability region C. The set C' that we specify next plays an essential role in the characterization of C since as it will be shown later $C' \subset C \subset \bar{C}'$ where $\bar{C}'$ is the closure of C'; the closure of C is well defined since C is a subset of R^{LJ}. The definition of C' involves deterministic flows in the graph G and the heuristic discussion that preceeds its definition provide some intuition.

Assume that the constraint queueing system is stable under some scheduling policy π and that it operates in steady state. Let f_{ij} be the rate with which customers of class j are served by server i. Since the system is in steady state, the rate with which customers of class j enter some queue l should be equal to the rate with which customers of the same class leave the queue l; that is the rates f_{ij} should satisfy the flow conservation equations in each network node. Consider a multicommodity arrival rate vector $\mathbf{a}$ and let $\mathbf{a}^j = (a_{lj} : l = 1, .., L)$ be the vector which contain the arrival rates of class j at all network queues for $j = 1, .., J$. The vector $\mathbf{f}^j = (f_{ij} : \ i = 1, .., N)$ that consists of nonnegative numbers and satisfy the flow conservation equations which are written in matrix form as

$$\mathbf{a}^j = -R^j \mathbf{f}^j, \tag{3.9}$$

is called $\mathbf{a}$-admissible flow vector for class j. The vector $\mathbf{f} = (f_{ij} : \ i = 1, .., N, \ j = 1, .., J)$ that consists of nonnegative numbers and is such that the corresponding vectors $\mathbf{f}^j$ satisfy (3.9) for $j = 1, .., J$ is an $\mathbf{a}$-admissible multicommodity flow vector. Let $F_{\mathbf{a}}$ be the set of all $\mathbf{a}$-admissible multicommodity flow vectors. Associated with a vector $\mathbf{f} \in F_{\mathbf{a}}$ is the vector $\hat{\mathbf{f}} = \sum_{j=1}^{J} \mathbf{f}^j$. The component of $\hat{\mathbf{f}}$ that corresponds to server i is the total rate of customers which are served by server i, irrespectively of their classes; therefore $\hat{\mathbf{f}}$ is called *total flow* vector. The set C' can be defined now as

$$C' = \{\mathbf{a} : \text{there exists } \mathbf{f} \in F_a,\ \mathbf{c} \in co(S) \text{ such that for the}$$

$$\text{corresponding } \hat{\mathbf{f}} \text{ we have } m_i^{-1}\ \hat{f}_i < c_i \text{ if } \hat{f}_i > 0 \text{ and } \hat{f}_i = 0 \text{ if } c_i = 0\ \}$$

where $co(S)$ is the convex hall of the constraint set S. The closure of C' is characterized in the following lemma.

Lemma 3.1: The closure $\bar{C}'$ of C' is

$$\bar{C}' = \{\mathbf{a} : \text{there exists an } \mathbf{f} \in F_a, \text{ and a } \mathbf{c} \in co(S), \text{ such that } M^{-1}\hat{\mathbf{f}} \leq \mathbf{c}\}$$

where M is the diagonal matrix with ith diagonal element equal to m_i, $i = 1, .., N$.

Proof: We denote by B the set which we want to show that is equal $\bar{C}'$. We show first that all points of B are points of closure of C' therefore $B \subset \bar{C}'$. Suppose that for the vector $\mathbf{a}$ there exist $\mathbf{f} \in F_{\mathbf{a}}, \mathbf{c} \in co(S)$ such that $M^{-1}\mathbf{f} \leq \mathbf{c}$. Consider the vectors $\mathbf{a}_n$, $n = 1, ...$ such that $\mathbf{a}_n = ((1 - \frac{1}{n})a_{ij} : j = 1, \ldots, J,\ j = 1, .., N)$ and the multicommodity flows $\mathbf{f}_n = ((1 - \frac{1}{n})f_{ij} : j = 1, \ldots, J,\ i = 1, .., N)$. We can easily verify that $\mathbf{f} \in F_a$, implies that $\mathbf{f}_n \in F_{\mathbf{a}_n}$. Furthermore since

$M^{-1}\hat{\mathbf{f}} \leq \mathbf{c}$ we get $(M^{-1}\hat{\mathbf{f}}(1-\frac{1}{n}))_i < c_i$ if $(M^{-1}\hat{\mathbf{f}})_i > 0$. Hence we have $\mathbf{a}_n \in C$ for every $n = 1, ...$ The limit of the sequence $\mathbf{a}_n$ is $\mathbf{a}$, therefore we have $\mathbf{a} \in \bar{C}'$.

Now we show that all points of closure of C' belong to B therefore $\bar{C}' \subset B$. Suppose that $\mathbf{a}, \in \bar{C}'$, then there exists a sequence $\mathbf{a}_n \quad n = 1, \ldots$ such that $\mathbf{a}_n \in C$ and $\lim_{n\to\infty} \mathbf{a}_n, = \mathbf{a}$. Since $\mathbf{a}_n \in C$, there exist $\mathbf{f}_n \in F_{\mathbf{a}_n}, \mathbf{c}_n \in co(S)$ such that $(M_n^{-1}\hat{\mathbf{f}}_n)_i < c_i$ if $(M_n^{-1}\hat{\mathbf{f}}_n)_i > 0$. We show that there exist $\mathbf{f} \in F_{\mathbf{a}},\ \mathbf{c} \in co(S)$ such that $M^{-1}\hat{\mathbf{f}} \leq \mathbf{c}$ which imply that $\mathbf{a}$ belongs to B. We can assume that for each class j, the server utilization vector $\mathbf{f}_n^j$ is *acyclic* in the sense that there is no sequence of queues $q_1, \ldots, q_n$ such that there exists a server i that directs traffic of class j from q_l to $q_{l+1} \quad l = 1 \ldots, n-1$, q_n to q_1 and $(\mathbf{f}_n^j)_l > 0$, $l = 1, .., n$. If some $\mathbf{f}_n^j$ is not acyclic, we can easily make it without violating the rest of the conditions that $\mathbf{f}_n$ satisfies. Note, furthermore, that if $\mathbf{f}_n^j$ is acyclic, then

$$\|\mathbf{f}_n^j\| \leq q\|\mathbf{a}_n^j\| \tag{3.10}$$

where $\|\cdot\|$ is the square norm of R^n and q depends only on the topological structure of the system that is numbers of servers, queues, customer classes, and the connectivity. Since $\mathbf{a}_n^j \to \mathbf{a}^j$ the sequence of flows $\mathbf{f}_n^j$ is bounded because of (3.10) therefore there exists a subsequence $\mathbf{f}_{n_k}^j$ that converges to some vector $\mathbf{f}^j$. Notice that $\mathbf{f}^j$is a flow vector for class j since

$$\|\mathbf{a}^j + R^j\mathbf{f}^j\| = \|\mathbf{a}^j + R^j\mathbf{f}^j - (\mathbf{a}_{n_k}^j + R^j\mathbf{f}_{n_k}^j)\|$$

$$\leq \|\mathbf{a}^j - \mathbf{a}_{n_k}^j\| + \|R^j(\mathbf{f}^j - \mathbf{f}_{n_k}^j)\| \to 0$$

therefore we have

$$\mathbf{a}^j = -R^j \mathbf{f}^j$$

Since the above holds for every $\mathbf{f}^j$ we conclude that there exist a subsequence of multicommodity flows $\mathbf{f}_{n_l}$ such that

$$\mathbf{f}_{n_l} \in F_{\mathbf{a}_{n_l}}, \ \mathbf{f}_{n_l} \rightarrow \mathbf{f}, \ M^{-1}\hat{\mathbf{f}}_{n_l} \leq \mathbf{c}_{n_l}. \tag{3.11}$$

Since $\mathbf{c}_{n_l} \in co(S), l = 1, \ldots$ and $co(S)$ is closed and bounded there exists a subsequence $\mathbf{c}_{n_{l_k}}, \ k = 1, ..$ that converges to a vector $\mathbf{c} \in co(S)$. From (3.11) we have $M^{-1}\hat{\mathbf{f}}_{n_{l_k}} \leq \mathbf{c}_{n_{l_k}}$ and by taking the limits in both sides of the inequality we get $M^{-1}\hat{\mathbf{f}} \leq \mathbf{c}$. ◇

2.3.4 Optimality Results

The optimality of π_0 and the characterization of C are stated in this section. Three lemmas preceed the theorem. In the following lemma we show that under π_0 the system is stable in C'. It shown that a quadratic function of the queue length vector satisfies the conditions (3.2), (3.3) therefore stability follows from theorem 3.1.

Lemma 3.2: Under policy π_0 the system is stable for every $\mathbf{a} \in C'$

$$C' \subset C_{\pi_0}.$$

Before we proceed in the proof of lemma 3.2 we state a property of the convex hall of the constraint set which is a consequence of assumption A.1.

Lemma 3.3: If a vector $\mathbf{c}$ belongs to $co(S)$, then any vector $\mathbf{a}$ such that $\mathbf{0} \leq \mathbf{a} \leq \mathbf{c}$ belongs to $co(S)$ as well.

Proof: The vector $\mathbf{c}$ can be written as

$$\mathbf{c} = \sum_{i=1}^{|S|} m_i \mathbf{c}_i$$

where $m_i \geq 0, \mathbf{c}_i \in S, i = 1, \ldots |S|$ and $\sum_{i=1}^{|S|} m_i \leq 1$. Notice that because of assumption A.1 vector $\mathbf{0}$ always belongs to S therefore the condition $\sum_{i=1}^{|S|} m_i \leq 1$ is sufficient for $\sum_{i=1}^{|S|} m_i \mathbf{I}_i$ to belong to $co(S)$ since we can always add to this sum the vector $(1 - \sum_{i=1}^{|S|} m_i)\mathbf{0}$ and make the sum of the coeficients to be equal to one.. Consider an element a_i of $\mathbf{a}$ such that $a_i < c_i$. We can find a vector $\mathbf{c}' \in co(S)$ such that $c'_j = c_j$ if $j \neq i$ and $c'_i = a_i$. We specify $m'_j, \quad j = 1, ..|S|$ such that $\mathbf{c}' = \sum_{i=1}^{|S|} m'_i \mathbf{c}_i$ For each indicator vector $\mathbf{c}_j$ which activates server i let $m'_j = m_j(\frac{a_i}{c_i})$; for the vector $\mathbf{c}_k$, which is similar to $\mathbf{c}_j$ except that its element that corresponds to server i is set to 0, let $m'_k = m_k + m_j(1 - \frac{a_i}{c_i})$; for all the other indicator vectors $\mathbf{c}_l$ let $m_l = m'_l$. Notice that $\sum_{i=1}^{|S|} m_i = \sum_{i=1}^{|S|} m'_i$ therefore the vector $\mathbf{c}'$ as defined belongs indeed to $co(S)$. Also $\mathbf{c}'$ is such that $\mathbf{a} \leq \mathbf{c}'$ but it differs from $\mathbf{a}$ in a smaller number of elements than $\mathbf{c}$ does. By repeating the process for each i such that $a_i < c_i$ we obtain an expression of $\mathbf{a}$ as convex combination of vectors from S and the lemma follows. ◇

Proof of lemma 3.2: For each vector $\mathbf{a} \in C'$ we show that the queue where $\mathbf{a}^T$ is the transpose of vector $\mathbf{a}$. The first term in the sum in the right-hand side of (3.14) can be bounded for all states $\mathbf{X}(t)$ by a constant, let say b_1, as we show in the following. By simple calculations we have

$$\sum_{j=1}^{J} E[(R^j\mathbf{M}(t+1)\mathbf{E}^j(t+1) + \mathbf{A}^j(t+1))^T(R^j\mathbf{M}(t+1)\mathbf{E}^j(t+1) + \mathbf{A}^j(t+1))|\mathbf{X}(t)] =$$

$$= \sum_{j=1}^{J}\sum_{l=1}^{L} E[(A_{lj}(t+1))^2 + 2\sum_{j=1}^{J}\sum_{l=1}^{L} E[A_{lj}(t+1)]E[(R^j\mathbf{M}(t+1)\mathbf{E}^j(t+1))_l|\mathbf{X}(t)]+$$

$$+ \sum_{j=1}^{J}\sum_{l=1}^{L} E[((R^j\mathbf{M}(t+1)\mathbf{E}^j(t+1))_l)^2|\mathbf{X}(t)] \tag{3.15}$$

where the notation $(\mathbf{a})_l$ denotes the lth element of vector $\mathbf{a}$ inside the parenthesis. The term $(R^j\mathbf{M}(t+1)\mathbf{E}^j(t+1))_l$ is upper bounded by the number of servers that direct traffic to queue l thus by N as well. Similarly, $((R^j\mathbf{M}(t+1)\mathbf{E}^j(t+1))_l)^2$ is upperbounded by N^2. Thus, from (3.15), we have

$$\sum_{j=1}^{J} E[(R^j\mathbf{M}(t+1)\mathbf{E}^j(t+1) + \mathbf{A}^j(t+1))^T(R^j\mathbf{M}(t+1)\mathbf{E}^j(t+1) + \mathbf{A}^j(t+1))|\mathbf{X}(t)] \leq$$

$$\leq \sum_{j=1}^{J}\sum_{l=1}^{L} E[(A_{lj}(t+1))^2] + 2N\sum_{j=1}^{J}\sum_{l=1}^{L} E[A_{lj}(t+1)] + LJN^2 = b_1. \tag{3.16}$$

For the second term of the sum in the right-hand side of (3.14) we have

$$\sum_{j=1}^{J} E[2(R^j\mathbf{M}(t+1)\mathbf{E}^j(t+1) + \mathbf{A}^j(t+1))^T \cdot \mathbf{X}^j(t)|\mathbf{X}(t)] =$$

$$= \sum_{j=1}^{J} 2(\mathbf{X}^j(t))^T E[R^j\mathbf{M}(t+1)\mathbf{E}^j(t+1)|\mathbf{X}(t)] + \sum_{j=1}^{J} 2(\mathbf{X}^j(t))^T E[\mathbf{A}^j(t+1)|\mathbf{X}(t)]$$

$$= \sum_{j=1}^{J} 2(\mathbf{X}^j(t))^T R^j M g_0^j(\mathbf{X}(t)) + \sum_{j=1}^{k} 2(\mathbf{X}^j(t))^T \mathbf{a}^j \tag{3.17}$$

where g_0 is the activation rule that corresponds to π_0 and $M = E[\mathbf{M}(t)]$. From (3.14), (3.16) and (3.17) the relation (3.13) follows. It remains to show (3.12). Notice that the ith element of the vector $\mathbf{X}^j(t)^T R^j M$ is equal to $-D_{ij}(t+1)$ where $D_{ij}(t+1)$ is as it has been defined in stage 1 of policy π_0. From the definition of π_0 we have for all $j = 1, .., J$

$$\mathbf{X}^j(t)^T R^j M g_0^j(\mathbf{X}(t)) = -(\mathbf{D}(t+1))^T g_0^j(\mathbf{X}(t))$$

therefore for the first term in the right hand side of (3.17) we have

$$\sum_{j=1}^{J} 2\mathbf{X}^j(t)^T R^j M g_0^j(\mathbf{X}(t)) = -2(\mathbf{D}(t+1))^T \sum_{j=1}^{J} g_0^j(\mathbf{X}(t)). \tag{3.18}$$

Since $\mathbf{a} \in C$, there exists a multiclass flow $\mathbf{f}$ with corresponding total flow vector $\hat{\mathbf{f}}$, and a vector $\mathbf{q} \in co(S)$ such that $\mathbf{f} \in F_a$ and $m_i^{-1}\hat{f}_i < q_i$ if $q_i > 0$, $f_i = 0$ if $q_i = 0$. Hence, we have

$$\mathbf{a}^j = -R^j \mathbf{f}^j \qquad j = 1, \ldots, J, \tag{3.19}$$

and there exist $\delta > 1$ such that for all $i = 1, .., N$

$$\delta m_i^{-1} \hat{f}_i < q_i \quad \text{ if } \quad q_i > 0. \tag{3.20}$$

Relation (3.20) together with lemma 3.3 imply that $\delta M^{-1}\mathbf{f} \in co(S)$. Thus, we have

$$\delta M^{-1}\hat{\mathbf{f}} = \sum_{i=1}^{|S|} \gamma_i \mathbf{c}_i$$

where $\mathbf{c}_i \in S, \gamma_i \geq 0$ for $i = 1, \ldots, |S|$ and $\sum_{i=1}^{|S|} \gamma_i \leq 1$. Alternatively, we have

$$M^{-1}\hat{\mathbf{f}} = \sum_{i=1}^{|S|} \lambda_i \mathbf{c}_i, \tag{3.21}$$

where $\lambda_i = \frac{\gamma_i}{\delta}$ that is $\lambda_i \geq 0, \sum_{i=1}^{|S|} \lambda_i < 1$. The second term of the sum in the right-hand side of (3.17), after substitutions from (3.19) and (3.21), becomes

$$\sum_{j=1}^{J} 2(\mathbf{X}^j(t))^T \mathbf{a}^j = \sum_{j=1}^{J} 2(\mathbf{X}^j(t))^T R^j \mathbf{f}^j =$$

$$\sum_{j=1}^{J} 2(\mathbf{D}^j(t+1))^T M^{-1} \mathbf{f}^j \leq 2 \max_{j=1,\ldots,J} ((\mathbf{D}^j(t+1))^T) M^{-1} \sum_{j=1}^{J} \mathbf{f}^j$$

$$= 2(\mathbf{D}(t+1))^T M^{-1} \hat{\mathbf{f}}. \tag{3.22}$$

By replacing $M^{-1}\hat{\mathbf{f}}$ in (3.22) from (3.21), we get

$$\sum_{j=1}^{J} 2(\mathbf{X}^j(t))^T \mathbf{a}^j \leq 2(\mathbf{D}(t+1))^T \sum_{i=1}^{|S|} \lambda_i \mathbf{c}_i. \tag{3.23}$$

From (3.17), (3.18) and (3.23) we get

$$\sum_{j=1}^{J} E[2(R^j \mathbf{M}(t+1)\mathbf{E}^j(t+1) + \mathbf{A}^j(t+1)^T)\mathbf{X}^j(t)|\mathbf{X}(t)]$$

$$\leq -2(\mathbf{D}(t+1))^T \sum_{j=1}^{J} g_0^j(\mathbf{X}(t)) + 2(\mathbf{D}(t+1))^T \sum_{i=1}^{|S|} \lambda_i \mathbf{c}_i. (3.24)$$

From the definition of the π_0 we have

$$\max_{\mathbf{c}\in S}\{(\mathbf{D}(t+1))^T \mathbf{c}\} \geq (\mathbf{D}(t+1))^T \sum_{j=1}^{J} g_0^i(\mathbf{X}(t))$$

$$\geq \max_{\mathbf{c}\in S}\{(\mathbf{D}(t+1))^T \mathbf{c}\} - N^2 \geq (\mathbf{D}(t+1))^T \mathbf{c} - N^2 \quad \forall\, \mathbf{c} \in S. \tag{3.25}$$

Relations (3.24) and (3.25) imply that

$$\sum_{j=1}^{J} E[2(R^j \mathbf{M}(t+1)\mathbf{E}^j(t+1) + a^j(t+1))^T \mathbf{X}^j(t)|\mathbf{X}(t)]$$

$$\leq -2 \max_{\mathbf{c}\in S}\{(\mathbf{D}(t+1))^T \mathbf{c}\} + N^2 + 2(\mathbf{D}(t+1))^T \sum_{i=1}^{|S|} \lambda_i \mathbf{c}_i$$

$$\leq -2(1 - \sum_{i=1}^{|S|} \lambda_i) \max_{\mathbf{c}\in S}\{(\mathbf{D}(t+1))^T \mathbf{c}\} + N^2. \tag{3.22}$$

The term $-2(1 - \sum_{i=1}^{|S|} \lambda_i) \max_{\mathbf{c} \in S}\{(\mathbf{D}(t+1))^T \mathbf{c}\}$ can be as small as we like if $V(\mathbf{X}(t))$ is sufficiently large. Note first that, as $V(\mathbf{X}(t))$ grows, the components of $\mathbf{X}(t)$ grow as well, that is if we have $V(\mathbf{X}(t)) \geq b$ then we get

$$\max_{\substack{l=1,\ldots,L \\ j=1,\ldots,J}} \{X_{lj}(t)\} \geq \sqrt{\frac{b}{JL}} \tag{3.27}$$

Let

$$(l_o, j_o) = \arg \max_{\substack{l=1,\ldots,L \\ j=1,\ldots,J}} \{X_{lj}(t)\}. \tag{3.28}$$

Consider a sequence of queues $l_o, l_1, \ldots l_n,\ n \leq L$ such that there is a server that directs traffic of class j_o from queue l_m to queue $l_{m+1}, 0 \leq m < n$ and from queue l_n out of the system; such a sequence exists since by assumption there is a path from any queue to the destination of any customer class. Then, we have

$$X_{l_o j_o}(t) = \sum_{m=0}^{n-1} (X_{l_m j_o}(t) - X_{l_{m+1} j_o}(t)) + X_{l_n j_0}(t). \tag{3.29}$$

From (3.27), (3.28), and (3.29), we get

$$\max_{m=0,\ldots,n-1} \{(X_{l_m j_o}(t) - X_{l_{m+1} j_o}(t)), X_{l_n j_0}(t)\} \geq \frac{X_{l_o j_o}(t)}{n} \geq \frac{X_{l_o j_o}(t)}{L} \geq \frac{1}{L}\sqrt{\frac{b}{JL}}. \tag{3.30}$$

From the definition of π_0 we have

$$\max_{\mathbf{c} \in S}\{(\mathbf{D}(t+1))^T \mathbf{c}\} \geq \min_{i=1,\ldots,N} m_i \max_{m=0,\ldots,n-1} \{(X_{l_m j_o}(t) - X_{l_{m+1} j_o}(t)), X_{l_n j_0}(t)\}$$

$$\geq \frac{1}{L}\sqrt{\frac{b}{JL}} \min_{i=1,\ldots,N} m_i. \tag{3.31}$$

From (3.26) and (3.31), we get

$$\sum_{j=1}^{J} E[2(R^j\mathbf{M}(t+1)\mathbf{E}^j(t+1)+\mathbf{A}^j(t+1))^T\mathbf{X}^j(t)|\mathbf{X}(t)] \leq$$

$$\leq -2(1-\sum_{i=1}^{|S|}\lambda_i)\frac{1}{L}\sqrt{\frac{b}{JL}}\min_{i=1,\ldots,N} m_i + N^2. \quad (3.32)$$

From (3.14), (3.16), and (3.32), we have

$$E[V(\mathbf{X}(t+1)) - V(\mathbf{X}(t))|\mathbf{X}(t)] \leq -2(1-\sum_{i=1}^{|S|}\lambda_i)\frac{1}{L}\sqrt{\frac{b}{JL}}\min_{i=1,\ldots,N} m_i + N^2 + b_1$$

$$\text{if} \qquad V(\mathbf{X}(t)) \geq b. \quad (3.33)$$

If we take

$$b = JL(\frac{L(\epsilon + b_1 + N^2)}{2(1-\sum_{i=1}^{|S|}\lambda_i)\min_{i=1,..,N} m_i})^2$$

then (3.12) follows. ◇

Lemma 3.4: If $\mathbf{a} \in (\bar{C}')^c$, then the queueing system is unstable for any policy in G.

Proof: Suppose that $\mathbf{a} \in (\bar{C}')^c$ and the system is stable under some policy $\tilde{\pi}$. There is a closed set of communicating states R_i such that all states in R_i are positive recurrent. For the rest of the proof we consider the Markov chain restricted in R_i. The restricted Markov chain is positive recurrent and therefore ergodic. We can easily see that since $\mathbf{X}(t)$ is ergodic Markov chain, $\mathbf{M}(t)$ an i.i.d. process and $\mathbf{M}(t)$ is independent of $(\mathbf{X}(0),..,\mathbf{X}(t-1))$, the process $(\mathbf{X}(t-1),\mathbf{M}(t))$ is a Markov chain which is ergodic as well. Consider the vector

$\mathbf{E}^j(t) = \mathbf{M}(t)\tilde{g}^j(\mathbf{X}(t-1))$ where $\tilde{g}$ is the activation rule of $\tilde{\pi}$. Its ith element is equal to 1 if during slot t a customer of class j which is served by server i completes service and moves from queue $q(i)$ to $h(i)$. The vector $\sum_{\tau=1}^{t} \mathbf{E}^j(\tau)$ indicates how many customers of class j have crossed each server during slots 1 to t. Since $(\mathbf{X}(t-1), \mathbf{M}(t))$ is ergodic, the normalized sum $1/t \sum_{\tau=1}^{t} \mathbf{E}(\tau)$ converges a.s. as $t \to \infty$ to a vector $\mathbf{f}^j$ which indicates the average number of class j customers that cross each server i. In each queue l and for each class j the average number of incoming customers should be equal to the average number of outgoing customers since otherwise $\mathbf{X}_{lj}(t)$ goes a.s. to infinity and the chain can not be positive recurrent. Hence we have $\mathbf{a}^j = -R^j\mathbf{f}^j$ and the vector $\mathbf{f} = (f_{ij} : i = 1, .., N,\ j = 1, .., J)$ belongs to $F_{\mathbf{a}}$ where f_{ij} is the ith element of vector $\mathbf{f}^j$. We show now that $\hat{\mathbf{f}} = \sum_{j=1}^{J} \mathbf{f}^j$ is such that $M^{-1}\hat{\mathbf{f}} \in co(S)$ therefore we get $\mathbf{a} \in \bar{C}'$ which is a contradiction. Consider the vector $\mathbf{E}(t) = \sum_{j=1}^{J} \mathbf{E}^j(t)$. We have

$$\lim_{t\to\infty} \frac{1}{t} \sum_{\tau=1}^{t} \mathbf{E}(\tau) = \lim_{t\to\infty} \frac{1}{t} \sum_{\tau=1}^{t} \sum_{j=1}^{J} \mathbf{E}^j(\tau) = \sum_{j=1}^{J} \mathbf{f}^j = \hat{\mathbf{f}}$$

and because of the ergodicity of $(\mathbf{X}(t-1), \mathbf{M}(t))$ we have

$$\hat{\mathbf{f}} = E[\mathbf{M}(t) \sum_{j=1}^{J} \tilde{g}(\mathbf{X}(t-1))] \tag{3.34}$$

where the expectation is taken with respect to the stationary probability distribution of $(\mathbf{X}(t-1), \mathbf{M}(t))$. Since for each slot t, $\mathbf{X}(t-1)$ and $\mathbf{M}(t)$ are independent, we have

$$E[\mathbf{M}(t) \sum_{j=1}^{J} \tilde{g}(\mathbf{X}(t-1))] = E[\mathbf{M}(t)]E[\sum_{j=1}^{J} \tilde{g}(\mathbf{X}(t-1))] \tag{3.35}$$

Since for all $\mathbf{X}(t-1)$ we have $\sum_{j=1}^{J} \tilde{g}(\mathbf{X}(t-1)) \in co(S)$, apparently $E[\sum_{j=1}^{J} \tilde{g}(\mathbf{X}(t-1))] \in co(S)$ as well and from (3.34), (3.35) we get that $M^{-1}\hat{\mathbf{f}}$ belongs to $co(S)$. ◇

Policy π_0 achieves indeed maximum throughput as it is stated in the following theorem which contains our main results.

Theorem 3.2: The set C' characterizes the system stability region in the sense

$$C' \subset C \subset \bar{C}'$$

Policy π_0 achieves maximum throughput

$$C' \subset C_{\pi_0} \subset C \subset \bar{C}_{\pi_0}$$

Proof: By definition of the system stability region we have $C_{\pi_0} \subset C$ and from lemma 3.2

$$C' \subset C_{\pi_0} \subset C \tag{3.36}$$

From lemmas 3.2, 3.4 we have

$$C \subset \bar{C}' \subset \bar{C}_{\pi_0} \tag{3.37}$$

The theorem follows from (3.36), (3.37). ◇

Remarks

1. From part a of theorem 3.2 we have $C' - C \subset \bar{C}' - C'$. Note that $\bar{C}' - C'$ is the boundary of C' which is a surface (has no volume) in the space where **a**

lies. Hence part a of theorem 3.2 determines C within a surface in the space where **a** lies, therefore provides complete characterization of the stability region for any practical purpose. Similarly part b implies C_{π_0} differs from C at most by a surface therefore π_0 achieves optimal throughput.

2. In the definition of C' the condition for a pair **a** of arrival and service rate vectors to belong to C' is an existential one. It is desirable to have an algorithm to decide if a particular pair **a** belongs to C. Whether an efficient algorithm exists or not for this problem depends highly on the structure of S. This problem has been studied in a different context in [Ar84], [HaS88] for two specific constraint sets. For a constraint queueing system that corresponds to a packet radio network with no secondary interference tolerance (in the next section both the radio network and the corresponding queueing system are specified), deciding whether an arrival rate vector **a** ($m_i = 1, \ i = 1,..,N$) belongs to C' or not is an NP-hard problem as it has been shown in [Ar84]. When secondary interference is tolerated the corresponding problem has been shown in [HaS88] to be solvable by an algorithm of polynomial time complexity.

2.4 Behavior of the system under nonstationary policies

In this section the behavior of the system when it is operated by nonstationary policies is studied. We focus on systems with a single class of customers and we show that for arrival rates in $(\bar{C}')^c$ the total number of customers in the system grows to infinity a.s. for any possible scheduling policy. Since there is a stationary policy that stabilizes the system within C' the above result implies that we do not gain anything in stability by considering nonstationary policies.

Consider a system with one class of incoming customers and assume that the service time of a customer is equal to one slot that is $M_i(t) = 1$ a.s. for $i = 1, ..., N$, $t = 1, 2, ...$ Let us denote by $\tilde{G}$ the class of all policies $\pi = \{g_t\}_{t=1}^{\infty}$ where g_t is some rule for selecting $\mathbf{E}(t)$ based on the whole history of queue lengths up to time t. Since we have just one class of customers, we will denote the unique arrival rate vector and queue length vector of the class by $\mathbf{a}$ and $\mathbf{X}(t)$ respectively in the following; the multiclass activation vector $\mathbf{E}(t)$ at slot t coincides with the activation vector for the unique customer class and a multicommodity flow coincides with the corresponding total flow vector and both vectors are denoted by $\mathbf{f}$. The following theorem is the main result of this section.

Theorem 4.1: For every policy $\pi \in \tilde{G}$ and arrival rate vector $\mathbf{a} \in \bar{C}^c$ the total number of customers in the system $\sum_{l=1}^{L} X_l(t)$ grows to infinity

$$\lim_{t \to \infty} \sum_{l=1}^{L} X_l(t) = \infty \qquad a.s. \tag{4.1}$$

In the proof of the theorem we use some results from deterministic network flow theory on a flow network that corresponds to the constrained queueing system. We present that next.

For each arrival rate vector $\mathbf{a}$ and flow vector $\mathbf{f}$, we consider a network $N_{\mathbf{af}}$ that consists of a graph $Y = (V, E)$, specifying the topology of the network and a capacity assignment to the edges $C_{\mathbf{af}} : E \rightarrow R^{+}$. Graph Y is very similar to the topology graph of the queueing network. The set of nodes V contains one node i for each queue i of the network, an *originator* node o and a *terminal* node d. The set of edges E contains one edge (i, j) for each server that serves queue i and directs traffic to queue j, one edge (i, d) for each server that serves queue i and directs traffic out of the system and one edge (o, i) for each queue i. The topology graph Y is the same for all vectors $\mathbf{a}$ and $\mathbf{f}$. The capacities of the edges depend on the vectors $\mathbf{a}$ and $\mathbf{f}$ as follows. Each edge that corresponds to server k has capacity f_k; each edge (o, i) has capacity a_i. The vector $\mathbf{q} = (q_i : i \in E)$ which is such that $0 \leq q_i \leq C_{\mathbf{af}}(i)$ and which satisfies the flow conservation equations

$$\sum_{\substack{i\text{:terminates}\\ \text{at } l}} q_i = \sum_{\substack{i\text{:originates}\\ \text{at } l}} q_i \qquad \text{for } l \in (V - \{o, d\}) \tag{4.2}$$

is a *feasible flow* vector for the network $N_{\mathbf{af}}$. Let $Q_{\mathbf{af}}$ be the set of feasible flows. The *flow transfer* q of a flow vector $\mathbf{q}$ is defined by $q = \sum_{i=1}^{L} q_{(o,i)}$. We need to consider the maximum flow transfer over all feasible flows in $Q_{\mathbf{af}}$. That is denoted by

$$q_{\mathbf{af}} = \max_{\mathbf{q} \in Q_{\mathbf{af}}} q \tag{4.3}$$

and is called maxflow in the following. An alternative characterization of the maxflow, which we need in the following, is given by the maxflow-mincut theorem. We need the notion of a *cut* to state that theorem. A cut (W, W') of the network $N_{\mathbf{af}}$ is a partition of V such that $0 \in W$ and $d \in W'$. The capacity $C_{\mathbf{af}}((W, W'))$ of the cut (W, W') is defined as the sum of the capacities of the edges which are directed from W to W' (We denote both the capacity of an edge and the capacity of a cut by $C_{\mathbf{af}}(\cdot)$). A *mincut* $(W, W')_{\mathbf{af}}$ of the network $N_{\mathbf{af}}$ is a cut of minimum capacity. In the following $(W, W')_{\mathbf{af}}$ denotes a mincut of $N_{\mathbf{af}}$ and $W_{\mathbf{af}}$, $W'_{\mathbf{af}}$ reffer to the sets W, W' respectively of $(W, W')_{\mathbf{af}}$.

Maxflow-mincut theorem ([PaS82]):

$$q_{\mathbf{af}} = C_{\mathbf{af}}((W, W')_{\mathbf{af}}).$$

Next lemma preceeds the proof of theorem 4.1.

Lemma 4.1: If $\mathbf{a} \in (\bar{C})^c$, then there exists $\mathbf{f}_o \in co(S)$ such that

$$\sum_{i=1}^{L} a_i - \max_{\mathbf{f} \in co(S)} C_{\mathbf{af}}((W, W')_{\mathbf{af}}) = \sum_{i=1}^{L} a_i - C_{a\mathbf{f}_o}((W, W')_{a\mathbf{f}_o}) > 0.$$

Proof: Since the set of edges $\{(o, i) : i = 1, \ldots, L\}$ is a cut with capacity $\sum_{i=1}^{L} a_i$, for every $\mathbf{f} \in co(S)$ we have

$$\sum_{i=1}^{L} a_i \geq \max_{\mathbf{f} \in co(S)} C_{\mathbf{af}}((W, W')_{\mathbf{af}}). \tag{4.4}$$

It is enough for the proof of the lemma to show that the equality in (4.4) does not hold and that the maximum is actually achieved. The capacity of a cut is a continuous function of $\mathbf{f}$. The capacity of a mincut $C_{\mathbf{af}}((W, W')_{\mathbf{af}})$ is continuous in $\mathbf{f}$ as a maximum of continuous functions. Since $C_{\mathbf{af}}((W, W')_{\mathbf{af}})$

is continuous in $\mathbf{f}$, its maximum value when $\mathbf{f} \in co(S)$ is achieved for some $\mathbf{f}_o \in co(S)$. It is enough to show that $\sum_{i=1}^{L} a_i > C_{\mathbf{af}_o}((W,W')_{\mathbf{af}_o}) > 0$. Assume that $C_{\mathbf{af}_o}((W,W')_{\mathbf{af}_o}) = \sum_{i=1}^{L} a_i$. Then, from the maxflow-mincut theorem, there exists $\mathbf{q}^o \in Q_{\mathbf{af}}$ such that

$$\sum_{i=1}^{L} q^o_{(o,i)} = C_{\mathbf{af}_o}((W,W')_{\mathbf{af}_o}) = \sum_{i=1}^{L} a_i. \tag{4.5}$$

Since $\mathbf{q}^o \in Q_{\mathbf{af}}$, we have $0 \leq q^o_{(o,i)} \leq a_i \quad i = 1,\ldots,L$ and in view of (4.5), we have

$$q^o_{(o,i)} = a_i \qquad i = 1,\ldots,L. \tag{4.6a}$$

From (4.5a) and the flow conservation equations (4.2) which should be satisfied by $\mathbf{q}^o$, we conclude that the elements of $\mathbf{q}^o$ that correspond to the edges of G that correspond to the servers, constitute a vector that belongs to $F_{\mathbf{a}}$. That vector belongs to $co(S)$ as well, as it is implied by the capacity constraints and the fact that $\mathbf{f}_o \in co(S)$. This is a contradiction since $\mathbf{a} \in (\bar{C})^c$. $\diamond$

Corollary 4.1: There exists an $\epsilon > 0$ such that for every $\mathbf{f} \in co(S)$, we have

$$\sum_{l \in W_{\mathbf{af}}} a_l - \epsilon \geq \sum_{\substack{i \in W_{\mathbf{af}},\ j \in W'_{\mathbf{af}} \\ i \neq o\ (i,j) \in E}} f_{(i,j)}.$$

Proof: From lemma 4.1 we have

$$\sum_{l=1}^{L} a_l - C_{\mathbf{af}}((W,W')_{\mathbf{af}}) \geq \sum_{l=1}^{L} a_l - \max_{\mathbf{f} \in co(S)} C_{\mathbf{af}}((W,W')_{\mathbf{af}}) = \epsilon > 0. \tag{4.6b}$$

In the left handside of (4.6b) the capacities of the forward edges of $(W,W')_{\mathbf{af}}$

that originate from o cancel out with the corresponding $a_l's$ and we have

$$\sum_{l=1}^{L} a_l - C_{\mathbf{af}}((W,W')_{\mathbf{af}}) = \sum_{l \in W_{\mathbf{af}}} a_l - \sum_{\substack{i \in W_{\mathbf{af}},\ j \in W'_{\mathbf{af}} \\ i \neq o\ (i,j) \in E}} f_{(i,j)} \geq \epsilon > 0$$

which completes the proof of the corollary. ◇

Now we proceed in the proof of theorem (4.1).

Proof of Theorem 4.1: We show first the following

$$\sum_{l=1}^{L} X_l(t) \geq \min_{Q \subset \{1,\dots,L\}} \{\sum_{\tau=0}^{t} (\sum_{l \in Q} (A_l(\tau) - a_l) + \epsilon)\} \tag{4.7}$$

For each $Q \subset \{1, ..., L\}$, from equations (2.1) we have

$$\sum_{l \in Q} X_l(t) = \sum_{l \in Q} X_l(t-1) + \sum_{l \in Q} (R\mathbf{E}(t))_l + \sum_{l \in Q} A_l(t) \tag{4.8}$$

Each edge which has both end nodes in Q contributes an 1 and a -1 in $\sum_{l \in Q}(R\mathbf{E}(t))_l$, each edge directed to a node in Q from a node outside of Q contributes an 1 and each edge directed from a node of Q to a node out of Q contributes a -1; hence we have

$$\sum_{l \in Q} (R\mathbf{E}(t))_l \geq - \sum_{\substack{l \in Q, j \notin Q \\ (l,j) \in E}} (E_{(l,j)}(t)) \tag{4.9}$$

where $E_{(l,j)}(t)$ denotes the component of $\mathbf{E}(t)$ that corresponds to the server that corresponds to link (l, j). From (4.8) and (4.9) after iterative substitutions we get

$$\sum_{l \in Q} X_l(t) \geq \sum_{\tau=1}^{t} (\sum_{l \in Q} A_l(\tau) - \sum_{\substack{l \in Q, j \notin Q \\ (l,j) \in E}} E_{(l,j)}(\tau)) \tag{4.10}$$

Consider the vector $\lambda(t) = \frac{1}{t}\sum_{\tau=1}^{t} \mathbf{E}(\tau)$ that belongs to $co(S)$ and the flow network $N_{\mathbf{a}\lambda(t)}$. From corollary (4.1) we get

$$\sum_{\substack{l \in W_{\mathbf{a}\lambda(t)}, j \in W'_{\mathbf{a}\lambda(t)} \\ l \neq o \ (l,j) \in E}} \lambda_{(l,j)}(t) \leq \sum_{l \in W_{\mathbf{a}\lambda(t)}} a_l - \epsilon \tag{4.11}$$

where ϵ is as defined there. From relations (4.10), (4.11) we get

$$\sum_{l \in W_{\mathbf{a}\lambda(t)}} X_l(t) \geq \sum_{\tau=0}^{t} \Big(\sum_{l \in W_{\mathbf{a}\lambda(t)}} (A_l(\tau) - a_l) + \epsilon \Big)$$

that shows (4.7). For any set $Q \subset \{1, ..., L\}$ the random variables $(\sum_{l \in Q}(A_l(\tau) - a_l) + \epsilon)$, $\tau = 1, 2, ..$ are i.i.d. with expected value $\epsilon > 0$ hence we have

$$\lim_{t \to \infty} \sum_{\tau=0}^{t} \Big(\sum_{l \in Q} (A_l(\tau) - a_l) + \epsilon \Big) = \infty \quad a.s. \quad \forall Q \in \{1, .., L\} \tag{4.12}$$

From (4.7), (4.11) we get (4.1). ◇

2.5 Implementable maximum throughput policy

The maximum throughput policy π_0 specified in section 2.3.2 requires the computation of a maximum weight activation set at each slot. That computation is complicated in several cases as we mentioned earlier. In this section we specify another link activation policy that we call π_1. This policy is, roughly speaking, an adaptive version of π_0; it achieves maximum throughput in the same sense as π_0 and is easily implementable. Let $\mathbf{I}(t) = \sum_{j=1}^{K} \mathbf{E}^j(t)$. Policy

π_1 computes the vector $\mathbf{E}(t)$ in three stages based on $\mathbf{X}(t-1)$ and $\mathbf{E}(t-1)$ as follows.

Stages 1,3. Those stages are the same to the corresponding stages of policy π_0.

Stage 2. An activation vector $\mathbf{c}$ is selected from S randomly according to some probability distribution $P[\mathbf{X}(t-1),\cdot]$ on S. Then we let

$$\hat{\mathbf{c}} = \arg \max_{\mathbf{d} \in \{\mathbf{c}, \mathbf{I}(t-1)\}} \{\mathbf{D}^T(t)\mathbf{d}\}$$

The distribution $P[\mathbf{X}(t-1),\cdot]$ should be such that the probability of selecting a maximum weight activation vector is bounded from below by some positive number a for all $\mathbf{X}(t-1) \in \mathcal{X}$. Under policy π_1 the process $\mathbf{Y}(t) = (\mathbf{X}(t-1), \mathbf{I}(t))$ is a Markov chain. The state space of $\mathbf{Y}$ is partitioned in the sets T, R_1, R_2,.. where R_j, $j = 1, 2, ..$ are closed sets of communicating states and T contains all the transient states For any $\mathbf{x} \in T$ assume that $\mathbf{Y}(0) = \mathbf{x}$ and consider the time

$$\tau_{\mathbf{x}} = \begin{cases} \infty, & \text{if } \mathbf{X}(t) \in T, \ \forall t > 0 \\ \min\{t > 0: \ \mathbf{X}(t) \notin T\}, & \text{otherwise} \end{cases} \tag{3.1}$$

The following theorem shows that π_1 achieves maximum throughput.

Theorem 5.1: If $\mathbf{a} \in C'$ and π_1 acts on the system then we have

$$P(\tau_{\mathbf{x}} < \infty) = 1 \ \ \forall \mathbf{x} \in T \tag{5.2}$$

and all states $\mathbf{x} \in \cup_{j=1}^{\infty} R_j$ are positive recurrent.

Proof: For each vector $\mathbf{a} \in C'$ we show that the queue length process satisfies the conditions of theorem 3.1. Let $I_{\max}(\mathbf{X}(t-1)) = \arg\max_{\mathbf{c}\in S}(\mathbf{D}^T(t)\mathbf{c})$; if more than one $\mathbf{c}$ achieve the maximum above then $I_{\max}$ is selected arbitrarily to be any of them. Consider the function $V(\mathbf{Y}(t)) = V_1(\mathbf{Y}(t)) + V_2(\mathbf{Y}(t))$ where $V_1(\mathbf{Y}(t)) = \sum_{i=1}^{L}\sum_{j=1}^{J}(X_{ij}(t-1))^2$ and $V_2(\mathbf{Y}(t)) = (\mathbf{D}^T(t)(I_{\max}(\mathbf{Y}(t)) - \mathbf{I}(t)))^2$. We show that if $\mathbf{a} \in C'$ and $\epsilon > 0$ there exists a positive number b which may be a function of ϵ, $\mathbf{a}$ and of the second order moments of the arrival process, such that

$$E[V(\mathbf{Y}(t+1)) - V(\mathbf{Y}(t))|\mathbf{Y}(t)] < -\epsilon \quad \text{if} \quad V(\mathbf{Y}(t)) \geq b. \tag{5.3}$$

Furthermore we show that

$$E[V(\mathbf{Y}(t+1)) - V(\mathbf{Y}(t))|\mathbf{Y}(t)] < \infty \ \ \forall\, \mathbf{Y}(t) \in \mathcal{X}. \tag{5.4}$$

Note that the set $S_b = \{\mathbf{y} : V(\mathbf{y}) < b\}$ is finite; therefore relations (5.3), (5.4) are the sufficient conditions for stability stated in theorem 3.1. Proceeding similarly to the proof of lemma 3.2 we have

$$E[V_1(\mathbf{Y}(t+1)) - V_1(\mathbf{Y}(t))|\mathbf{Y}(t)] \leq -2(1-\sum_{i=1}^{|S|}\lambda_i)\frac{1}{L}\sqrt{\frac{b - V_2(\mathbf{Y}(t))}{JL}} \min_{i=1,\ldots,N} m_i +$$

$$N^2 + b_1 + 2\mathbf{D}^T(t)(I_{\max}(\mathbf{Y}(t)) - \mathbf{I}(t)) \tag{5.5}$$

where $\sum_{i=1}^{|S|}\lambda_i < 1$ and b_1 is a constant. We also have

$$E[V_2(\mathbf{Y}(t+1)) - V_2(\mathbf{Y}(t))|\mathbf{Y}(t)]$$

$$= -(\mathbf{D}^T(t)(I_{\max}(\mathbf{Y}(t)) - \mathbf{I}(t)))^2 P(\mathbf{I}(t+1) = I_{\max}(\mathbf{Y}(t+1))$$

$$+E[V_2(\mathbf{Y}(t+1)) - V_2(\mathbf{Y}(t))|\mathbf{Y}(t), \mathbf{I}(t+1) \neq I_{\max}(\mathbf{Y}(t+1))]$$

$$P(\mathbf{I}(t+1) = I_{\max}(\mathbf{Y}(t+1)) + b_2$$

where b_2 is a constant. From the assumption about the probability distribution with which we select the activation vector in stage 2 we get

$$E[V_2(\mathbf{Y}(t+1)) - V_2(\mathbf{Y}(t))|\mathbf{Y}(t)] \leq -V_2(\mathbf{Y}(t))a$$

$$+((\mathbf{D}^T(t+1)(I_{\max}(\mathbf{Y}(t+1)) - \mathbf{I}(t+1)))^2 - (\mathbf{D}^T(t)(I_{\max}(\mathbf{Y}(t)) - \mathbf{I}(t)))^2)(1-a) + b_2$$

and after some calculations we get

$$E[V_2(\mathbf{Y}(t+1)) - V_2(\mathbf{Y}(t))|\mathbf{Y}(t)] \leq -V_2(\mathbf{Y}(t))a$$

$$+b_3\sqrt{V_2(\mathbf{Y}(t))}(1-a) + b_2 \tag{5.6}$$

where $b_3 > 0$. From (5.5), (5.6) the inequality (5.4) easily follows. We proceed now to show (5.3). After some calculations we have

$$E[V(\mathbf{Y}(t+1)) - V(\mathbf{Y}(t))|\mathbf{Y}(t)]$$

$$\leq -m_1\sqrt{b - V_2(\mathbf{Y}(t))} - V_2(\mathbf{Y}(t))a + m_2\sqrt{V_2(\mathbf{Y}(t))} + m_3 \tag{5.7}$$

where $m_1 > 0$. From (5.7) we get

$$E[V(\mathbf{Y}(t+1)) - V(\mathbf{Y}(t))|\mathbf{Y}(t)] \leq -V_2(\mathbf{Y}(t))a + m_2\sqrt{V_2(\mathbf{Y}(t))} + m_3$$

hence we can select a θ such that for all b we have

$$-V_2(\mathbf{Y}(t))a + m_2\sqrt{V_2(\mathbf{Y}(t))} + m_3 \leq -\epsilon \text{ if } \sqrt{V_2(\mathbf{y})} \geq \theta \tag{5.8}$$

If $\sqrt{V_2(\mathbf{y})} \leq \theta$ then we have

$$-m_1\sqrt{b - V_2(\mathbf{Y}(t))} - V_2(\mathbf{Y}(t))a + m_2\sqrt{V_2(\mathbf{Y}(t))} + m_3$$
$$\leq m_2\theta + m_3 - m_1\sqrt{\frac{(b-\theta^2)^+}{M}}. \tag{5.9}$$

From (5.7), (5.8) and (5.9) we get (5.3) ◇

2.6 Other applications

In this section we present two applications for which the constrained queueing network is an appropriate model. Before we proceed to specific examples of constrained queueing systems, we discuss one class of activation constraints which are encountered in several practical systems; those are the constraints of the *conflicting pair* type. In that kind of constraints certain pairs of servers, the conflicting pairs, are specified; no two servers that constitute a conflicting pair can be activated simultaneously. Activation set is any set of servers that does not include any conflicting pair of servers. In this case the constraint set has a nice representation. Consider an undirected graph $G = (V, E)$ where V is the set of servers and E contains a link (i, j) if servers i and j are a conflicting pair. The constraint set contains all independent sets of nodes that is all sets such that no two nodes of the set are connected by a link. For example if the constraints are of the conflicting pair type then the solution of the optimization problem 3.8 is equivalent to the computation of the maximum weighted independent set of the graph that represents the constraints. In other cases, depending on S, the optimization problem 3.8 is reduced in different combinatorial optimization problems.

2.6.1 Databases with concurrency control

In databases where concurrent processing of several transactions is possible a control mechanism is needed to prevent conflicting transactions (transactions which may try to alter the same items of the database) of being executed simultaneously. The constrained queueing model that we are considering provides a model for concurrent processing in databases and the constraints in the simultaneous server activation captures the constraint in the simultaneous processing of conflicting transactions; furthermore the maximum throughput policy π_0 that we have specified earlier provides a concurrency control mechanism that achieves maximum throughput. The following model for databases with concurrency control has been considered in [Ke85], [MiW84], [Mi85].

The database consists of N items. The processing of a transaction requires a set of the items of the database; some of these items need to be exclusively allocated to the transaction while the rest may be used by several transactions simultaneously as long as no transaction demands them exclusively. A transaction j is completely specified by two disjoint sets of items W_j and R_j where W_j is the set of items that should be exclusively allocated (locked) during the processing of j and R_j the set of items that need not be exclusively locked by j. Two transactions j and l may be processed simultaneously if no transaction needs to lock exclusively items which are needed by the other transaction; that is the two transactions may be processed simultaneously if

$$(W_j \cap W_l) \cup (W_j \cap R_l) \cup (W_l \cap R_j) = \emptyset. \tag{6.1}$$

There are J different transaction classes. Each class is characterized by the set of items that the transactions need to lock exclusively and nonexclusivelly. Transactions of each class are generated according to Poisson point processes. A transaction may be queued for processing if it can not be processed at the time that it is generated. Assume that the processing time of a transaction is constant and the same for all classes. The processing of all transactions is synchronized to start at the same time. At the time instant that a new processing phase is initiated a decision is taken which set of nonconflicting transactions should be selected; this decision can be based on the number of transactions of each class which are in the system at that time. The above database model corresponds to a constrained queueing system with J parallel queues, J servers one for each queue and J customer (transaction) classes. Each queue i receives customers of class i only and a served customer is always routed out of the system. Activation set is any set of servers that serve nonconflicting transaction classes. Note that the constraints in this case are of the conflicting pair type. The policy π_0 selects for processing at each slot the set of transaction classes for which the sum of queue lengths is maximum. The stability region of the system is equal to the convex hall of the constraint set S.

2.6.2 Parallel Processing

The generalized multiserver queue has been proposed in [BaW90] as a model for certain parallel processing systems. The multiserver queue has N servers; the customers arrive with rate λ; each customer requests to engage a random number k of servers (processors) for its service; the arrival rate of customers that request k servers is λp_k where $\sum_{j=1}^{N} p_k = 1$. The total number of servers requested by the customers which are served simultaneously should be less than or equal to N at each time instant t.

The multiserver queue as specified above corresponds to the following constrained queueing system. There are N classes of arriving customers and N queues. Customers of class j arrive exclusively in queue j with rate λp_k and they correspond to the customers of the multiserver queue that need to engage j servers. There are N servers at each queue. After service completion a customer leaves the system. The element of an activation vector $\mathbf{i}$ that correspond to server m of queue l is denoted by i_{lm}. The necessary and sufficient condition for a binary vector with N^2 elements to be an activation vector is

$$\sum_{l=1}^{N} l(\sum_{m=1}^{N} i_{lm}) \leq N.$$

In [BaW90], under the assumptions of stationarity and ergodicity of the arrival processes and the service times a scheduling policy that stabilize the queue is obtained. This scheduling policy depends on the parameters $(p_1, .., p_N)$.

Under the assumption of Poisson arrivals and constant service times, the policy π_0 that we propose here stabilizes the system as well. The assumption about the statistics of the arrival and service processes are more restrictive in the latter case. The corresponding policy π_0 though stabilizes the system without knowledge of the parameters $(p_1, .., p_N)$. The knowledge of these parameters is necessary for the stabilization of the system by the policy proposed in [BaW90].

CHAPTER 3

Scheduling for minimum delay in tandem radio networks

3.1 Introduction

In this chapter we address the issue of queueing delay. We consider a tandem radio network with a single transceiver per node. Under two different assumptions about the traffic we obtain optimal link activation scheduling policies and necessary optimality conditions. In the proofs we use pathwise arguments that enable us to make strong statements about the optimality of certain scheduling policies. More specifically the optimality will be in the sense of stochastic order which is strictly stronger than that in the sense of of expected values. In the following we give the definition of stochastic order and a theorem that will be used later (for more details on the notion of stochastic order the reader is referred to [St83]).

Consider the discrete time processes $X = \{X(t)\}_{t=1}^{\infty}$, $Y = \{Y(t)\}_{t=1}^{\infty}$ and the space of all real valued sequences $\mathcal{R} = R^{Z_+}$. We say that the process X is stochastically smaller than the process Y, and write $X \leq_{st} Y$ if $P\{f(X) > z\} \leq P\{f(Y) > z\}$ for every $z \in R$, where $f : \mathcal{R} \to R$ is measurable and $f(x) \leq f(y)$ for every $x, y \in \mathcal{R}$ such that $x(t) \leq y(t)$ for $t \in Z_+$. The next theorem provides alternative characterizations of the stochastic ordering relationship between two processes.

Theorem 1.1 ([St83]): The following three statements are equivalent:

1) $X \leq_{st} Y$

2) $P(g(X(t_1), ..., X(t_n)) > z) \leq P(g(Y(t_1), ..., Y(t_n)) > z)$ for all $(t_1, ..., t_n)$, all z, all n, and for all $g : R^n \rightarrow R$, measurable and such that $x_j \leq y_j$, $1 \leq j \leq n$ implies $g(x_1, ..., x_n) \leq g(y_1, ..., y_n)$.

3) There exist two stochastic processes $X' = \{X'(t)\}_{t=1}^{\infty}$, $Y' = \{Y'(t)\}_{t=1}^{\infty}$ on a common probability space with the same probability laws as X and Y respectively such that $X'(t) \leq Y'(t)$ a.s. for every $t \in Z_+$.

Note that if the process of total number of packets in the system under policy π_0 is stochastically smaller than the corresponding process under some other policy π then the average number of packets in the system under π_0 is smaller than that under π (if the averages are well defined). By Little's law ([Wa88]) it is implied that the average delay under π_0 is smaller than that under π. Hence optimality in the stochastic ordering sense is stronger than average delay optimality and implies the latter.

3.2 Tandem radio networks

We consider a tandem radio network consisting of $N + 1$ nodes indexed from 0 to N. There is a radio link directed from each node i to node $i - 1$ and it is denoted by i. There is a single transceiver at each node i, therefore at most one link of those adjacent to node i may transmit at each time instant without conflicts. We assume that the transmissions of neighboring nonadjacent links do not interfere. A set of links constitutes a transmission set if and only if no two links in the set are incident at the same node. The time is considered slotted. The slot length is equal to one time unit; the slot t is the time interval $(t-1,\ t]$. The packets have constant length equal to the length of the slot. The transmissions are synchronized to start at the beginnings of the slots.

Each node i at each slot t receives $A_i(t)$ exogeneous arrivals. The vector of arrivals at all network nodes during slot t is denoted by $A(t)$. Exogeneous arriving packets, as well as packets which are forwarded to node i from neighboring nodes are queued for transmission; let $X_i(t)$ denote the length of the queue of packets at node i by the end of slot t; the corresponding queue length vector is denoted by $\mathbf{X}(t)$ and it lies in Z_+^N which is denoted by $\mathcal{X}$. The queue length process $\{\mathbf{X}(t)\}_{t=1}^{\infty}$ is denoted by $\mathbf{X}$. We are making two assumptions about the traffic.

A1. All packets have the same eventual destination which is node 0.

A2. The packets which enter the network at node i have as destination node $i-1$ from where they leave the system.

In fig. 3 we see the two queuing systems that correspond to the above assumptions about the traffic. The servers correspond to the links and the noninterference constraints require that no two servers that correspond to neighboring queues should be activated simultaneously. When the assumption A1 holds the radio network is represented by a tandem queueing system;

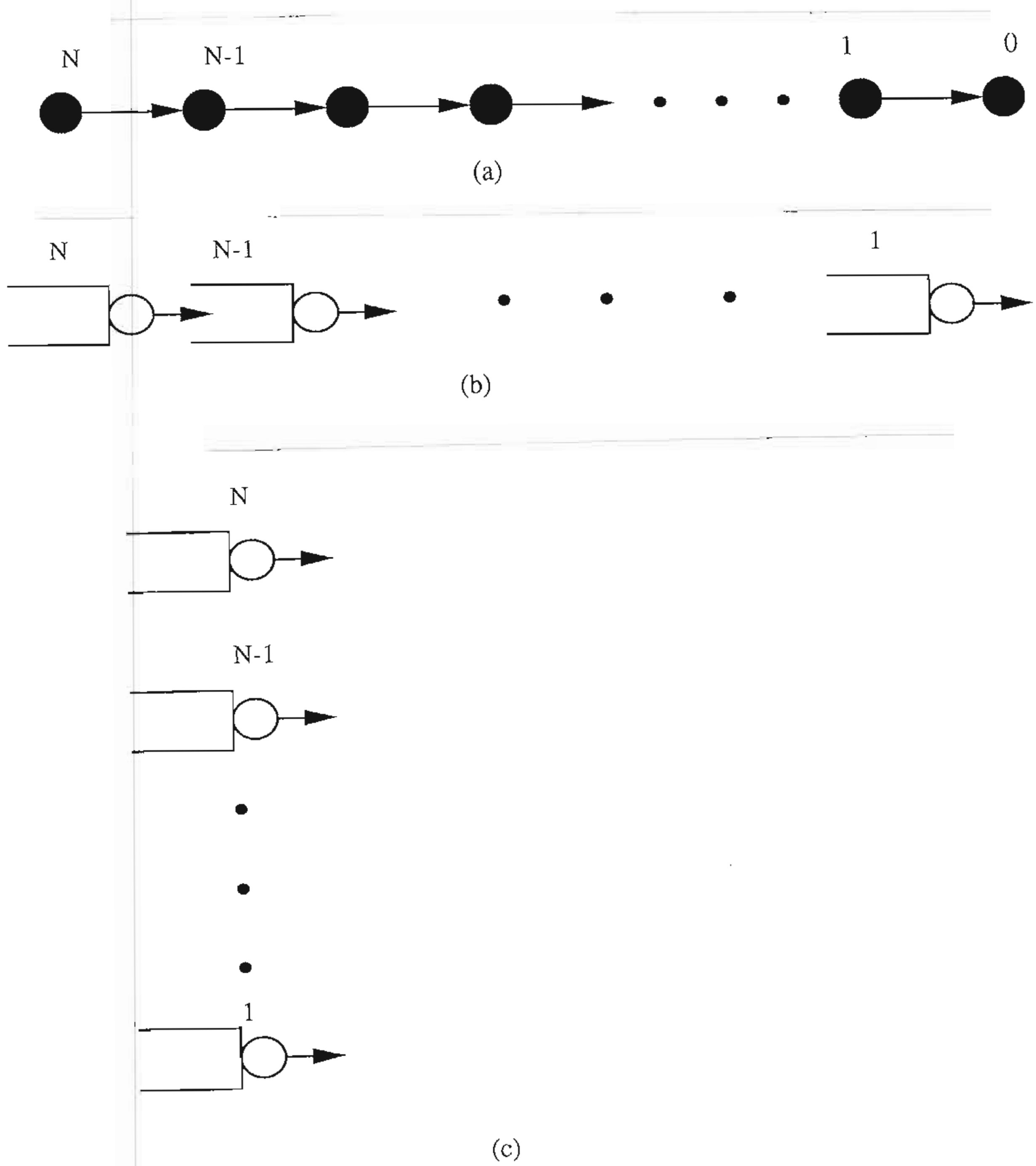

Figure 3. In (a) we see the diagram of a packet radio network. In (b) and (c) we see the queueing models of this radio network under traffic assumptions A1 and A2 respectively.

the queues that correspond to different links interact both because there is traffic forwarded from one queue to the other and because the servers that correspond to different queues are dependent. When assumption A2 holds, the system is represented by a set of parallel queues; in this case the queues interact only because their servers are dependent. Let $\mathbf{I}(t)$ be the transmission vector which is activated at slot t. We assume in the following that a link is activated only if its origin node is nonempty. Under assumption A1 the queue length vector evolves according to the equation

$$\mathbf{X}(t+1) = \mathbf{X}(t) + R\mathbf{I}(t+1) + A(t+1) \tag{2.1}$$

where R is an $N \times N$ matrix with elements

$$r_{ij} = \begin{cases} 1, & \text{if } j = i+1 \\ -1, & \text{if } i = j \\ 0, & \text{otherwise.} \end{cases}$$

Under assumption A2 the queue length vector evolves according to the equation

$$\mathbf{X}(t+1) = (\mathbf{X}(t) - \mathbf{I}(t+1))^{+} + A(t+1). \tag{2.2}$$

The decision of whether each link j will be activated or not, that is the value of the jth element of $\mathbf{I}(t)$, is taken by a central controller which selects the whole vector $\mathbf{I}(t)$ at each slot. The selection is based on the queue lengths at all network nodes. In the following two sections we study the delay optimal link activation scheduling problem under the two traffic assumptions.

3.3 Single eventual destination

In this section we focus on the case where all packets have the same eventual destination which is one of the end nodes of the tandem. Consider the stationary policy π_o which at slot t selects the transmission vector $\mathbf{I}(t) = g_0(\mathbf{X}(t-1))$ where $g_0 : \mathcal{X} \to S$ is defined next. Let $\mathbf{i} = g_0(\mathbf{x})$ and i_j, x_j be the jth elements of vectors $\mathbf{i}$ and $\mathbf{x}$ respectively; the vector $\mathbf{i}$ is defined recursively by the following equations

$$i_1 = \begin{cases} 1, & \text{if } x_1 > 0 \\ 0, & \text{if } x_1 = 0 \end{cases}$$

$$i_j = \begin{cases} 1, & \text{if } x_j > 0 \text{ and } i_{j-1} = 0 \\ 0, & \text{otherwise} \end{cases} \qquad j = 2, ..., N-1.$$

In fig. 4 we see the transmission vector which is selected by π_0 for the particular state of the network in the picture. Notice that in order to implement π_0 we just need to know whether each queue is empty or not and we do not need the exact queue length. Let G be the class of all possible activation policies. The optimality of π_0 is stated in the next theorem.

Theorem 3.1: Consider the evolution of the system under policy π_0 and an arbitrary policy $\pi \in G$. Let the arrival processes be identical under both policies π and π_0 and assume that the system starts from the same initial state under both policies. Let $\mathbf{X}(t)$,$\mathbf{X}^0(t)$ be the queue length processes under π and π_0 respectively. For all $t = 0, 1, ...$ we have

$$\sum_{i=1}^{N} X_i^0(t) \leq \sum_{i=1}^{N} X_i(t) \quad a.s. \tag{3.1}$$

The proof of the theorem follows after a few definitions and lemmas.

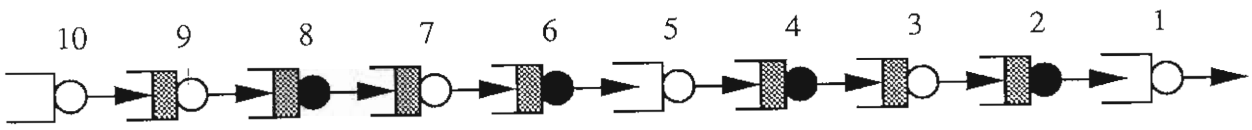

Figure 4. In this picture we see the servers (black) which are activated by policy π_0 when the state of the system is a s indicated in the figure (the shadowed queues are nonempty and the white are empty).

The total number of packets in the system when the state is $\mathbf{x}$ is denoted by $l(\mathbf{x}) = \sum_{i=1}^{N} x_i$. The following partial ordering is essential in the proof of the theorem.

Definition 3.1: Consider two vectors $\mathbf{x}, \mathbf{y} \in \mathcal{X}$. Let $\mathbf{X}(t)$, $\mathbf{Y}(t)$ be the queue length processes when the initial queue length vectors are $\mathbf{X}(0) = \mathbf{x}$, $\mathbf{Y}(0) = \mathbf{y}$ respectively, there are no exogeneous arrivals and policy π_0 schedules link transmissions. We say that the vectors $\mathbf{x}$ and $\mathbf{y}$ are related with the partial ordering $\prec$ and we write $\mathbf{x} \prec \mathbf{y}$ if for all $t = 0, 1, \ldots$ we have

$$l(\mathbf{X}(t)) \leq l(\mathbf{Y}(t)) \tag{3.2}$$

Notice that $\mathbf{x} \prec \mathbf{y}$ implies $l(\mathbf{x}) \leq l(\mathbf{y})$. We prove theorem 3.1 by showing that if at time $t = 0$ we have

$$\mathbf{X}^0(t) \prec \mathbf{X}(t) \tag{3.3}$$

and processes $\mathbf{X}(t)$, $\mathbf{X}^0(t)$ are as in theorem 3.1 then relation 3.3 holds at any time $t > 0$. The propagation of the partial ordering is shown by forward induction. We need an alternative characterization of the partial ordering in the proof of the theorem. To each state $\mathbf{x}$ we associate the *departure times* $t_i^{\mathbf{x}}$, $i = 1 \ldots, l(\mathbf{x})$ and the *positions* $d_i^{\mathbf{x}}$, $i = 1, \ldots, l(\mathbf{x})$ which are defined as follows.

Definition 3.2: Assume that the system is initially in state $\mathbf{x}$, ($\mathbf{X}(0) = \mathbf{x}$) there are no exogeneous arrivals and policy π_0 schedules link activations. Let

$\{\mathbf{X}(t)\}_{t=1}^{\infty}$ be the corresponding queue length process. The time $t_i^{\mathbf{x}}$ is defined by

$$t_i^{\mathbf{x}} = \min\{t : \ t > 0, \ l(\mathbf{X}(t)) \leq l(\mathbf{x}) - i\} \quad i = 1, .., l(\mathbf{x})$$

and the position $d_i^{\mathbf{x}}$ is defined by

$$d_i^{\mathbf{x}} = \max\{j + 1 : \ \sum_{l=1}^{j} \mathbf{X}_l(t) < i\} \quad i = 1, .., l(\mathbf{x}).$$

The departure times and positions as defined have the following interpretation. Index the packets by an index i that denotes the order in which the packets reach node 0 when the system is in state $\mathbf{x}$ at $t = 0$, π_0 schedules link activation and there are no exogeneous arrivals. The departure time $t_i^{\mathbf{x}}$ is the slot by the end of which packet i reaches node 0 and the position $d_i^{\mathbf{x}}$ the node where packet i was residing at $t = 0$. For a state $\mathbf{x}$ the departure times and the positions are related as stated in the following lemma.

Lemma 3.1: For all states $\mathbf{x} \in \mathcal{X}$ we have

$$t_i^{\mathbf{x}} = \begin{cases} d_i^{\mathbf{x}} & \text{if } i = 1, \\ i & \text{if } d_i^{\mathbf{x}} = 1, \\ \max\{t_{i-1}^{\mathbf{x}} + 2, \ d_i^{\mathbf{x}}\} & \text{if } i > 1, \ d_i^{\mathbf{x}} > 1. \end{cases} \tag{3.4}$$

Proof: Consider the system operated under policy π_0, with initial state $\mathbf{x}$ and without arrivals. The first packet is forwarded towards the destination by one node at each slot. Hence we have $t_1^{\mathbf{x}} = d_1^{\mathbf{x}}$ and (3.4) is true for $i = 1$. At each slot one packet is forwarded from node 1 to node 0 until the time that node 1 becomes empty for the first time. If $d_i^{\mathbf{x}} = 1$ then the *ith* packet will reach the destination at the end of slot i; hence if $d_i^{\mathbf{x}} = 1$ then $t_i^{\mathbf{x}} = i$ and (3.4) is true for i such that $d_i^x = 1$. If $i > 1$ and $d_i^x > 1$ then we distinguish the following cases.

A. $d_i^x - t_{i-1}^x \geq 2$.

Notice that at any slot $t < t_{i-1}^x$, packet $i-1$ should reside in a node j such that $j \leq t_{i-1}^x - t$ since it should reach the destination in $t_{i-1}^x - t$ slots and can not be forwarded faster than one hop per slot. Packet i should reside at time t in a node m such that $m \geq d_i^x - t$ since it can not move faster towards the destination than one hop per slot. Hence we have $m \geq d_i^x - t \geq t_{i-1}^x - t + 2 \geq j + 2$ which implies that packet $i-1$ will be, at each slot t, at least 2 nodes closer to the destination than packet i. Therefore packet i will be the first packet in its queue and the next node towards the destination will be empty. Because of that packet i will be forwarded by one node towards the destination at each slot (since packet $i-1$ will never prevent it from doing so) hence it will reach the destination by the end of slot d_i^x, that is $t_i^x = d_i^x$ which agrees with (3.4).

B. $d_i^x - t_{i-1}^x \leq 1$.

Notice first that if $i > 1$, $d_i^x > 1$ then $t_i^x \geq t_{i-1}^x + 2$. This is so because any packet, which is not placed initially at node 1, may reach node 1 only when this node is empty (because if it is not the transfer of any packet to that node is prevented from the activation of link 1). Hence at the slot in which packet $i-1$ leaves the system, packet i will be in node 2 or further away from the destination; hence it needs at least two additional slots in order to reach the destination. We show in the following that $t_i^x = t_{i-1}^x + 2$ which agrees with equation (3.4) in this case. If packet i is forwarded towards the destination by one node at each slot then it will reach the destination by slot $d_i^{\mathbf{x}}$; but this is impossible since $d_i^{\mathbf{x}} - t_{i-1}^{\mathbf{x}} \leq 1$ and as we just argued we should have $t_i^{\mathbf{x}} - t_{i-1}^{\mathbf{x}} \geq 2$. Hence at some slot packet i is not forwarded from its node. This may happen only because packet $i-1$ at that slot is either in the same node with i or in the node in front of i towards the destination. Because of that, at the slot at which i is not forwarded and at all subsequent slots until the time that packet $i-1$ leaves the system, packets i and $i-1$ can not be in two nodes $j,\ m$ such that $j - m > 2$. Hence two slots after the time packet $i-1$ reaches node 0, packet i reaches node 0 as well, that is $t_i^{\mathbf{x}} = t_{i-1}^{\mathbf{x}} + 2$ as we have claimed. $\diamond$

The ordering $\prec$ between two vectors $\mathbf{x},\ \mathbf{y}$ implies certain relations on the departure times associated with those two vectors. The next lemma provides an equivalent characterization of the partial ordering between $\mathbf{x}$ and $\mathbf{y}$ in terms of the departure times associated with the vectors.

Lemma 3.2: For two vectors $\mathbf{x},\ \mathbf{y} \in \mathcal{X}$ we have $\mathbf{x} \prec \mathbf{y}$ if and only if

$$t_i^{\mathbf{x}} \leq t_{i+k}^{\mathbf{y}}, \quad i = 1, .., l(\mathbf{x}) \qquad (3.4a)$$

where $k = l(\mathbf{y}) - l(\mathbf{x})$.

Proof: Let $\mathbf{X}(t),\ \mathbf{Y}(t),\ t = 0, 1, ...$ be the queue length processes when the initial queue length vectors are $\mathbf{X}(0) = \mathbf{x},\ \mathbf{Y}(0) = \mathbf{y}$ respectively, there are no exogeneous arrivals and π_0 schedules link activations. If $t_{i+k}^{\mathbf{y}} < t_i^{\mathbf{x}}$ then by the end of slot $t^{\mathbf{x}}_{i+k}$ exactly $i+k$ packets have departed from the system when the initial state is $\mathbf{y}$ while less than i packets have departed from the system when the initial state is $\mathbf{x}$. Hence we have

$$l(\mathbf{Y}(t_{i+k})) = l(\mathbf{y}) - i - k = l(\mathbf{x}) - i < l(\mathbf{X}(t_{i+k}))$$

which contradicts the fact $\mathbf{x} \prec \mathbf{y}$ and the necessity of (3.4a) follows.

Next we show the sufficiency of (3.4a). For an arbitrary slot t let j be the packet most recently departed from the system when the initial state is $\mathbf{y}$. If $j \leq k$ apparently (3.4a) is satisfied at t. If $j > k$ then, since $t^{\mathbf{x}}_{j-k} \leq t^{\mathbf{y}}_{j}$, by time t at least $j-k$ packets have departed from the system with initial state $\mathbf{x}$. Hence we have

$$l(\mathbf{X}(t)) \leq l(\mathbf{x}) - j + k = l(\mathbf{y}) - j = l(\mathbf{Y}(t))$$

and the sufficiency of (3.4a) follows. ◇

After the two preliminary lemmas relating the partial ordering we defined on $\mathcal{X}$, the departure times and the positions we proceed to the proof of theorem 3.1. The following two lemmas are essential for the induction step in the proof of the propagation of the partial ordering. Next lemma implies that the partial ordering propagates if there are no exogeneous arrivals.

Lemma 3.3: If we have $\mathbf{x} \prec \mathbf{y}$ for $\mathbf{x},\ \mathbf{y} \in \mathcal{X}$ and $\mathbf{i}$ is an arbitrary transmission vector then for the states $\mathbf{u} = \mathbf{x} - Rg_0(\mathbf{x})$ and $\mathbf{z} = \mathbf{y} - R\mathbf{i}$ we have $\mathbf{u} \prec \mathbf{z}$.

Proof: We show that for all $i = 1,..,l(\mathbf{x})$ we have $t_i^{\mathbf{u}} \leq t_{i+l(\mathbf{z})-l(\mathbf{y})}^{\mathbf{u}}$ and from lemma 3.2 we conclude $\mathbf{u} \prec \mathbf{z}$. Let $l(\mathbf{y}) - l(\mathbf{x}) = k$. We distinguish the following cases.

A. $l(\mathbf{u}) = l(\mathbf{x}),\ l(\mathbf{z}) = l(\mathbf{y})$.

In this case we need to show that for all $i = 1, ..., l(\mathbf{u})$ we have

$$t_i^{\mathbf{u}} \leq t_{i+k}^{\mathbf{z}}. \tag{3.5}$$

From the definition of the departure times we can easily see that

$$t_i^{\mathbf{u}} = t_i^{\mathbf{x}} - 1 \tag{3.6}$$

since $\mathbf{u} = \mathbf{X}(1)$ in the definition of the departure times. We show by induction on i that

$$t_i^{\mathbf{y}} \geq t_i^{\mathbf{z}} \geq t_i^{\mathbf{y}} - 1. \tag{3.7}$$

For $i = 1$ we have $t_1^{\mathbf{y}} = d_1^{\mathbf{y}} \leq d_1^{\mathbf{z}} + 1 = t_1^{\mathbf{z}} + 1$; therefore 3.7 holds for $i = 1$. Suppose that (3.7) holds for i. We show that it holds for $i+1$ as well. If $d_{i+1}^{\mathbf{z}} = 1$ then we have $t_{i+1}^{\mathbf{z}} = t_{i+1}^{\mathbf{y}} = i+1$ and (3.7) holds for $i+1$. If $d_{i+1}^{\mathbf{z}} > 1$ then we have $t_{i+1}^{\mathbf{z}} = \max\{t_i^{\mathbf{z}} + 2,\ d_{i+1}^{\mathbf{z}}\}$. If packet $i+1$ is forwarded by one node because of the activation of vector $\mathbf{i}$ then we have $d_{i+1}^{\mathbf{z}} = d_{i+1}^{\mathbf{y}} - 1$; otherwise we have $d_{i+1}^{\mathbf{z}} = d_{i+1}^{\mathbf{y}}$. We distinguish the following cases. If either $d_{i+1}^{\mathbf{z}} = d_{i+1}^{\mathbf{y}}$ or $d_{i+1}^{\mathbf{z}} = d_{i+1}^{\mathbf{y}} - 1$ and $t_i^{\mathbf{z}} \geq t_i^{\mathbf{y}} - 1$ we can easily see that

$$\max\{t_i^{\mathbf{y}} + 2,\ d_{i+1}^{\mathbf{y}}\} - 1 \leq \max\{t_i^{\mathbf{z}} + 2,\ d_{i+1}^{\mathbf{z}}\} \leq \max\{t_i^{\mathbf{y}} + 2,\ d_{i+1}^{\mathbf{y}}\}. \tag{3.7a}$$

therefore 3.7 holds for $i+1$. If $d^{\mathbf{z}}_{i+1} = d^{\mathbf{y}}_{i+1} - 1$ and $t^{\mathbf{z}}_i = t^{\mathbf{y}}_i$ then

$$\max\{t^{\mathbf{y}}_i + 2,\ d^{\mathbf{y}}_{i+1}\} \geq \max\{t^{\mathbf{z}}_i + 2,\ d^{\mathbf{z}}_{i+1}\} \geq \max\{t^{\mathbf{y}}_i + 2,\ d^{\mathbf{y}}_{i+1}\} - 1. \quad (3.7b)$$

therefore (3.7) holds for $i+1$. If $d^{\mathbf{z}}_{i+1} = d^{\mathbf{y}}_{i+1}$ and $t^{\mathbf{z}}_i = t^{\mathbf{y}}_i$ then apparently $t^{\mathbf{z}}_{i+1} = t^{\mathbf{y}}_{i+1}$ and 3.7 holds for $i+1$. Relations 3.6, 3.7 and the fact that $\mathbf{x} \prec \mathbf{y}$ imply that 3.5 holds for all $i = 1,..,l(\mathbf{u})$

B. $l(\mathbf{u}) = l(\mathbf{x}) - 1,\ l(\mathbf{z}) = l(\mathbf{y})$.

In this case we need to show that for all $i = 1,...,l(\mathbf{u})$ we have

$$t^{\mathbf{u}}_i \leq t^{\mathbf{z}}_{i+k+1}. \quad (3.8)$$

The $(i+1)th$ packet in state $\mathbf{x}$ becomes ith packet in state $\mathbf{u}$; hence we have

$$t^{\mathbf{u}}_i = t^{\mathbf{x}}_{i+1} - 1. \quad (3.9)$$

For the state $\mathbf{z}$ the situation is identical to that of case A; hence (3.7) holds. Equations (3.7), (3.9) immediatelly imply (3.8).

C. $l(\mathbf{u}) = l(\mathbf{x}) - 1,\ l(\mathbf{z}) = l(\mathbf{y}) - 1$.

In this case we need to show that for all $i = 1,...,l(\mathbf{u})$ we have

$$t^{\mathbf{u}}_i \leq t^{\mathbf{z}}_{i+k}. \quad (3.10)$$

For the departure times of state $\mathbf{u}$ (3.9) holds as it it has been argued in case B. We show by induction that

$$t^{\mathbf{y}}_{i+1} \geq t^{\mathbf{z}}_i \geq t^{\mathbf{y}}_{i+1} - 1. \quad (3.10a)$$

When $i = 1$ we have $t_1^{\mathbf{z}} = d_1^{\mathbf{z}} \leq d_2^{\mathbf{y}} \leq t_2^{\mathbf{y}}$; if $t_2^{\mathbf{y}} = d_2^{\mathbf{y}}$ then $t_1^{\mathbf{z}} = d_1^{\mathbf{z}} = d_2^{\mathbf{y}} - 1 = t_2^{\mathbf{y}} - 1$; if $t_2^{\mathbf{y}} = d_2^{\mathbf{y}} + 1$ then $t_1^{\mathbf{z}} = d_1^{\mathbf{z}} = d_2^{\mathbf{y}} = t_2^{\mathbf{y}} - 1$. Hence (3.10a) holds for $i = 1$. Assume that it holds for i; we show that it holds for $i + 1$ as well. We show first that $t_{i+2}^{\mathbf{y}} \geq t_{i+1}^{\mathbf{z}}$. If $t_{i+1}^{\mathbf{z}} = d_{i+1}^{\mathbf{z}}$ then we have $t_{i+1}^{\mathbf{z}} = d_{i+1}^{\mathbf{z}} \leq d_{i+2}^{\mathbf{z}} \leq t_{i+1}^{\mathbf{y}}$. If $t_{i+1}^{\mathbf{z}} = t_i^{\mathbf{z}} + 2$ then from the induction hypothesis we have $t_{i+1}^{\mathbf{z}} = t_i^{\mathbf{z}} + 2 \leq t_{i+1}^{\mathbf{y}} + 2 \leq t_i^{\mathbf{y}} + 2$. Now we show that $t_{i+1}^{\mathbf{z}} \geq t_{i+2}^{\mathbf{y}} - 1$. If $t_{i+2}^{\mathbf{y}} = d_{i+2}^{\mathbf{y}}$ then we have $t_{i+2}^{\mathbf{y}} = d_{i+2}^{\mathbf{y}} = d_{i+1}^{\mathbf{z}} + 1 = t_{i+1}^{\mathbf{z}} + 1$. If $t_{i+2}^{\mathbf{y}} = t_{i+1}^{\mathbf{y}} + 2$ then we have $t_{i+1}^{\mathbf{z}} \geq t_i^{\mathbf{z}} + 2 \geq t_{i+1}^{\mathbf{y}} + 2 - 1 = t_{i+2}^{\mathbf{y}} - 1$. If $t_{i+2}^{\mathbf{y}} = d_{i+2}^{\mathbf{y}}$ then we have $t_{i+2}^{\mathbf{y}} = d_{i+2}^{\mathbf{y}} = d_{i+1}^{\mathbf{z}} + 1 = t_{i+1}^{\mathbf{z}} + 1$. That completes the proof of the induction step. Relations (3.9) and (3.10a) immediatelly imply (3.10).

D. $l(\mathbf{u}) = l(\mathbf{x}),\ l(\mathbf{z}) = l(\mathbf{y}) - 1$.

In this case we need to show that for all $i = 1, ..., l(\mathbf{u})$ we have

$$t_i^{\mathbf{u}} \leq t_{i+k-1}^{\mathbf{z}}. \tag{3.11}$$

For the departure times of $\mathbf{u}$, (3.6) hold as we have shown in case A while for the departure times of $\mathbf{z}$, (3.10a) hold as we have shown in case C; hence (3.11) follows. ◇

The ordering $\prec$ between two states is preserved after a packet arrives at any network node. More specifically let $\mathbf{e}_j$ be the vector which has all its elements equal to 0 except of the element j which is equal to 1. Then we have the following.

Lemma 3.4: If we have $\mathbf{x} \prec \mathbf{y}$ for $\mathbf{x},\ \mathbf{y} \in \mathcal{X}$ then for all $j = 1, ..., N$ we also have $\mathbf{x} + \mathbf{e}_j \prec \mathbf{y} + \mathbf{e}_j$.

Proof: Let $\mathbf{u} = \mathbf{x} + \mathbf{e}_j$, $\mathbf{z} = \mathbf{y} + \mathbf{e}_j$. Since $\mathbf{x} \prec \mathbf{y}$ we have that $l(\mathbf{y}) - l(\mathbf{x}) = k \geq 0$ which implies that $l(\mathbf{z}) - l(\mathbf{u}) = k$. We show in the following that for all $i = 1, ..., l(\mathbf{x}) + 1$ we have

$$t_i^{\mathbf{u}} \leq t_{i+k}^{\mathbf{z}} \tag{3.12}$$

which in view of lemma (3.2) completes the proof. Let

$$m = \sum_{l=1}^{j} x_l + 1, \qquad n = \sum_{l=1}^{j} y_l + 1.$$

The newly arrived packet is the mth packet of state $\mathbf{u}$ and the nth packet of state $\mathbf{z}$. We consider the following cases.

A. $i < m,\ i + k < n$.

The transmissions of all packets in nodes 1 to j, which are preceeding the new packet that has arrived in node j, are not affected by the presence of that packet which joins the system in the end of queue j. Hence we have $t_i^{\mathbf{u}} = t_i^{\mathbf{x}}$, $t_{i+k}^{\mathbf{z}} = t_{i+k}^{\mathbf{y}}$ and (3.12) follows.

B. $i \geq m,\ i + k < n$.

Notice first that for all i such that $1 \leq i \leq l(\mathbf{x})$ we have

$$d_i^{\mathbf{u}} \leq d_i^{\mathbf{x}} \tag{3.13}$$

since for $i < m$ $d_i^{\mathbf{u}} = d_i^{\mathbf{x}}$, for $m = 1 \leq l(\mathbf{x})$ we have $d_m^{\mathbf{u}} \leq d_m^{\mathbf{x}}$ and for $m < i \leq l(\mathbf{x})$ we have $d_i^{\mathbf{u}} = d_{i-1}^{\mathbf{x}} \leq d_i^{\mathbf{x}}$. We show by induction in the following that for all i such that $1 \leq i \leq l(\mathbf{x})$ we have

$$t_i^{\mathbf{u}} \leq t_i^{\mathbf{x}}. \tag{3.14}$$

For $i = 1$ we have from (3.13) $t_1^{\mathbf{u}} = d_1^{\mathbf{u}} \leq d_1^{\mathbf{x}} = t_1^{\mathbf{x}}$. If (3.14) holds for some i we show that it holds for $i+1$ as well. If $t_{i+1}^{\mathbf{u}} = d_{i+1}^{\mathbf{u}}$ and $t_{i+1}^{\mathbf{x}} = d_{i+1}^{\mathbf{x}}$ then, in view of (3.13), (3.14) holds. If $t_{i+1}^{\mathbf{u}} = t_i^{\mathbf{u}} + 2$ and $t_{i+1}^{\mathbf{x}} = t_i^{\mathbf{x}} + 2$ then (3.14) holds by the induction hypothesis. If $t_{i+1}^{\mathbf{x}} = t_i^{\mathbf{x}} + 2$ and $t_{i+1}^{\mathbf{u}} = d_{i+1}^{\mathbf{u}}$ then we have $t_i^{\mathbf{x}} + 2 \geq d_{i+1}^{\mathbf{x}} \geq d_{i+1}^{\mathbf{u}} = t_{i+1}^{\mathbf{u}}$ and (3.14) holds. If $t_{i+1}^{\mathbf{x}} = d_{i+1}^{\mathbf{x}}$ and $t_{i+1}^{\mathbf{u}} = t_i^{\mathbf{u}} + 2$ then we have $d_{i+1}^{\mathbf{x}} \geq t_i^{\mathbf{x}} + 2 \geq t_i^{\mathbf{u}} + 2$ and (3.14) holds. From case A above we have $t_{i+k}^{\mathbf{y}} = t_{i+k}^{\mathbf{z}}$ whenever $i + k < n$ which together with (3.14) imply (3.12).

$C.\ i \leq m,\ i + k \geq n.$

We show (3.12) by contradiction in this case. Assume that

$$t_i^{\mathbf{u}} > t_{i+k}^{\mathbf{z}}. \tag{3.15}$$

We claim that if (3.15) holds and $m \geq i > 1$, $i + k \geq n$ then we have

$$t_{i-1}^{\mathbf{u}} > t_{i+k-1}^{\mathbf{z}}. \tag{3.16}$$

Since $i + k \geq n,\ i < m$ and because of lemma 3.1 we have

$$t_{i+k}^{\mathbf{z}} \geq d_{i+k}^{\mathbf{z}} \geq j \geq d_i^{\mathbf{u}}. \tag{3.17}$$

Hence if (3.15) holds and because of (3.17) we conclude that

$$t_i^{\mathbf{u}} = t_{i-1}^{\mathbf{u}} + 2 > d_i^{\mathbf{u}}. \tag{3.18}$$

From (3.15) and (3.18) we get

$$t_{i-1}^{\mathbf{u}} + 2 > t_{i+k}^{\mathbf{z}} \geq t_{i+k-1}^{\mathbf{z}} + 2. \tag{3.19}$$

Equation (3.19) implies (3.16).

By iteratively substituting i with $i-1$ in (3.15) at some point we we will have either $i = 1$ or $i + k < n$. In the first case from lemma 3.1 we have $t_1^{\mathbf{u}} = d_1^{\mathbf{u}}$ which contradicts (3.18). In the second case, as we argued in case A, we have $t_{i+k}^{\mathbf{z}} = t_{i+k}^{\mathbf{y}}$ which in view of (3.15) and since $t_i^{\mathbf{u}} = t_i^{\mathbf{x}}$ for $i < m$ contradicts the fact that $\mathbf{x} \prec \mathbf{y}$.

D. $i > m,\ i + k \geq n$.

If $i > m$ then the ith packet of state $\mathbf{u}$, is the same with the $i-1$ packet of state $\mathbf{x}$ hence we have

$$d_i^{\mathbf{u}} = d_{i-1}^{\mathbf{x}}. \tag{3.20}$$

Similarly if $i + k > n$ we have

$$d_{i+k}^{\mathbf{z}} = d_{i+k-1}^{\mathbf{y}}. \tag{3.21}$$

If $i + k = n < l(\mathbf{z})$ then we have $d_n^{\mathbf{z}} \leq d_n^{\mathbf{y}}$ which implies

$$t_n^{\mathbf{z}} = \max\{t_{n-1}^{\mathbf{y}} + 2,\ d_n^{\mathbf{z}}\} \leq \max\{t_{n-1}^{\mathbf{y}} + 2,\ d_n^{\mathbf{y}}\} = t_n^{\mathbf{y}}. \tag{3.22}$$

We can easily show by induction that

$$t_{i+k}^{\mathbf{z}} \geq t_{i+k-1}^{\mathbf{y}} \text{ if } \ i + k \geq n. \tag{3.23}$$

For $i + k = n$, since $t_{i+k}^{\mathbf{z}} \geq t_{i+k-1}^{\mathbf{z}} = t_{i+k-1}^{\mathbf{y}}$, 3.23 holds. Assume that it holds for $i + k = l > n$. Then from (3.21) and the induction hypothesis we obtain

$$t_{l+1}^{\mathbf{z}} = \max\{t_l^{\mathbf{z}} + 2,\ d_{l+1}^{\mathbf{z}}\} \geq \max\{t_{l-1}^{\mathbf{y}} + 2,\ d_l^{\mathbf{y}}\} = t_l^{\mathbf{y}}. \tag{3.24}$$

We show (3.12) by contradiction. Assume that

$$t_i^{\mathbf{u}} > t_{i+k}^{\mathbf{z}}. \tag{3.25}$$

When (3.25) holds, we can not have $t_i^{\mathbf{u}} = d_i^{\mathbf{u}}$ since in that case and because of (3.20), (3.23)

$$t_{i-1}^{\mathbf{x}} \geq d_{i-1}^{\mathbf{x}} = d_i^{\mathbf{u}} = t_i^{\mathbf{u}} > t_{i+k}^{\mathbf{z}} \geq t_{i+k-1}^{\mathbf{y}}$$

which contradicts $\mathbf{x} \prec \mathbf{y}$. Hence we have

$$t_i^{\mathbf{u}} = t_{i-1}^{\mathbf{u}} + 2. \tag{3.26}$$

Notice that if $d_{i+k}^{\mathbf{z}} = 1$ we should have $i + k = n$ which imply that $i = m$; therefore we have $d_{i+k}^{\mathbf{z}} > 1$ which imply $t_{i+k}^{\mathbf{z}} \geq t_{i+k-1}^{\mathbf{z}} + 2$ and from (3.25), (3.26) we obtain $t_{i-1}^{\mathbf{u}} > t_{i+k-1}^{\mathbf{z}}$. By applying the same argument several times (like in case C) we reach a point where

$$t_i^{\mathbf{u}} > t_{i+k}^{\mathbf{z}} \tag{3.26a}$$

and either $i = m$ or $i + k < n$. If $i = m$ then (3.26a) contradicts either case B or C depending on whether $i + k < n$ or $i + k \geq n$. If $i + k < n$ then (3.26a) contradicts case B. Hence (3.25) can not hold and (3.12) should hold in this case also. ◇

We proceed to the proof of the theorem.

Proof of theorem 3.1: We show that

$$\mathbf{X}^0(t) \prec \mathbf{X}(t) \qquad t = 0, 1, 2, ... \tag{3.27}$$

Hence (3.1) is implied. We use induction to show (3.27). For $t = 0$ (3.27) holds trivially since $\mathbf{X}^0(0) = \mathbf{X}(0)$. Assume that (3.27) holds for some t; we will show that it holds for $t+1$ as well. Let $\mathbf{I}(t+1)$ be the activation vector under π at $t+1$. Then from lemma (3.3) we have

$$(\mathbf{X}^0(t) + Rg_0(\mathbf{X}^0(t))) \prec \mathbf{X}(t) + R\mathbf{I}(t+1) \qquad (3.28).$$

The arrival vector $\mathbf{A}(t+1)$ can be written as

$$\mathbf{A}(t+1) = \sum_{i=1}^{N} A_i(t+1)\mathbf{e}_i.$$

Hence from lemma (3.4) and equation (3.28) we can easily see that

$$\mathbf{X}^0(t+1) = \mathbf{X}^0(t) + Rg_0(\mathbf{X}^0(t)) + \sum_{i=1}^{N} A_i(t+1)\mathbf{e}_i$$

$$\prec \mathbf{X}(t) + R\mathbf{I}(t+1) + \sum_{i=1}^{N} A_i(t+1)\mathbf{e}_i = \mathbf{X}(t+1).$$

◇

Remark

Notice that we do not pose any restriction on the policies of class G. That is a policy in G may select the transmission vector $\mathbf{I}(t)$ based on the knowledge of the whole arrival sample path. Hence G contains even nonanticipative policies which may use for decision making information about the future evolution of the system.

3.4 Single hop transmission requirements

Let $S(\mathbf{x})$ be the set of all transmission vectors which are such that if the servers are activated according to any of those transmission vectors and the system state is $\mathbf{x}$ then the maximum number of nonempty queues is served. Our main result in this section is that we do not loose anything with respect to delay optimality if at each slot t we consider for transmission the vectors in $S(\mathbf{X}(t-1))$ only. More specifically we show that for every policy π there exists a policy π' which achieves smaller delay than π and is such that the transmission vector $\mathbf{I}'(t)$ selected by π' at t belongs to $S(\mathbf{X}(t-1))$. We give first an explicit characterization of the set $S(\mathbf{x})$. Let $k = k(\mathbf{x})$ be such that $\frac{k}{2}$ is the number of groups of consecutive nonempty queues and $j_1 = j_1(\mathbf{x}), ..., j_k = j_k(\mathbf{x})$ are the nonempty queues which are neighboring with one empty and one nonempty queue or they are in the end of the tandem. The numbers $j_1, .., j_N$ are called the boundary indices of $\mathbf{x}$ and they are uniquely defined by the following conditions

1. All queues j such that $j > j_k$ or $j < j_1$ are empty.
2. All queues j such that $j_{2m-1} \leq j \leq j_{2m}$ $m = 1, ..., \frac{k}{2}$ are nonempty.
3. All queues j such that $j_{2m} \leq j \leq j_{2m+1}$ $m = 1, ..., \frac{k}{2} - 1$ are empty.

In fig. 5 the boundary indices are illustrated. The following lemma provides necessary and sufficient conditions for an transmission vector to belong to $S(\mathbf{x})$.

Lemma 4.1: An transmission vector $\mathbf{i}$ belongs to $S(\mathbf{x})$ if it satisfies the following conditions.

1. If $j_{2m} - j_{2m-1}$ is an even number then for all links j, $j_{2m-1} \leq j \leq j_{2m}$ we

have

$$i_j = \begin{cases} 1 & \text{if } j - j_{2m-1} \text{ is even,} \\ 0 & \text{if } j - j_{2m-1} \text{ is odd.} \end{cases}$$

for all $m = 1, ..., \frac{k}{2}$.

2. If $j_{2m} - j_{2m-1}$ is an odd number then $\mathbf{i}$ should satisfy one of the following conditions.

2a.

$$i_j = \begin{cases} 1 & \text{if } j - j_{2m-1} \text{ is even } j_{2m-1} \leq j \leq j_{2m}, \\ 0 & \text{otherwise,} \end{cases}$$

2b.

$$i_j = \begin{cases} 1 & \text{if } j - j_{2m-1} \text{ is odd } j_{2m-1} \leq j \leq j_{2m}, \\ 0 & \text{otherwise,} \end{cases}$$

2c. There exists an l such that

$$i_j = \begin{cases} 1 & \text{if } j - j_{2m-1} \text{ is even and } j_{2m-1} \leq j < l \\ & \text{or } j_{2m} - j \text{ is even and } j_{2m} \geq j > l + 1, \\ 0 & \text{otherwise} \end{cases}$$

for all $m = 1, ..., \frac{k}{2}$.

3. If $j_1 = 1$ then for all links j, $j_{2m-1} \leq j \leq j_{2m}$ we have

$$i_j = \begin{cases} 1 & \text{if } j_2 - j \text{ is even } j_1 \leq j \leq j_2, \\ 0 & \text{otherwise.} \end{cases}$$

for all $m = 1, ..., \frac{k}{2}$. Similarly for the case where $j_2 = N$

Proof: When an transmission vector satisfies the conditions 1-3 above for every group of consecutive nonempty queues the maximum number of queues are served. If $j_{2m} - j_{2m-1}$ is an even number then all vectors in $S(\mathbf{x})$ serve $\frac{(j_{2m} - j_{2m-1})}{2} + 1$ queues of those with indices j such that $j_{2m-1} \leq j \leq j_{2m}$ when they are activated. No other vector can activate more queues of this group of consecutive nonempty queues since the neighboring queues of each one which is activated, should not be activated. If $j_{2m} - j_{2m-1}$ is an odd number then all vectors in $S(\mathbf{x})$ activate $\frac{(j_{2m}-j_{2m-1}+1)}{2}$ queues of those with indices j such that $j_{2m-1} \le j \le j_{2m}$. No other vector can activate more queues of this group of consecutive nonempty queues for the same reason as above. $\diamond$

The set $S(\mathbf{x})$ for some state $\mathbf{x}$ is illustrated in fig. 5. Consider the class of policies $\tilde{G}$ that contains any policy π such that the transmission vector $\mathbf{I}(t)$ selected by π at t belongs to $S(\mathbf{X}(t-1))$. For each policy π in G there exists a policy $\tilde{\pi}$ in $\tilde{G}$ which performs better than π. We define next the policy $\tilde{\pi}$ that corresponds to π and it has the above property. Consider the mapping $J : S \times \mathcal{X} \to S(\mathbf{x})$ defined next. Let $\mathbf{i}' = J(\mathbf{i}, \mathbf{x})$; consider the boundary indices $j_1, ..., j_k$ for the state $\mathbf{x}$. Since $\mathbf{i}'$ belongs to $S(\mathbf{x})$ its elements i'_j are uniquely specified for all $j's$ other than those such that for some m, $j_{2m} \ge j \ge j_{2m-1}$ where $j_{2m} - j_{2m-1}$ is an odd number. For those $j's$, i'_j is defined as follows:

1. If $i_{j_{2m-1}} = 0$ then

$$i'_j = \begin{cases} 1 & \text{if } j - j_{2m-1} \text{ is an odd number } j_{2m-1} \le j \le j_{2m}, \\ 0 & \text{if } j - j_{2m-1} \text{ is an odd number.} \end{cases}$$

2. If $i_{j_{2m-1}} \ne 0$ and $i_{j_{2m}} = 0$ then

$$i'_j = \begin{cases} 1 & \text{if } j - j_{2m-1} \text{ is an even number,} \\ 0 & \text{if } j - j_{2m-1} \text{ is an odd number.} \end{cases}$$

3. If $i_{j_{2m}} \ne 0$ $i_{j_{2m-1}} \ne 0$ then let l be the smallest number greater than j_{2m-1}

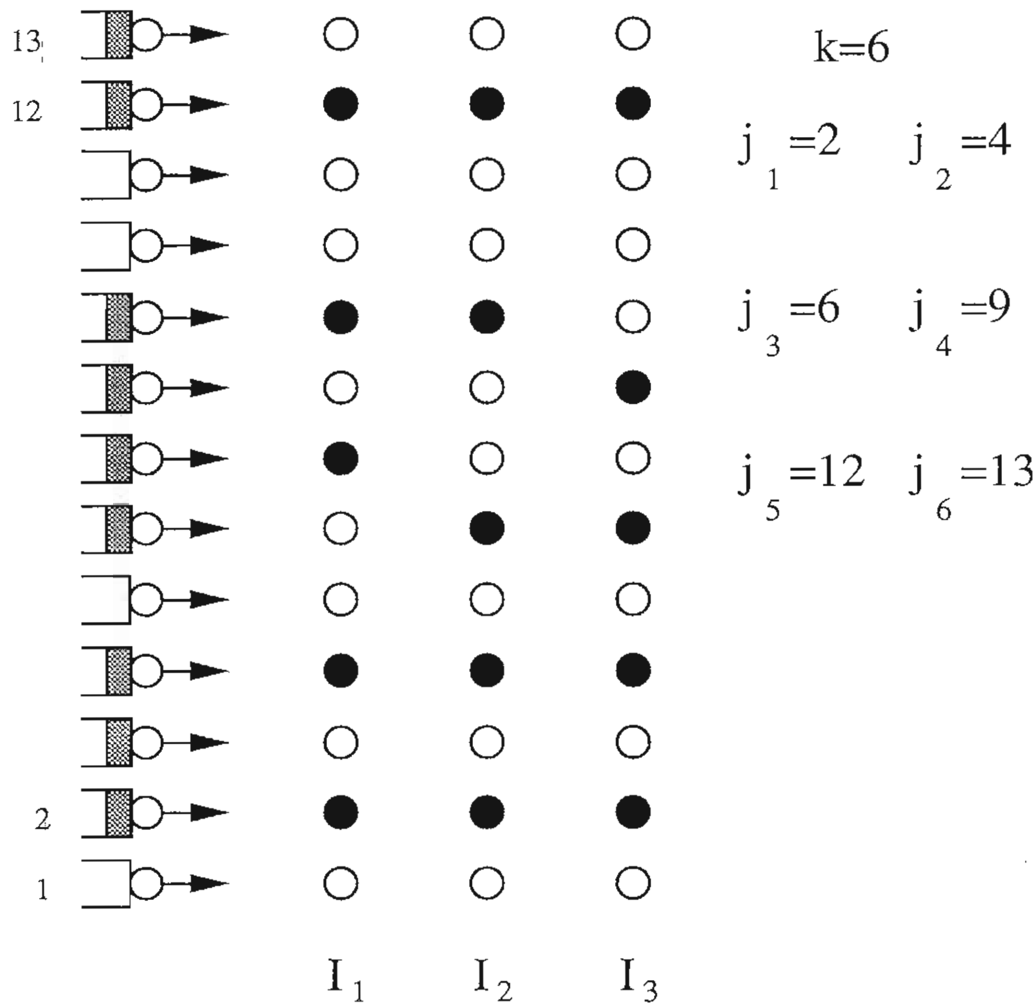

Figure 5. In this picture we see the queueing system that corresponds to a radio network with 13 nodes and traffic assumption A.2. The shadowed queues are nonempty while the others are empty. the boundary indices and the activation vectors of the set $S(\mathbf{x})$ for this particular state are indicated. The activation vectors are represented by columns of circles where the black circles correspond to activated servers.

such that $i_l = i_{l+1} = 0$. We have

$$i'_j = \begin{cases} 1 & \text{if } j - j_{2m-1} \text{ is an even number and } j_{2m-1} \leq j < l \\ & \text{or } j_{2m} - j \text{ is an even number and } j_{2m} \geq j > l+1, \\ 0 & \text{otherwise} \end{cases}$$

It easy to check that such a number l as defined above in 3 does exists and also that the $\mathbf{i}'$ belongs to $S(\mathbf{x})$. Policy $\tilde{\pi}$ at slot t selects the vector $\tilde{I}(t) = J(\mathbf{I}(t),\ \mathbf{X}(t-1))$. In the following we denote the policy in $\tilde{G}$ that corresponds to a policy $\pi \in G$ by putting a tilde over the symbol of the policy. Policy $\tilde{\pi}$ as defined above is better than π in the stochastic ordering sense.

Theorem 4.1: For each policy $\pi \in G$ the corresponding policy $\tilde{\pi} \in \tilde{G}$ is such that if the system starts from the same initial state and the arrivals have the same statistics under both policies π, $\tilde{\pi}$, then for the corresponding processes of total numbers of packets in the system $Q(t) = \sum_{i=1}^{N} X_i(t)$, $\tilde{Q}(t) = \sum_{i=1}^{N} \tilde{X}_i(t)$ we have

$$\tilde{Q} \leq_{st} Q \tag{4.1}$$

The proof of the theorem follows after the next lemma.

Lemma 4.1: Consider a policy $\pi \in G$ and its corresponding policy $\tilde{\pi} \in \tilde{G}$. There exists another policy $\pi' \in G$ which acts similarly to $\tilde{\pi}$ at $t = 1$, it is appropriately defined at $t > 1$ and satisfies the following. If the system starts from the same state $\mathbf{x}$ at $t = 0$ under both policies π, π', then for the corresponding queue length processes $\mathbf{X}(t)$, $\mathbf{X}'(t)$ and for all $t = 0, 1, \ldots$ we have

$$\sum_{i=1}^{N} X'_i(t) \leq \sum_{i=1}^{N} X_i(t), \quad a.s. \tag{4.2}$$

Proof: We construct π' and we show that (4.2) is satisfied. We show first that at $t = 1$ the queue lengths satisfy the following:

a. $X'_l(t) \leq X_l(t) + 1, \quad l = 1, ..., N$.

b. If $X'_l(t) = X_l(t) + 1$ and $l < N$ then $X'_{l+1}(t) = X_{l+1}(t) - 1$.

c. If $X'_l(t) = X_l(t) + 1$ and $l > 1$ then $X'_{l-1}(t) = X_{l-1}(t) - 1$.

d. If $j_1 = 1$, $j_2 = N$ and N is odd then $X'_1(t) \leq X_1(t)$ and $X'_N(t) \leq X_N(t)$.

Condition a is obvious. For the conditions b, c we argue as follows. For a queue l we have $X'_l(1) = X_l(1) + 1$ if and only if the queue is served by $\mathbf{I}(1)$ while it is not served by $\mathbf{I}'(1)$; where $\mathbf{I}(1)$, $\mathbf{I}'(1)$ are the transmission vectors selected by π, π' respectively. If $j_{2m-1} \leq l \leq j_{2m}$ and $j_{2m} - j_{2m-1}$ is even then, by definition of $\tilde{\pi}$, the links $l + 1$ and $l - 1$ are activated by $\mathbf{I}'(t)$ (if $l < N$ and $l > 1$ respectively) while the same links are not activated by $\mathbf{I}(t)$, since link l were activated by the latter transmission vector. Therefore relations b and c follow. If $j_{2m-1} \leq l \leq j_{2m}$ and $j_{2m} - j_{2m-1}$ is odd then, by definition of $\tilde{\pi}$, the links $l + 1$ and $l - 1$ are activated by $\mathbf{I}'(t)$ (if $l < N$ and $l > 1$ respectively) while the same links are not activated by $\mathbf{I}(t)$, since link l were activated by the latter transmission vector. Therefore relations b and c follow. If $j_1 = 1$, $j_2 = N$ and N is odd number then by definition of $\tilde{\pi}$ the links 1 and N are activated therefore d follows. It is easy to see that if conditions a-d hold for some t then (4.2) follows.

For $t > 1$ the transmission vector $\mathbf{I}'(t)$ is defined based on $\mathbf{I}(t)$ and $\mathbf{X}(t-1)$. Let $\mathbf{I}(t)$ be the transmission vector selected by π at slot t. At the same slot policy π' selects $\mathbf{I}'(t)$ such that all queues l for which

$$X_l'(t-1) = X_l(t-1) + 1 \tag{4.3}$$

are served; furthermore all queues, which are served by $\mathbf{I}(t)$ and are not conflicting with any queue l for which (4.3) is satisfied, are served as well.

We show in the following that if conditions a-d are satisfied at t then they are satisfied at $t+1$ as well. Then (4.2) follows for all t by induction. Apparently condition a is satisfied at $t+1$ since, by definition of π', any queue l for which at t we have $X_l'(t) = X_l(t) + 1$ is served. For the conditions b and c we argue as follows. Assume that at $t+1$ we have $X_l'(t+1) = X_l(t+1) + 1$, ($l < N$, $l > 1$). Apparently at time t we can not have $X_l'(t) < X_l(t)$. Notice that we can not have $X_l'(t) = X_l(t)$ since in that case queue l can not be adjacent to any queue m for which $X_m'(t) = X_m(t) + 1$; therefore if l is activated by π it is activated by π' as well. Hence we should have $X_l'(t) = X_l(t) + 1$. In this case $X_{l-1}'(t) = X_{l-1}(t) - 1$ ($X_{l+1}'(t) = X_{l+1}(t) - 1$) and since queue $l-1$ ($l+1$) is not served by neither π nor π' at $t+1$ we also have $X_{l-1}'(t+1) = X_{l-1}(t+1) - 1$ ($X_{l+1}'(t+1) = X_{l+1}(t+1) - 1$). For condition d we have the following. If $X_1'(t) \leq X_1(t) - 1$, ($X_N'(t) \leq X_N(t) - 1$) then d holds for $t+1$. If $X_1'(t) = X_1(t)$, ($X_N'(t) = X_N(t)$) then queue 1 (N) is activated by π if if and only if it is activated by π' therefore condition d follows. ◇

Now we can prove the theorem.

Proof of theorem 4.1: We will show that the policy $\tilde{\pi} = J(\pi)$ has the property claimed in the theorem. Consider a sequence of policies π_1, π_2, ... defined as follows. Policy π_1 is the same as policy π' constructed in lemma 4.1, when policies π in the lemma and the theorem 4.1 are the same. Policies π_τ, $\tau > 1$ are defined inductively as follows. Consider the construction of policy π' in lemma 4.1 in terms of π. Let π be such that at time t it takes the same action as policy $\pi_{\tau-1}$ at time $\tau - 1 + t$. Let π_τ at times $t = 1, ..., \tau$ take the same actions as $\tilde{\pi}$ while at times $t > \tau$ takes the same actions as the policy π' at times $t - \tau$. Where π' is constructed as in lemma 4.1 when π is as above. We denote by $\mathbf{X}^\tau$ the queue length processes under π_τ for $\tau = 1, ...$ By definition of the policies, for all τ we have

$$\tilde{\mathbf{X}}(t) = \mathbf{X}^\tau(t), \quad t = 1, ..\tau \tag{4.15}$$

From lemma 4.1 and from the construction of policy π_τ, for all τ we have

$$l(\mathbf{X}(t)) \geq l(\mathbf{X}^1(t)) \geq ... \geq l(\mathbf{X}^\tau(t) \geq ... \tag{4.16}$$

Consider the time slots $t_1, t_2, ...t_n$ and a function g as in part 2 of theorem 4.1. Consider also the policy π_{t_n} defined as above. By construction the variables $l(\mathbf{X}^{t_1}(t_1)), ..., l(\mathbf{X}^{t_n}(t_n))$ have the same joint probability distribution with the variables $l(\tilde{\mathbf{X}}(t_1)), ..., l(\tilde{\mathbf{X}}(t_n))$. Hence for all z we have

$$P(g(l(\mathbf{X}^{t_n}(t_1)), ..., l(\mathbf{X}^{t_n}(t_n))) > z) = P(g(l(\tilde{\mathbf{X}}(t_1)), ..., l(\tilde{\mathbf{X}}(t_n))) > z). \tag{4.17}$$

Since $l(X^{t_n}(t)) \leq l(\mathbf{X}(t))$ *a.s.* for all $t = 0, 1, ..., t_n$, we have

$$P(g(l(\mathbf{X}^{t_n}(t_1)), ..., l(\mathbf{X}^{t_n}(t_n))) > z) \leq P(g(l(\mathbf{X}(t_1)), ..., l(\mathbf{X}(t_n))) > z). \tag{4.18}$$

Equations (4.17) and (4.18) and part 2 of theorem 4.1 complete the proof. $\diamond$

Remark

For each queue j, $j_{2m-1} \leq j \leq j_{2m}$ where $j_{2m} - j_{2m-1}$ is an even number or $j_{2m-1} = 1$ or $j_{2m} = N$, the corresponding elements of all transmission vectors in $S(\mathbf{x})$ are identical. Therefore the necessary optimality condition specifies uniquely the transmission vector at slot t up to the elements that correspond to groups of consecutive nonempty queues with even number of queues. If $N = 3$ then $\tilde{G}$ contains exactly one policy; that policy minimizes in the stochastic ordering sense the process of total number of packets in the system for any arrival process.

CHAPTER 4

Time varying connectivity

4.1 Introduction

In all previous chapters we have considered systems where the topology is fixed. This is not always the case since time varying connectivity is inherent in several types of communication networks. In this chapter we depart from the fixed topology assumption. We focus on a single hop network the connectivity of which varies randomly with time. Its queueing model consists of a single server and N parallel queues (fig. 6). The time is slotted. At slot t each queue i may be either *connected* to the server or not; that is denoted by the binary variable $C_i(t)$ which is equal to 1 and 0 respectively. It is called the connectivity variable of queue i. The connectivity varies randomly with time. There are exogeneous arrivals at each queue. At each slot t the server is either allocated to one of the queues or idles; the *control variable* $U(t)$ indicates the queue served during slot t or is equal to e if the server idles. If the queue i at which the server is allocated is disconnected then no service is provided. If it is connected then service is provided and the served packet completes its service requirements and leaves the system with some probability; if the packet does not complete service it remains in the queue.

The server allocation is controlled. The lengths of the connected queues are available to the controller for decision making. The allocation decision at slot t may be based on the history of the observations and the past allocation decisions. When the buffers have unlimited capacity, depending on the allocation policy and the statistics of the arrivals, services and connectivities, two things may happen. The system either reaches a steady state behavior or the queue lengths start growing without bound. In the former case the system is stable while in the latter unstable. We obtain necessary and sufficient conditions on the arrivals, service and connectivity statistics for the existence of an allocation policy under which the system is stable. We also give a policy under which the system is stable if there exists some policy that stabilizes it. The performance of the system with respect to queueing delay is studied then. In a symmetric system, the allocation policy that minimizes the delay is obtained. The problem of optimal server allocation in a changing connectivity system with a single buffer per node is studied last. In that case, if an arriving packet at some node i finds the buffer full then is blocked from admission into the system. A policy that maximizes the throughput and at the same time minimizes the delay is obtained.

Radio networks with meteor-burst communication channels and cellular networks with mobile users and small cell sizes are two among the several examples of systems with time varying connectivity mentioned above. In the first case there is a central station (the server) and N users (the queues) each one of which is connected to the station through a meteor burst communication channel. These channels have the property that can not be used continuously, but only during time intervals of random duration which occur at random time instants (whenever there exists a meteor burst) ([CNR89], [Ya90]). At each time slot a user may communicate with the central station if its channel is active; hence a subset of the users (those with active channels) are competing for the attention of the station at each slot.

As the cell size in cellular network decreases, (that is the tendency in the future cellular networks in order to maximize the spatial spectrum reutilization ([Go90], [StP85])), the variability in the distance between a mobile user and the station of the cell results in variation of their radio connectivity. At each time slot only the users which are within a certain distance from the cell station may communicate with it. The model of a single server with parallel queues of time varying connectivity arises in this case as well.

One special case of the model studied in this chapter is when the connectivities are fixed and equal to one at all slots. In this case all queues are connected to the server at all times and the model is reduced to that of allocating a server to a set of parallel queues. That is a well known problem of optimal queueing control ([Wa88]) and has been studied extensively in the past ([BMM85], [BVW85]). The time varying connectivity makes the server allocation problem considerably more difficult than the case where all queues are available for service all the time. That is made clear as we present the results that we obtain for the system with time varying connectivity in contrast with what is known for systems with fixed connectivity.

This chapter is organized as follows. In section 4.2 we specify the model. In section 4.3 the stability properties of the system are investigated. The issue of queueing delay is studied in section 4.4. In section 4.5 we study throughput and delay performance in a system with a single buffer per node.

4.2 System dynamics

During slot t there are $A_i(t)$ exogeneous arrivals at queue i. When queue i is connected and the server is allocated at that queue, the service is completed with some probability. That is represented at slot t by the binary random variable $M_i(t)$ which is equal to 1 if the service is completed and to 0 otherwise. The stability and delay optimality results are obtained under different assumptions on the statistics of the arrival, service and connectivity processes. Those assumption are stated as needed later. Let $X_i(t)$ be the number of packets in the ith queue by the end of slot t (or the beginning of slot $t+1$). Until section 4.5 we study the system under the assumption of unlimited buffer capacity. Under this assumption the number of packets at queue i evolves with time according to the equation

$$X_i(t) = (X_i(t-1) - 1\{U(t) = i\}C_i(t)M_i(t))^+ + A_i(t), \quad t = 1, .. \qquad (2.1)$$

We assume that the controller which allocates the server is informed at the beginning of each slot about the connectivity at that slot as well as about

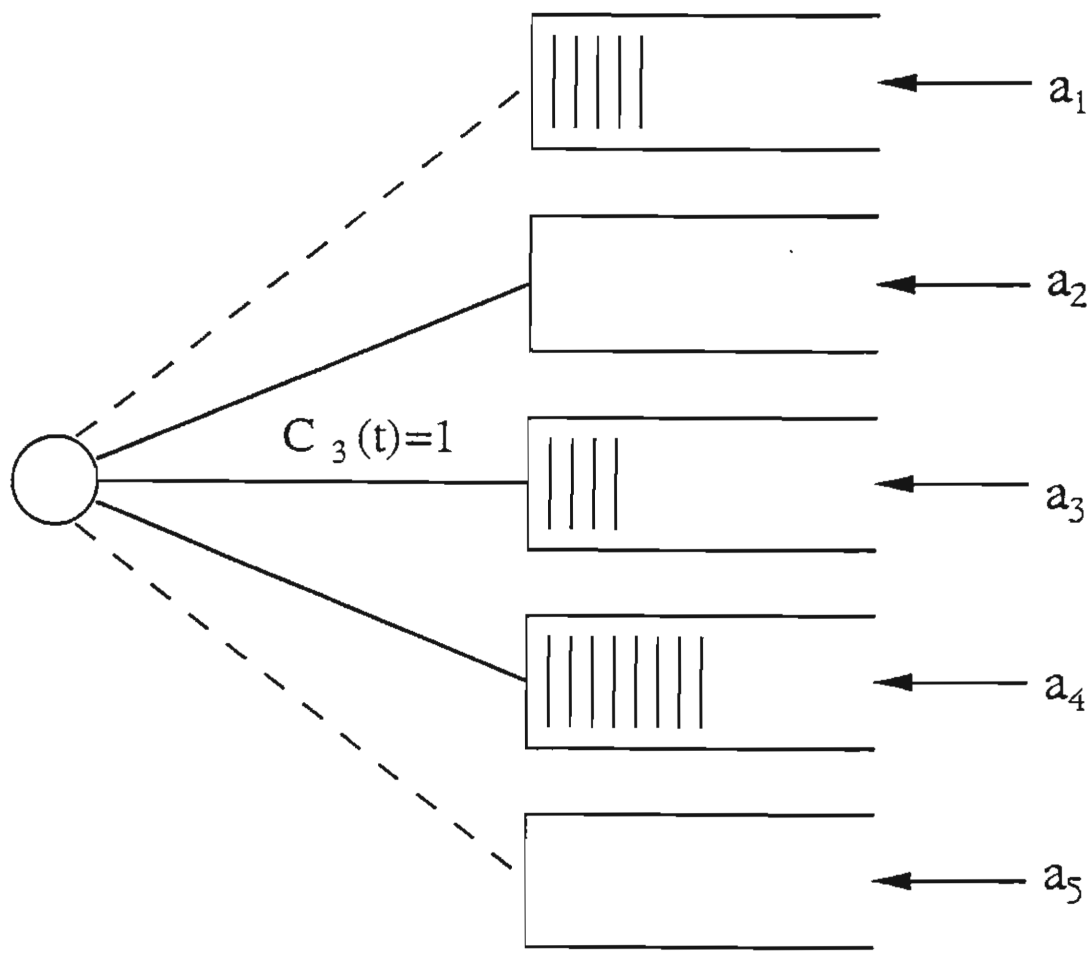

Figure 6. A single-hop network with time varying connectivity. The solid line between a queue and the server denotes that the queue is connected to the server (it may receive service). The dashed line denotes that the queue is disconnected

the lengths of the queues which are connected. This information is represented by $\mathbf{Y}(t) = (\mathbf{X}(t-1) \bigotimes \mathbf{C}(t),\ \mathbf{C}(t))$ where $\mathbf{X}(t) = (X_i(t) : i = 1,..,N)$ is the vector of queue lengths at slot t, $\mathbf{C}(t) = (C_i(t) : i = 1,..,N)$ is the vector of the connectivities at slot t and $\bigotimes$ denotes the pointwise product* between vectors. The server is allocated based on the available information $\mathbf{Y}(t)$. We study the stability properties and the delay performance of the system under policies that base their decisions on the available control information.

Remark

A single hop radio network with a central station and several radio nodes which need to communicate with the station, corresponds to the above model as follows. The server corresponds to the central station and the queues to the radio nodes. The packets have constant length equal to one slot; each time a packet is transmitted it is received successfully with some probability. Unsuccessful transmissions are due to channel errors and not to collisions since the transmissions are scheduled. The variable $M_i(t)$ in this case indicates whether a transmission of node i at time t was successful or not (if node i was transmitting at slot t). If a transmission is unsuccessful then the packet remains at node i.

4.3 System stabilizability

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* If $\mathbf{a} = (a_i : i = 1,..,N)$, $\mathbf{b} = (b_i : i = 1,..,N)$ and $\mathbf{c} = \mathbf{a} \bigotimes \mathbf{b}$ then $\mathbf{c} = (a_i b_i : i = 1,..,N)$

As in chapter 2 we consider the system to be stable if in the long run approaches a stationary behavior, that is the backlog in the nodes does not grow to infinity. We study system stability under some independence assumptions on the arrival, service and connectivity processes. More specifically we assume that the processes $\{A_i(t)\}_{t=1}^{\infty}$, $\{C_i(t)\}_{t=1}^{\infty}$, $\{M_i(t)\}_{t=1}^{\infty}$, $i = 1, ..., N$ are i.i.d. and independent; furthermore we assume that $E[A_i^2(t)], E[M_i^2(t)], E[C_i^2(t)], < \infty$. Consider the class of stationary policies G that allocate the server at slot t based on the available information $\mathbf{Y}(t)$. A policy in G is specified by a function $g : \mathcal{Y}^1 \rightarrow \{1, .., N, e\}$ where $\mathcal{Y}^1$ is the space at which $\mathbf{Y}(t)$ lies. The allocation decision at slot t is $U(t) = g(\mathbf{Y}(t))$. Under any policy in G and because of the independence assumptions on the arrivals, services and connectivities, the queue length process $\mathbf{X} = \{\mathbf{X}(t)\}_{t=1}^{\infty}$ is a time homogeneous Markov chain with state space $\mathcal{X} = Z_+^N$. The definition of stability that we consider in this chapter is somehow more restrictive than that in chapter 2.

Definition 3.1: The system is defined to be stable under some allocation policy in G if the Markov chain $\mathbf{X}$ is irreducible and the probability distribution of $\mathbf{X}(t)$ converges in the sense that

$$\lim_{t\rightarrow\infty} P[\mathbf{X}(t) \leq \mathbf{b}] = F(\mathbf{b}) \quad \forall \mathbf{b} \in \mathcal{X} \tag{3.1}$$

where $F(\cdot)$ is a probability distribution on $\mathcal{X}$. Similarly with chapter 2, the system is called *stabilizable* if there exists an allocation policy in G under which it is stable.

The necessary and sufficient stabilizability conditions involve the expectations of $A_i(t)$, $C_i(t)$, $M_i(t)$ which are denoted by $a_i = E[A_i(t)]$, $p_i = E[C_i(t)]$ and $m_i = E[M_i(t)]$. Consider the class G_0 of stationary policies that base the allocation decision at slot t on the lengths of all queues $\mathbf{X}(t-1)$ and not only of the connected ones. Under any such policy $\mathbf{X}(t)$ is a Markov chain. Next lemma provides a condition that is necessary for stabilizability of the system even if the queue lengths of the disconnected queues are observable at each slot.

Lemma 3.1: If there exists a policy π in G_0 under which the system is stable, then

$$\sum_{i \in Q} \frac{a_i}{m_i} < 1 - \prod_{i \in Q}(1 - p_i), \quad \forall Q \subset \{1, ..., N\}. \tag{3.2}$$

Proof: Assume that the system is operated under some policy in G_0 and is stable. Definition 3.1 implies that the Markov chain $\mathbf{X}$ is ergodic and possesses a stationary distribution. We start the system with its stationary distribution therefore the queue length process is stationary and ergodic. Let $h_j(t)$ be the indicator variable that is equal to 1 if queue j is connected and receives service at slot t and to 0 otherwise. The departure process from queue j is $\{h_j(t)M_j(t)\}_{t=1}^{\infty}$ and is stationary and ergodic. The departure rate from queue j is

$$E[h_j(t)M_j(t)] = m_j E[h_j(t)].$$

Since the system is stationary and ergodic, in each queue the departure rate should be equal to the arrival rate; that is

$$m_j E[h_j(t)] = a_j. \tag{3.3}$$

Hence from (3.3), for any set of queues Q we have

$$\sum_{j \in Q} \frac{a_j}{m_j} = \sum_{j \in Q} E[h_j(t)]. \tag{3.3a}$$

The sum in the right hand side of (3.3a) can be written as

$$\sum_{j \in Q} E[h_j(t)] = E[E[\sum_{j \in Q} h_j(t) | C_j(t), X_j(t-1), j \in Q]]. \tag{3.4}$$

Consider the partition of the probability space into the events

$$B_1 = \{C_j(t) = 0, j \in Q\},$$

$$B_2 = \{C_j(t) = 0, j \in Q\}^c \cap \{X_j(t-1) = 0, j \in Q\},$$

$$B_3 = \{C_j(t) = 0, j \in Q\}^c \cap \{X_j(t-1) = 0, j \in Q\}^c$$

where A^c is the complementary set of A. Notice that

$$E[\sum_{j \in Q} h_j(t) | C_j(t), X_j(t-1), j \in Q; B_l] = 0, \;\; l = 1, 2$$

$$E[\sum_{j \in Q} h_j(t) | C_j(t), X_j(t-1), j \in Q; B_3] \leq 1$$

hence we have

$$E[E[\sum_{j \in Q} h_j(t) | C_j(t), X_j(t-1), j \in Q]]$$

$$= E[\sum_{l=1}^{3} E[\sum_{j \in Q} h_j(t) | C_j(t), X_j(t-1), j \in Q; B_l] P[B_l]]$$

$$\leq 1 - P[B_1] - P[B_2]. \tag{3.5}$$

Since the Markov chain $\mathbf{X}$ is irreducible and ergodic, under the stationary distribution we have $P[X_j(t) = 0,\ j \in Q] > 0$ for any $Q \subset \{1, ..., N\}$; hence we have

$$P[B_2] = P[C_j(t) = 0, j \in Q]P[X_j(t-1) = 0, j \in Q] > 0. \tag{3.5a}$$

and because of the independence of the connectivity processes that correspond to different queues we have

$$P[B_1] = \prod_{i \in Q}(1 - p_i) \tag{3.5b}$$

Relations (3.5,3.5a,3.5b) imply

$$E[E[\sum_{j \in Q} h_j(t) | C_j(t), X_j(t), j \in Q]] < 1 - \prod_{i \in Q}(1 - p_i). \tag{3.6}$$

Equations (3.3a,3.4,3.6) imply (3.2). ◇

Note that $\sum_{i \in Q} \frac{a_i}{m_i}$ is the rate with which work (in the form of service slots) is entering the set Q of queues and $1 - \prod_{i \in Q}(1 - p_i)$ is the proportion of slots at which at least one queue of Q is connected and can receive service; hence the necessity of 3.2 for stability can be visualized. The sufficiency though of 3.2 for stability can not be seen easily in advance since the rate at which service is provided to the queues within set Q is strictly less than $1 - \prod_{i \in Q}(1 - p_i)$. That is because the connected queues of the set Q at each slot t may be either empty or have length less than that of another connected queue out of Q. In the next lemma it is shown that conditions 3.2 are sufficient for stabilizability as well. Consider the policy $\pi_0 \in G$ which during slot t allocates the server according to the function $g_0 : \mathcal{Y}^1 \to \{e, 1, ..., N\}$ defined by

$$g_0(\mathbf{x}, \mathbf{c}) = \begin{cases} e, & \text{if } x_i c_i = 0,\ i = 1, ..., N \\ \arg\max_{i=1,...,N}\{x_i c_i\}, & \text{otherwise.} \end{cases}$$

That is π_0 allocates the server at slot t to the connected queue i ($C_i(t) = 1$) with maximum length. Policy π_0 is shown next to stabilize the system as long as there exists a policy in G_0 under which it is stable. In the following we let $h_j(t) = 1\{g_0(\mathbf{X}(t-1), \mathbf{C}(t)) = j\}$.

Lemma 3.2: The system is stable under π_0 if

$$\sum_{i \in Q} \frac{a_i}{m_i} < 1 - \prod_{i \in Q}(1 - p_i), \quad \forall Q \subset \{1, ..., N\}.$$

Proof: Under π_0, $\mathbf{X}$ is apparently irreducible. We use Foster's criterion for ergodicity of a Markov chain ([As87]) to show that $\mathbf{X}$ is ergodic under the condition of the lemma; from ergodicity 3.1 is implied. Consider the function V defined on the state space $\mathcal{X}$ of the chain by $V(\mathbf{x}) = \sum_{i=1}^{N} m_i^{-1} x_i^2$. For all $\mathbf{x} \in \mathcal{X}$ we have

$$E[V(\mathbf{X}(t+1))|\mathbf{X}(t) = \mathbf{x}] = E[\sum_{i=1}^{N} m_i^{-1} X_i^2(t+1)|\mathbf{X}(t) = \mathbf{x}] \leq$$

$$E[\sum_{i=1}^{N} m_i^{-1}(x_i + A_i(t+1))^2|\mathbf{X}(t) = \mathbf{x}]$$

$$= V(\mathbf{X}(t)) + 2\sum_{i=1}^{N} m_i^{-1} a_i x_i + \sum_{i=1}^{N} m_i^{-1} E[A_i^2(t+1)] < \infty. \tag{3.6a}$$

We show that if condition (3.2) is satisfied then for a fixed $\epsilon > 0$ there exists a number b, which may be a function of the first and second moments of the arrival, service and connectivity processes, for which we have

$$E[V(\mathbf{X}(t+1)) - V(\mathbf{X}(t))|\mathbf{X}(t)] < -\epsilon \quad if \quad V(\mathbf{X}(t)) > b. \tag{3.7}$$

Notice that the set

$$V_b = \{\mathbf{x} : V(\mathbf{x}) \leq b, \quad \mathbf{x} \in Z_+^N\}$$

has finite cardinality for all b. From (3.6a), (3.7) we can conclude that $\mathbf{X}(t)$ is ergodic. We proceed now to show 3.7. By simple calculations we get

$$E[V(\mathbf{X}(t+1)) - V(\mathbf{X}(t))|\mathbf{X}(t)]$$

$$= E[\sum_{i=1}^{N} m_i^{-1}(X_i(t+1) - X_i(t))(X_i(t+1) - X_i(t) + 2X_i(t))|\mathbf{X}(t)] =$$

$$E[\sum_{i=1}^{N} 2m_i^{-1}X_i(t)(X_i(t+1)-X_i(t))|\mathbf{X}(t)] + E[\sum_{i=1}^{N} m_i^{-1}(X_i(t+1)-X_i(t))^2|\mathbf{X}(t)]. \tag{3.8}$$

The second term of the sum in the right hand side of (3.8) can be upper bounded as follows

$$E[\sum_{i=1}^{N} m_i^{-1}(X_i(t+1) - X_i(t))^2|\mathbf{X}(t)] \leq E[\sum_{i=1}^{N} m_i^{-1}(A_i(t+1))^2|\mathbf{X}(t)] + 1$$

$$= \sum_{i=1}^{N} m_i^{-1}E[A_i^2(t)] + 1. \tag{3.9}$$

For the first term of the sum in the right hand side of (3.8) we have

$$E[\sum_{i=1}^{N} 2m_i^{-1}X_i(t)(X_i(t+1) - X_i(t))|\mathbf{X}(t)] =$$

$$= E[\sum_{i=1}^{N} 2m_i^{-1} X_i(t) A_i(t+1) | \mathbf{X}(t)] - E[\sum_{i=1}^{N} 2m_i^{-1} X_i(t) M_i(t+1) h_i(t+1) | \mathbf{X}(t)]. \tag{3.10}$$

The first term of the sum in the right hand side of (3.10) can be calculated to give

$$E[\sum_{i=1}^{N} 2m_i^{-1} X_i(t) A_i(t+1) | \mathbf{X}(t)] = 2 \sum_{i=1}^{N} X_i(t) \frac{a_i}{m_i}. \tag{3.11}$$

We need to introduce some notation before we manipulate the second term of the sum in (3.10). Consider a permutation e_i $i = 0, ..., N$ of the integers 0 to N which is such that $e_0 = 0$, $X_{e_i}(t) \geq X_{e_{i-1}}(t)$, for $i = 2, ..., N$ and if $X_{e_i}(t) = X_{e_{i-1}}(t)$ then $e_i > e_{i-1}$. Consider also a partition of the probability space into the events D_i, $i = 0, ..., N$ defined by

$$D_0 = \{\mathbf{C}(t+1) = \mathbf{0}\},$$

$$D_i = \{C_{e_i}(t+1) = 1,\ C_{e_j}(t+1) = 0 \ for \ N \geq j > i\} \ \ for \ \ i = 1, ..., N.$$

The probabilities of the events D_i are

$$P[D_0] = \prod_{i=1}^{N} (1 - p_i), \ \ P[D_i] = p_{e_i} \prod_{j=i+1}^{N} (1 - p_{e_j}) \ \ i = 1, ..., N. \tag{3.12}$$

Apparently the permutation as well as the events D_i depend on the state $\mathbf{X}(t)$ and the connectivity vector $\mathbf{C}(t)$ at each slot t. Now we can calculate the second term of the sum in the right hand side of (3.10) to be

$$E[\sum_{i=1}^{N} 2m_i^{-1} X_i(t) M_i(t+1) h_i(t+1) | \mathbf{X}(t)] = E[\sum_{i=1}^{N} 2X_i(t) h_i(t+1) | \mathbf{X}(t)]$$

$$= E[\sum_{i=1}^{N} 2X_{e_i}(t)h_{e_i}(t+1)|\mathbf{X}(t)]$$

$$= \sum_{j=0}^{N} E[\sum_{i=1}^{N} 2X_{e_i}(t)h_{e_i}(t+1)|\mathbf{X}(t), D_j]P(D_j). \tag{3.13}$$

Notice that from the definition of the policy, in the event D_j queue e_j is served if it is not empty. If it is empty then every other connected queue is empty as well. Therefore we have

$$E[\sum_{i=1}^{N} 2X_{e_i}(t)h_{e_i}(t+1)|\mathbf{X}(t), D_j] = 2X_{e_j}(t). \tag{3.13a}$$

From (3.12, 3.13, 3.13a) we get

$$E[\sum_{i=1}^{N} 2m_i^{-1}X_i(t)M_i(t+1)h_i(t+1)|\mathbf{X}(t)] = \sum_{i=1}^{N} 2X_{e_i}(t)p_{e_i} \prod_{j=i+1}^{N}(1-p_{e_j}). \tag{3.14}$$

where $\prod_{j=N+1}^{N}(\cdot) = 1$. By a simple calculation in the right side of (3.14) we get

$$E[\sum_{i=1}^{N} 2m_i^{-1}X_i(t)M_i(t+1)h_i(t+1)|\mathbf{X}(t)] =$$

$$= 2\sum_{j=2}^{N}(X_{e_j}(t) - X_{e_{j-1}}(t))(1 - \prod_{i=j}^{N}(1-p_{e_i})) + 2X_{e_1}(t)(1 - \prod_{i=1}^{N}(1-p_{e_i})). \tag{3.15}$$

Using the permutation we defined earlier and after some calculations (3.11) can be written as

$$E[\sum_{i=1}^{N} 2m_i^{-1}X_i(t)A_i(t+1)|\mathbf{X}(t)]$$

$$= 2\sum_{j=2}^{N}(X_{e_j}(t) - X_{e_{j-1}}(t))\sum_{i=j}^{N}\frac{a_{e_i}}{m_{e_i}} + 2X_{e_1}(t)\sum_{i=1}^{N}\frac{a_{e_i}}{m_{e_i}}. \tag{3.16}$$

From (3.10),(3.15) and (3.16) we get

$$E[\sum_{i=1}^{N} 2m_i^{-1}X_i(t)(X_i(t+1) - X_i(t))|\mathbf{X}(t)]$$

$$= 2\sum_{j=2}^{N}(X_{e_j}(t) - X_{e_{j-1}}(t))\sum_{i=j}^{N}\frac{a_{e_i}}{m_{e_i}} + 2X_{e_1}(t)\sum_{i=1}^{N}\frac{a_{e_i}}{m_{e_i}}$$

$$-2\sum_{j=2}^{N}(X_{e_j}(t) - X_{e_{j-1}}(t))(1 - \prod_{i=j}^{N}(1 - p_{e_i})) - 2X_{e_1}(t)(1 - \prod_{i=1}^{N}(1 - p_{e_i})) =$$

$$= 2\sum_{j=2}^{N}(X_{e_j}(t) - X_{e_{j-1}}(t))(\sum_{i=j}^{N}\frac{a_{e_i}}{m_{e_i}} - 1 + \prod_{i=j}^{N}(1 - p_{e_i}))$$

$$+2X_{e_1}(t)(\sum_{i=1}^{N}\frac{a_{e_i}}{m_{e_i}} - 1 + \prod_{i=1}^{N}(1 - p_{e_i})). \qquad (3.17)$$

We define

$$c = \max_{Q\subset\{1,\ldots,N\}}\{\sum_{i\in Q}\frac{a_i}{m_i} - 1 + \prod_{i\in Q}(1 - p_i)\}. \qquad (3.17a)$$

From condition (3.2) we have $c < 0$. From (3.17, 3.17a) we get

$$E[\sum_{i=1}^{N} 2m_i^{-1}X_i(t)(X_i(t+1) - X_i(t))|\mathbf{X}(t)] \leq$$

$$2\sum_{j=2}^{N}(X_{e_j}(t) - X_{e_{j-1}}(t))c + 2X_{e_1}(t)c = 2X_{e_N}(t)c. \qquad (3.18)$$

From (3.8),(3.9) and (3.18) we get

$$E[V(\mathbf{X}(t+1)) - V(\mathbf{X}(t))|\mathbf{X}(t)] \leq \sum_{i=1}^{N} E[(A_i(t))^2] + 1 + 2X_{e_N}(t)c. \qquad (3.19)$$

If $V(\mathbf{X}(t)) \geq b$ then $X_{e_N}(t) \geq \sqrt{\frac{b}{N}}$ and from 3.2 $c < 0$ therefore we have from (3.19)

$$E[V(\mathbf{X}(t+1)) - V(\mathbf{X}(t))|\mathbf{X}(t)] \leq \sum_{i=1}^{N} E[(A_i(t))^2] + 1 + 2c\sqrt{\frac{b}{N}}. \qquad (3.20)$$

If $b = N(\frac{\epsilon+1+\sum_{i=1}^{N} E[(A_i(t))^2]}{2c})^2$ then the right hand side of (3.20) equals to $-\epsilon$ and the proof is complete. ◇

The next theorem provides the necessary and sufficient stabilizability conditions and it follows from lemmas 3.1, 3.2.

Theorem 3.1: The necessary and sufficient stabilizability condition is

$$\sum_{i \in Q} \frac{a_i}{m_i} < 1 - \prod_{i \in Q}(1 - p_i), \quad \forall Q \subset \{1, ..., N\}. \tag{3.20a}$$

Furthermore policy π_0 stabilizes the system as long as it is stabilizable.

Corollary 3.1: When the arrival and service rates as well as the connectivity probabilities of all queues are the same and equal to a, m and p respectively then the necessary and sufficient stabilizability condition 3.2 is equivalent to

$$\frac{a}{m} < \frac{1 - (1 - p)^N}{N}. \tag{3.21}$$

Proof: Since all nodes are identical, for any set Q with k nodes, condition 3.2 is written as

$$\frac{a}{m} < \frac{1 - (1 - p)^k}{k}. \tag{3.22}$$

When Q includes all nodes of the network then (3.2) is identical to (3.21). To show that (3.21) implies (3.22) for all k it is enough to show

$$\frac{1 - (1 - p)^k}{k} \geq \frac{1 - (1 - p)^{k+1}}{k + 1}, \quad k = 1, 2, ..$$

which is true since

$$\frac{1 - (1 - p)^k}{k} \geq \frac{1 - (1 - p)^{k+1}}{k + 1} \Leftrightarrow$$

$$(k + 1)p \sum_{i=0}^{k-1}(1 - p)^i \geq kp \sum_{i=0}^{k}(1 - p)^i \Leftrightarrow \sum_{i=0}^{k-1}(1 - p)^i \geq k(1 - p)^k. \qquad \diamond$$

For a symmetric system like that considered in the corollary the total throughput is equal to $1-(1-p)^N$ and the performance degradation due to time varying connectivity is equal to $(1-p)^N$, which is the probability that all nodes are disconnected during a particular slot.

When the system has fixed connectivities ($M_i(t) = 1$ *a.s.*, $i = 1,..,N$, $t = 1,..$) it is well known ([Wa88]) that the necessary and sufficient stabilizability condition is

$$\sum_{i=1}^{N} \frac{a_i}{m_i} < 1$$

Furthermore, under the necessary and sufficient stabilizability condition the system is stabilized by any work conserving policy that is for any policy which never idles the server if there are packets in the system. When the connectivities are time varying a policy is defined to be *work conserving* if it does not idle the server when there is a nonempty connected queue. Any work conserving policy in the latter case does not necessarily stabilizes the system even if it is stabilizable. This is demonstrated in the following counterexample.

Counterexample 3.1. Consider a system with two queues which have Bernoulli arrivals with rates a_1 and a_2 respectively. The server provides deterministic service to both queues, ($m_1 = m_2 = 1$); queue 1 is constantly available for service ($p_1 = 1$) while queue 2 is available with probability $p_2 < 1$. The stability condition (3.2) in this case is equivalent to the following

$$a_1 + a_2 < 1 \quad , \qquad a_2 < p_2. \tag{3.22}$$

Consider the nonidling policy π' that always give priority to queue 1. We claim that (3.22) is not sufficient for stability of the system under π'. Assume that the system starts with queue 1 being empty. At slot t, queue 2 may receive service if it is connected and no packet arrived at queue 1 during slot $t-1$. Consider the binary variable

$$d(t) = \begin{cases} 0, & \text{if } a_1(t-1) = 0 \text{ and } p_2(t) = 1 \\ 1, & \text{otherwise.} \end{cases}$$

The process $\{d(t)\}_{t=1}^{\infty}$ is Bernoulli and such that $P[d(t) = 1] = 1 - p_2(1 - a_1)$. Consider a hypothetical queue, let say 3, with deterministic service and arrivals at each slot t equal to $a_2(t) + d(t)$. We can easily see that the length of queue 2 is always greater than or equal to the length of queue 3 minus 1. Hence stability of queue 3 is necessary for stability of queue 2. The necessary stability condition for queue 3 is that

$$a_2 + 1 - p_2(1 - a_1) < 1. \tag{3.23}$$

Apparently we can find nonnegative numbers a_1, a_2, p_2 that satisfy (3.22) but do not satisfy (3.23); hence (3.22) is not sufficient for stability under π'. $\diamond$

Remarks

1. In order to verify the stabilizability condition 3.2 we have to verify the inequality in 3.2 for all subsets Q of the set $\{1, .., N\}$. The number of subsets of that set is 2^N. Hence, for a large number of queues, verifying whether the system is stabilizable for certain arrival, service and connectivity rates becomes

an intractable task. It is of interest to find an efficient algorithm, if there exists one, that verifies stabilizability in polynomial time.

2. Condition 3.2 is necessary for the existence of a policy in G under which the system is stable and sufficient for stability of the system under π_0. The policies in G_0 may base their decision on the lengths of all queues in the system irrespectively of whether they are connected or not while π_0 base its decisions on the lengths of the connected queues only. Hence the additional information on which the polices in G_0 may base their decisions, that is the lengths of the unobserved queues, is irrelevant to the stability performance of the system.

3. The independence among the processes $\{A_i(t)\}_{t=1}^{\infty}, \{M_i(t)\}_{t=1}^{\infty}, \{C_i(t)\}_{t=1}^{\infty}$, $i = 1,..,N$ has not been used in the proof of theorem 3.1. The stability result in that theorem holds under the more general assumption that the variables $A_i(t), M_i(t), C_i(t),\ i = 1,..,N$ are independent in different slots and identically distributed. Under that more general assumption the theorem holds if the term $1 - \prod_{i \in Q}(1 - p_i)$ in the right side of relationship (3.20a) is replaced by $P[\sum_{i \in Q} C_i(t) > 0]$.

4.4 Optimal server allocation

In this section we study the problem of delay optimal server allocation. We consider a symmetric system in which the arrival, service and connectivity processes in different queues have identical statistics. We assume furthermore that the variables $A_i(t),\ i = 1, ..., N,\ t = 1, ...$ are binary. No further assumptions are made about the statistics of the system. The policy π_0 defined in section 4.3, which allocates the server at each slot to the longest connected queue is shown to be optimal; more specifically it minimizes, in the stochastic ordering sense, the process of total number of packets in the system. In section 3.1 the concept of stochastic ordering has been briefly introduced.

4.4.1 Optimality of π_0

Consider the class of policies $\tilde{G}$ that take an action at slot t based on the entire history of the past observations and control actions. A policy in $\tilde{G}$ is specified by a sequence of functions $\{g_t(\cdot)\}_{t=1}^{\infty}$, $g_t : \mathcal{Y}^t \times \{e, 1, ..., N\}^{t-1} \to \{e, 1, ..., N\}$ where $\mathcal{Y}^t$ is the space where $\mathbf{Y}^t(t) = (\mathbf{Y}(1), ..., \mathbf{Y}(t))$ lies. The allocation decision at slot t is $U(t) = g_t(\mathbf{Y}^t(t), \mathbf{U}^t(t))$ where $\mathbf{U}^t(t) = (U(1), ..., U(t-1))$. Apparently $\tilde{G}$ is a bigger class of policies than G. We show that π_0 is optimal within $\tilde{G}$. We need notation to consider the process of total number of packets in the system $Q = \{Q(t)\}_{t=1}^{\infty}$ where $Q(t) = \sum_{i=1}^{N} X_i(t)$. Next theorem states that π_0 minimizes in the stochastic ordering sense the process of total number of packets in the system.

Theorem 4.1: Let Q be the process of total number of packets in the system when the initial state is $\mathbf{x}_0$ and policy $\pi \in \tilde{G}$ acts on it and Q_0 the corresponding process when π_0 acts on the system. We have

$$Q_0 \leq_{st} Q. \tag{4.1}$$

We need the following lemma in the proof of the theorem.

Lemma 4.1: For every policy $\pi \in \tilde{G}$ there exists a policy $\tilde{\pi} \in \tilde{G}$ which acts similarly to π_0 at $t = 1$ and is such that when the system is in state $\mathbf{x}_0$ at $t = 0$ and policies π, $\tilde{\pi}$ act on it the corresponding processes Q, $\tilde{Q}$ of total number of packets in the system can be constructed so

$$\tilde{Q}(t) \leq_{st} Q(t) \ a.s., \quad t = 0, 1, 2, ... \tag{4.2}$$

Proof: We construct $\tilde{\pi}$ and we couple the queue length realizations under π and $\tilde{\pi}$ appropriately so that (4.2) holds. Let X and $\tilde{X}$ be the queue length processes under policies π and $\tilde{\pi}$ respectively. At slot $t = 1$ give the same connectivity variables under the two policies to the same queues.

If π and $\tilde{\pi}$ take the same action at $t = 1$ then let the arrival, service, and connectivity variables be the same at the same queues under both policies for every subsequent slot and take $\tilde{\pi}$ to coincide with π for $t = 2, ...$ Then the queue length processes are identical under both policies and (4.2) follows immediately.

If π idles at $t = 1$ while $\tilde{\pi}$ serves queue j then give to the same queues, the same arrival service and connectivity variables under both policies at all subsequent slots. At $t = 1$ we have

$$X_l(t) = \tilde{X}_l(t) \ \text{ if } \ l \neq j \ , \quad \tilde{X}_j(t) \leq X_j(t). \tag{4.3}$$

Let policy $\tilde{\pi}$ be identical to π at all subsequent slots $t = 2, 3, ...$ If (4.3) holds at t we can easily see that it holds at $t + 1$ as well and (4.2) follows by induction.

If π serves queue k while $\tilde{\pi}$ serves queue j at $t = 1$ then at that time slot give the same service variable at queues k and j under π and $\tilde{\pi}$ respectively. At each time $t \geq 1$ consider the indicator variables $s(t)$, $\tilde{s}(t)$, $l(t)$, $\tilde{l}(t)$ defined as follows

$$s(t) = \arg \min_{m=j,k} \{X_m(t)\}, \qquad \tilde{s}(t) = \arg \min_{m=j,k} \{\tilde{X}_m(t)\},$$

$$l(t) = \arg \max_{m=j,k} \{X_m(t)\}, \qquad \tilde{l}(t) = \arg \max_{m=j,k} \{\tilde{X}_m(t)\}.$$

If we have $X_j(t) = X_k(t)$ then we take $s(t) = \min\{j, k\}$. Similarly for the rest indicator variables. In the following we write $X_s(t)$ instead of $X_{s(t)}(t)$. The same for the rest of the above indicator variables. We distinguish the following cases.

Case 1. $X_k(0) = X_j(0)$

Give the same arrival variables at $t = 1$ to the queues j and k under $\tilde{\pi}$ and π respectively. Similarly for the queues k and j under $\tilde{\pi}$ and π respectively. Give the same arrival variables under both policies to each one of the rest of the queues. Then at $t = 1$ the queue lengths satisfy the following relationships

$$X_s(t) = \tilde{X}_{\tilde{s}}(t) \quad , \quad X_l(t) = \tilde{X}_{\tilde{l}}(t) \quad , \quad X_i(t) = \tilde{X}_i(t) \ , \ i \neq k, j. \tag{4.4}$$

Case 2. $X_k(0) < X_j(0)$

At slot $t = 1$ give the same arrival variables to the same queues under π and $\tilde{\pi}$. If the service at $t = 1$ is not completed then the queue lengths at $t = 1$ satisfy (4.4). If the service is completed we distinguish the following cases.

A. $X_k(0) < X_j(0) - 1$

In this case we can easily verify that the queue lengths satisfy the following relationships

$$X_s(t) = \tilde{X}_{\tilde{s}}(t) - 1 \quad , \quad X_l(t) = \tilde{X}_{\tilde{l}}(t) + 1 \quad , \quad X_i(t) = \tilde{X}_i(t) \ , \ i \neq k, j. \quad (4.5)$$

for $t = 1$.

B. $X_k(0) = X_j(0) - 1$

In this case the queue lengths are as follows depending on the arrivals. If during slot 1 a packet arrives only at queue k (under both policies) then the queue lengths at the end of slot 1 satisfy (4.4); otherwise the queue lengths satisfy relations (4.5).

The cases 1 and 2 above cover all the possibilities since it is not possible to have $X_k(0) > X_j(0)$ given that π_0 serves the longest queue. Hence at the end of slot 1 the queue lengths under π and $\tilde{\pi}$ satisfy either (4.4) or (4.5). Note that in both cases we have

$$\sum_{i=1}^{N} X_i(t) = \sum_{i=1}^{N} \tilde{X}_i(t) \qquad (4.5a)$$

Hence if at each slot either 4.4 or 4.5 hold then 4.5a holds at all slots and (4.2) is satisfied at $t = 1$. We show in the following that if the queue lengths at slot t satisfy either (4.4) or (4.5) then we can couple the processes $\mathbf{X}$ and $\tilde{\mathbf{X}}$ by choosing appropriately the connectivity, arrival and service variables at slot $t+1$, and define $\tilde{\pi}$ such that at slot $t+1$ one of the relations (4.4) and (4.5) is satisfied again. From induction we can conclude that there exists a $\tilde{\pi}$ such that the queue length processes under π and $\tilde{\pi}$ satisfy either (4.4) or (4.5) at any t; hence (4.2) holds and the lemma follows. We distinguish the following cases for $\mathbf{X}(t)$.

Case 1'. Relations (4.4) hold at t.

Let at slot $t+1$ the queues $l(t)$, $\tilde{l}(t)$ have the same connectivity arrival and service variables under π and $\tilde{\pi}$ respectively. Similarly for the queues $s(t)$ and $\tilde{s}(t)$. Let all queues, other than k, j, have the same connectivity, arrival and service variables at $t+1$ under π and $\tilde{\pi}$ respectively. If π serves queue $l(t)$ at slot $t+1$ let $\tilde{\pi}$ serve $\tilde{l}(t)$; if π serves queue $s(t)$ let $\tilde{\pi}$ serve $\tilde{s}(t)$. Let $\tilde{\pi}$ be identical to π otherwise. Then we can easily check that at $t+1$, (4.4) are satisfied

Case 2'. Relations (4.5) hold at t and $\tilde{X}_{\tilde{s}}(t) < \tilde{X}_{\tilde{l}}(t)$.

Let the connectivity, arrival and service variables at slot $t+1$ as well as the policy $\tilde{\pi}$ be as in case 1' above. If we have $\tilde{X}_{\tilde{s}}(t) \leq \tilde{X}_{\tilde{l}}(t)+2$ then (4.5) hold at slot $t+1$. If we have $\tilde{X}_{\tilde{s}}(t) = \tilde{X}_{\tilde{l}}(t)+1$ the following may hold. If queues $s(t)$ and $\tilde{s}(t)$ are served under π and $\tilde{\pi}$ respectively then at slot $t+1$ (4.5) hold. If instead queues $l(t)$ and $\tilde{l}(t)$ are served then if the service is not completed, (4.5) hold at slot $t+1$. If the service is completed and we have an arrival at queues $s(t)$, $\tilde{s}(t)$ and no arrivals at $l(t)$, $\tilde{l}(t)$ then (4.4) hold at slot $t+1$. If the service is completed and the arrivals are not as above then (4.5) hold at $t+1$.

Case 3'. Relations (4.5) hold at t and $\tilde{X}_{\tilde{s}}(t) = \tilde{X}_{\tilde{l}}(t)$.

Let the connectivity variables at slot $t+1$ be as in case $1'$ above. If π serves queue $l(t)$ at slot $t+1$ let $\tilde{\pi}$ serve $\tilde{l}(t)$; if π serves queue $s(t)$ let $\tilde{\pi}$ serve $\tilde{s}(t)$. Let $\tilde{\pi}$ be identical to π otherwise. We distinguish the following cases.

A'. Queues $l(t)$, $\tilde{l}(t)$ are served under π, $\tilde{\pi}$ respectively.

Let the service variables of $l(t)$, $\tilde{l}(t)$ be identical under π, $\tilde{\pi}$. Let all queues $i \neq j, k$ have the same arrivals under both policies. If service is not completed let queues $l(t)$, $\tilde{l}(t)$ have the same arrivals under π, $\tilde{\pi}$ respectively and similarly for queues $s(t)$, $\tilde{s}(t)$. If there is an arrival at $s(t)$, $\tilde{s}(t)$ and no arrival at $l(t)$, $\tilde{l}(t)$ at $t+1$ then (4.4) hold at $t+1$, otherwise (4.5) hold. If service is completed let queues $s(t)$, $\tilde{l}(t)$ have the same arrivals at $t+1$ under π, $\tilde{\pi}$ and similarly for $l(t)$, $\tilde{s}(t)$. Then (4.4) hold at $t+1$.

B'. Queues $s(t)$, $\tilde{s}(t)$ are served under π, $\tilde{\pi}$ respectively.

Let the service variables of $s(t)$, $\tilde{s}(t)$ be identical under π, $\tilde{\pi}$. Let queues $l(t)$, $\tilde{l}(t)$ have the same arrivals under π, $\tilde{\pi}$ respectively and similarly for the queues $s(t)$, $\tilde{s}(t)$. Let all queues $i \neq j, k$ have the same arrivals under the two policies. If service is completed then (4.5) hold at $t+1$. If service is not completed then either (4.4) or (4.5) hold depending on whether there are arrivals at $s(t)$, $\tilde{s}(t)$ and no arrivals at $l(t)$, $\tilde{l}(t)$, or not.

C'. Queue $i \neq j, k$ is served under π, $\tilde{\pi}$ respectively.

Let queues $l(t)$, $\tilde{l}(t)$ under π, $\tilde{\pi}$ respectively have the same arrivals and similarly for the queues $s(t)$, $\tilde{s}(t)$. Let all queues $i \neq j, k$ have the same arrivals under the two policies. Let queue i have the same service variables under π, $\tilde{\pi}$. If there is an arrival at $s(t)$, $\tilde{s}(t)$ and no arrival at $l(t)$, $\tilde{l}(t)$ at $t+1$ then (4.4) hold at $t+1$, otherwise (4.5) hold. ◇

We proceed now in the proof of the theorem.

Proof of Theorem 4.1: From lemma 4.1 we have that for any policy π we can construct a policy π_1 which is similar to π_0 at $t = 1$ and such that for the corresponding total number of packets processes Q, Q_1 we have

$$Q_1(t) \leq Q(t) \ a.s., \quad t = 0, 1, ..$$

By repeating the construction we can show that there exists a policy π_2 which agrees with π_1 at the first slot, agrees with π_0 in the second slot and is such that for the corresponding process Q_2 we have

$$Q_2(t) \leq Q_1(t) \ a.s., \quad t = 0, 1, ..$$

If we repeat the argument k times we obtain policies π_i, $i = 1, ..., k$ such that policy π_i agrees with π_0 at the first ith slots and for the corresponding processes we have

$$Q_k(t) \leq Q_{k-1}(t) \leq ... \leq Q_1(t) \leq Q(t) \ a.s., \quad t = 0, 1, .. \tag{4.6}$$

Consider the time slots $t_1, t_2, ...t_n$ and a function g as in part 2 of theorem 4.1. Consider also the policy π_{t_n} defined as above. By construction the variables $Q_{t_n}(t_1), ..., Q_{t_n}(t_n)$ have the same joint probability distribution with the variables $Q_0(t_1), ..., Q_0(t_n)$ where $Q_0, ..., Q_{t_n}$ are the processes of total number of packets in the system under the policies π_0, π_{t_n} respectively. Hence for all z we have

$$P(g(Q_{t_n}(t_1), ..., Q_{t_n}(t_n)) > z) = P(g(Q_0(t_1), ..., Q_0(t_n)) > z). \quad (4.7)$$

From 4.6 $Q_{t_n}(t) \leq Q(t)$ $a.s.$ for all $t = 0, 1, ...$ therefore we have

$$P(g(Q_{t_n}(t_1), ..., Q_{t_n}(t_n)) > z) \leq P(g(Q(t_1), ..., Q(t_n)) > z). \quad (4.8)$$

Equations (4.7) and (4.8) and part 2 of theorem 4.1 completes the proof. ◇

Remark

The fact that policy π in lemma 4.1 base its decisions on the lengths of the connected queues only is not essential in the proof of the lemma. That proof goes through even if π is any policy that base its decision on the history of the lengths of all queues in the system in addition to the connectivities and past control actions. Therefore π_0 is optimal within the class of policies that base their decisions on the complete system history.

When the connectivites are fixed ($C_i(t) = 1,\ i = 1, .., N,\ t = 1, ..$) then in the symmetric system any work conserving policy minimizes the delay. Furthermore in the general case (asymmetric system), if the service processes are i.i.d. (geometric service requirements) the optimal policy is known to be the one that serves the nonempty queue with largest m_i. In the case of varying connectivities work conservation is not enough for optimality. Serving the queue that suffers most is essential for optimal system performance. This is made more plausible by our result in the next section where it is stated that serving the queue that suffers less at each slot maximizes the delay in the class of work conserving policies.

4.4.2 Worst performance within the work conserving policies

Consider the policy $\pi' \in G$ which during slot t allocates the server according to the function $g' : \mathcal{Y}^1 \to \{e, 1, ..., N\}$ defined by

$$g'(\mathbf{x}, \mathbf{c}) = \begin{cases} e, & \text{if } x_i c_i = 0,\ i = 1, ..., N \\ \arg\min_{\substack{i=1,...,N \\ x_i c_i > 0}} \{x_i c_i\}, & \text{otherwise.} \end{cases}$$

That is π' allocates the server at slot t to the connected, nonempty queue i ($C_i(t) = 1$) with minimum length. The following theorem states that policy π' maximizes in the stochastic order sense the process of total number of packets in the system within the class of work conserving policies.

Theorem 4.2: If Q is the process of total number of packets in the system when the initial state is $\mathbf{x}_0$ and a work conserving policy π acts on it and Q' the corresponding process when π' acts on the system then we have

$$Q \leq_{st} Q'. \tag{4.9}$$

The following lemma plays a role in the proof of theorem 4.2 analogous to that of lemma 4.1 in the proof of theorem 4.1.

Lemma 4.2: For every work conserving policy $\pi \in \tilde{G}$ there exists a work conserving policy $\tilde{\pi} \in \tilde{G}$ which acts similarly to π_1 at $t = 1$ and is such that when the system is in state $\mathbf{x}_0$ at $t = 0$ and policies π, $\tilde{\pi}$ act on it the corresponding processes Q, $\tilde{Q}$ of total number of packets in the system can be constructed so

$$\tilde{Q}(t) \geq_{st} Q(t) \ a.s., \quad t = 0, 1, 2, ... \tag{4.10}$$

Proof: By induction we construct $\tilde{\pi}$ and we couple the realizations under π and $\tilde{\pi}$ appropriately such that (4.10) holds. Let $\mathbf{X}, \tilde{\mathbf{X}}$ be the queue length processes under π, $\tilde{\pi}$ respectively. At slot $t = 1$ give the same connectivity variables under the two policies to the same queues.

If π and $\tilde{\pi}$ take the same action at $t = 1$ then let the arrival, service, and connectivity variables be the same at the same queues under both policies in every subsequent slot and let $\tilde{\pi}$ coincide with π for $t = 2, ...$ Policy π is work conserving therefore $\tilde{\pi}$ is work conserving as well. The queue length processes are identical under both policies and (4.10) follows immediately.

If π serves queue k while $\tilde{\pi}$ serves queue j at $t = 1$ then at that time slot give the same service variable at queues k and j under π and $\tilde{\pi}$ respectively. Let the indicator variables $s(t)$, $\tilde{s}(t)$ $l(t)$, $\tilde{l}(t)$ be as in the proof of lemma 4.1. We distinguish the following cases.

Case 1. $X_k(0) = X_j(0)$

Give the same arrival variables at $t = 1$ to the queues j and k under $\tilde{\pi}$ and π respectively. Similarly for the queues k and j under $\tilde{\pi}$ and π respectively. Give the same arrival variables under both policies to each one of the rest of the queues. Then at $t = 1$ the queue lengths satisfy the following relationships

$$X_s(t) = \tilde{X}_{\tilde{s}}(t) \quad , \quad X_l(t) = \tilde{X}_{\tilde{l}}(t) \quad , \quad X_i(t) = \tilde{X}_i(t) \ , \ i \neq k, j. \tag{4.11}$$

Case 2. $X_k(0) > X_j(0)$

At slot $t = 1$ give the same arrival variables to the same queues under π and $\tilde{\pi}$. If the service at $t = 1$ is not completed then the queue lengths at $t = 1$ satisfy (4.11). If the service is completed we distinguish the following cases.

A. $X_k(0) - 1 > X_j(0)$

In this case we can easily verify that the queue lengths satisfy the following relationships

$$X_s(t) = \tilde{X}_{\tilde{s}}(t) + 1 \quad , \quad X_l(t) = \tilde{X}_{\tilde{l}}(t) - 1 \quad , \quad X_i(t) = \tilde{X}_i(t) \ , \ i \neq k, j. \tag{4.12}$$

B. $X_k(0) - 1 = X_j(0)$

In this case the queue lengths are as follows depending on the arrivals. If during slot 1 a packet arrives only at queue j (under both policies) then the queue lengths at the end of slot 1 satisfy (4.11); otherwise the queue lengths satisfy relations (4.12).

The cases 1 and 2 above cover all the possibilities because it is not possible to have $X_k(0) < X_j(0)$ since π' serves the shortest queue. Hence at the end of slot 1 the queue lengths under π and $\tilde{\pi}$ satisfy either (4.11) or (4.12). Note that in both cases we have

$$\sum_{i=1}^{N} X_i(t) = \sum_{i=1}^{N} \tilde{X}_i(t)$$

and (4.2) is satisfied at $t = 1$. We show in the following that if the queue lengths at slot t satisfy either (4.11) or (4.12) then we can couple the processes X and $\tilde{X}$ by choosing appropriately the connectivity, arrival and service variables at slot $t+1$, and define $\tilde{\pi}$ without violating the work conservation requirement, such that at slot $t+1$ one of the relations (4.11) and (4.12) is satisfied. Then from induction we can conclude that there exists a $\tilde{\pi}$ such that the queue length processes under π and $\tilde{\pi}$ satisfy either (4.11) or (4.12) for any t; hence (4.2) holds and the lemma follows. We distinguish the following cases for $\mathbf{X}(t)$.

Case 1'. Relations (4.11) hold at t.

Let at slot $t+1$ the queues $l(t)$, $\tilde{l}(t)$ have the same connectivity arrival and service variables under π and $\tilde{\pi}$ respectively. Similarly for the queues $s(t)$ and $\tilde{s}(t)$. Let all queues, other than k, j, have the same connectivity, arrival and service variables at $t+1$ under π and $\tilde{\pi}$ respectively. If π serves queue $l(t)$ at slot $t+1$ let $\tilde{\pi}$ serve $\tilde{l}(t)$; if π serves queue $s(t)$ let $\tilde{\pi}$ serve $\tilde{s}(t)$. Let $\tilde{\pi}$ be identical to π otherwise. Then we can easily check that at $t+1$, (4.11) are satisfied

Case 2'. Relations (4.12) hold at t and $X_s(t) < X_l(t)$.

Let the connectivity, arrival and service variables at slot $t+1$ be as in case 1' above. If π serves $l(t)$ at $t+1$ let $\tilde{\pi}$ serve $\tilde{l}(t)$. If π serves $s(t)$ at $t+1$ let $\tilde{\pi}$ serve $\tilde{s}(t)$ if it is not empty, otherwise $l(t)$. Let $\tilde{\pi}$ be identical to π otherwise. If we have $X_s(t) \leq X_l(t) + 2$ then 4.12 hold at $t+1$.

If we have $\tilde{X}_{\tilde{s}}(t) \leq \tilde{X}_{\tilde{l}}(t) + 2$ then (4.12) hold at slot $t + 1$. If we have $\tilde{X}_{\tilde{s}}(t) = \tilde{X}_{\tilde{l}}(t) + 1$ the following may happen. If queues $s(t)$ and $\tilde{s}(t)$ are served under π and $\tilde{\pi}$ respectively then at slot $t+1$ (4.12) hold. If instead queues $l(t)$ and $\tilde{l}(t)$ are served then if the service is not completed, (4.12) hold at slot $t+1$. If the service is completed and we have an arrival at queues $s(t)$, $\tilde{s}(t)$ and no arrivals at $l(t)$, $\tilde{l}(t)$ then (4.11) hold at slot $t+1$. If the service is completed and the arrivals are not as above then (4.12) hold at $t + 1$. If queues $s(t)$ and $\tilde{l}(t)$ are served by π, $\tilde{\pi}$ respectively then 4.12 hold at $t + 1$.

Case 3′. Relations (4.12) hold at t and $X_s(t) = X_l(t)$.

Let the connectivity variables at slot $t+1$ be as in case 1′ above. If π serves one of the queues $l(t)$, $s(t)$ at slot $t + 1$ let $\tilde{\pi}$ serve $\tilde{l}(t)$. Let $\tilde{\pi}$ be identical to π otherwise. We distinguish the following cases.

A′. One of the queues $l(t)$, $s(t)$ is served under π.

Without loss of generality assume that queue $l(t)$ is served under π. Let the service variables of $l(t)$, $\tilde{l}(t)$ be identical under π, $\tilde{\pi}$. Let all queues $i \neq j, k$ have the same arrivals under both policies. If service is not completed let queues $l(t)$, $\tilde{l}(t)$ have the same arrivals under π, $\tilde{\pi}$ respectively and similarly for queues $s(t)$, $\tilde{s}(t)$. If there is an arrival at $s(t)$, $\tilde{s}(t)$ and no arrival at $l(t)$, $\tilde{l}(t)$ at $t+1$ then (4.11) hold at $t+1$, otherwise (4.12) hold. If service is completed let queues $s(t)$, $\tilde{l}(t)$ have the same arrivals at $t+1$ under π, $\tilde{\pi}$ and similarly for $l(t)$, $\tilde{s}(t)$. Then (4.4) hold at $t + 1$.

B'. Queue $i \neq j, k$ is served under π, $\tilde{\pi}$ respectively.

Let queues $l(t)$, $\tilde{l}(t)$ under π, $\tilde{\pi}$ respectively have the same arrivals and similarly for the queues $s(t)$, $\tilde{s}(t)$. Let all queues $i \neq j, k$ have the same arrivals under the two policies. Let queue i have the same service variables under π, $\tilde{\pi}$. If there is an arrival at $s(t)$, $\tilde{s}(t)$ and no arrival at $l(t)$, $\tilde{l}(t)$ at $t+1$ then (4.11) hold at $t+1$, otherwise (4.12) hold. $\diamond$

We proceed now to the proof of the theorem.

Proof of Theorem 4.2: From lemma 4.2 we have that for any work conserving policy π we can construct a work conserving policy π_1 which is similar to π' at $t = 1$ and such that for the corresponding total number of packets processes Q, Q_1 we have

$$Q_1(t) \geq Q(t) \; a.s., \quad t = 0, 1, ..$$

By repeating the construction we can show that there exists a work conserving policy π_2 which agrees with π_1 at the first slot, agrees with π' in the second slot and is such that for the corresponding process Q_2 we have

$$Q_2(t) \geq Q_1(t) \; a.s., \quad t = 0, 1, ..$$

If we repeat the argument k times we obtain policies π_i, $i = 1, ..., k$ such that policy π_i agrees with π_0 at the first ith slots and for the corresponding processes we have

$$Q_k(t) \geq Q_{k-1}(t) \geq ... \geq Q_1(t) \geq Q(t) \; a.s., \quad t = 0, 1, .. \tag{4.13}$$

Consider the time slots $t_1, t_2, ...t_n$ and a function g as in part 2 of theorem 4.1. Consider also the policy π_{t_n} defined as above. By construction the variables $Q_{t_n}(t_1), ..., Q_{t_n}(t_n)$ have the same joint probability distribution with the variables $Q'(t_1), ..., Q'(t_n)$ where $Q', Q_1, ..., Q_{t_n}$ are the processes of total number of packets in the system under the policies $\pi', \pi_1, ..., \pi_{t_n}$ respectively. Hence for all z we have

$$P(g(Q_{t_n}(t_1), ..., Q_{t_n}(t_n)) > z) = P(g(Q'(t_1), ..., Q'(t_n)) > z). \tag{4.14}$$

From 4.13 $Q_{t_n}(t) \geq Q(t)$ $a.s.$ for all $t = 0, 1, ...$ therefore we have

$$P(g(Q_{t_n}(t_1), ..., Q_{t_n}(t_n)) > z) \geq P(g(Q(t_1), ..., Q(t_n)) > z). \tag{4.15}$$

Equations (4.14) and (4.15) and part 2 of theorem 4.1 completes the proof. $\diamond$

Apparently π' has no practical significance since it maximizes the delay. The result though in theorem 4.2 emphasizes the fact that serving the more suffering queues improves the delay. If we consider an hierarchy of the work conserving policies with respect to how close they follow the rule to serve the queues that suffer most at each slot then π_0 is in the top of this hierarchy and π' in the bottom. It is intuitively appealing the fact that their delay performances are the best and worst respectively within the class of work conserving policies.

4.4.3 Discussion

Assume that the arrival service and connectivity processes are i.i.d. In this case the problem of minimizing the delay can be casted as a discrete time

Markov Decision Process. Consider the cost function defined by

$$J_\pi(\mathbf{x}_0) = \lim_{T\to\infty} \sup E^{\pi}_{\mathbf{x}_0}[\frac{1}{T}\sum_{t=0}^{T}\sum_{i=1}^{N} X_i(t)] \tag{4.16}$$

where π is a policy in $\tilde{G}$, $\mathbf{x}_0$ is the initial system state and the expectation is taken with respect to the probability measure induced by π when the system starts from $\mathbf{x}_0$. Minimizing the delay is equivalent to minimizing (4.16) within $\tilde{G}$. The latter minimization problem falls within the category of discrete time Markov Decision Processes (MDP) with partial observations ([KuV86]). The controlled Markov chain is $(\mathbf{X}(t-1), \mathbf{C}(t))$, the control action is $\mathbf{U}(t)$ and the evolution of the chain is governed by (2.1). The observation at time t is $\mathbf{Y}(t) = (\mathbf{X}(t-1) \bigotimes \mathbf{C}(t), \mathbf{C}(t))$. The optimal policies in MDP with partial observations are in general nonstationary since the action taken at slot t is a function of all past observations. Those policies are usually hard to specify. The optimality result obtained in section 4.1 implies that π_0 minimizes (4.16). Therefore in asymmetric systems the policy that minimizes (4.16) is stationary. In a general asymmetric system the optimization of (4.16) remains an open problem. We conjecture that the optimal server allocation policy is stationary in the general case as well. Nevertheless we believe that the allocation decisions are a complicated function of the state and the policy is difficult to be specified completely.

In our study we have assumed that the connectivities become available for decision making in the beginning of each slot. An interesting case is when the connectivities are not observable and the server at slot t is allocated based on the queue lengths $\mathbf{X}(t-1)$ only. If the connectivity processes are i.i.d. then the changing connectivity model is reduced to one with fixed connectivities where the service variable for queue i at slot t is $C_i(t)M_i(t)$.

4.5 Optimization of throughput and delay in a finite buffer system

When the buffers in the nodes have finite length then an arriving packet is blocked from admission when it finds the buffers full. The stability of the system is not an issue in this case since the queues can not grow to infinity. The number of packets which are successfully transmitted, that is the throughput of the system, is an important performance measure in addition to delay. In this section we study both throughput and delay performance in a finite buffer system with one buffer per node. A policy is obtained which is both throughput and delay optimal.

When there is a single buffer per node an arriving packet at node i during slot t is accepted if the buffer is empty in the beginning of the slot ($X_i(t-1) = 0$) or node i is the one selected for service at slot t in which case its packet is forwarded from its buffer in the beginning of the slot and the buffer is empty. The queue length vector in this case belongs to $\{0,1\}^N$; the queue length at node i evolves according to the equation

$$X_i(t) = A_i(t) + X_i(t-1)(1 - A_i(t))(1 - 1\{U(t) = 1\}C_i(t)M_i(t)) \qquad (5.1)$$

where the variables $X_i(t)$, $U(t)$, $C_i(t)$, $M_i(t)$, $A_i(t)$ are as defined in section

4.2. We assume that at each slot there can be at most one arrival at each node that is the variables $A_i(t),\ i = 1, .., N$ are binary; furthermore we assume that the arrival and service processes have identical statistics at different nodes. The number of packets blocked from admission into the system during slot t is

$$B(t) = \sum_{i=1}^{N} A_i(t) X_i(t-1)(1 - 1\{U(t) = i\} C_i(t) M_i(t))$$

Note that the number of packets blocked from admission into the system plus the number of packets which are admitted in the system during slot t and they are finally served is equal to the number of packets arrived during slot t. Therefore maximizing the throughput of the system is equivalent to minimizing the number of blocked packets. Consider the policy $\hat{\pi} \in G$ which during slot t allocates the server according to the function $\hat{g} : \mathcal{Y}^1 \to \{e, 1, ..., N\}$ defined by

$$\hat{g}(\mathbf{x}, \mathbf{c}) = \begin{cases} e, & \text{if } x_i c_i = 0,\ i = 1, ..., N \\ \arg\min_{\substack{i=1,\dots,N \\ x_i c_i > 0}} \{p_i x_i c_i\}, & \text{otherwise.} \end{cases}$$

That is $\hat{\pi}$ allocates the server at slot t to the connected nonempty queue i ($C_i(t) = 1$) with the smallest probability of being connected. Policy $\hat{\pi}$ minimizes in the stochastic ordering sense both the process of blocked packets and the process of total number of packets in the system.

Theorem 5.1: Consider an arbitrary policy $\tilde{\pi} \in G$ and let policies $\hat{\pi}$ and $\tilde{\pi}$ schedule transmissions starting from the same initial state $\mathbf{x}$ at $t = 0$. Let Q, B be the processes of total number of packets in the system and of blocked packets respectively under $\hat{\pi}$; let $\tilde{Q}$, $\tilde{B}$ be the corresponding processes under $\tilde{\pi}$.

Then we have

$$Q \leq_{st} \tilde{Q}, \tag{5.2}$$

$$B \leq_{st} \tilde{B} \tag{5.3}$$

Proof: We construct the queue length processes $\mathbf{X}$, $\tilde{\mathbf{X}}$ under $\hat{\pi}$, $\tilde{\pi}$ respectively by appropriate coupling of the arrivals services and connectivities such that

$$Q(t) \leq \tilde{Q}(t) \;\; a.s., \;\; t = 0, 1, ... \tag{5.4}$$

$$B(t) \leq \tilde{B}(t) \; a.s., \;\; t = 0, 1, ... \tag{5.5}$$

Hence (5.2), (5.3) follow.

We show that a particular partial ordering (defined next) holds between the system states under the two policies at every slot. This partial ordering implies relations (5.4), (5.5). Assume that the queues are indexed such that $p_i \leq p_{i+1}, \; i = 1, .., N-1$. We say $\mathbf{x} \prec \mathbf{y}, \; \mathbf{x}, \mathbf{y} \in \{0, 1\}^N$ if

$$\sum_{i=1}^{j} x_i \leq \sum_{i=1}^{j} y_i, \; j = 1, .., N \tag{5.6}$$

We construct the queue length processes such that for all $\tau = 0, 1, ..$ we have

$$\mathbf{X}(\tau) \prec \tilde{\mathbf{X}}(\tau) \tag{5.7}$$

We use forward induction. At $\tau = 0$ we have $\mathbf{X}(0) = \tilde{\mathbf{X}}(0)$ therefore for $\tau = 0$ (5.7) follows. Assume that (5.7) is true for $\tau = t$; we show that it is true for $\tau = t + 1$ as well. Let $A_i(t+1), \; C_i(t+1), \; M_i(t+1), \; i = 1, .., N, \; U(t+1)$

be the arrival, connectivity, service, and control variables under $\hat{\pi}$ and $\tilde{A}_i(t+1)$, $\tilde{C}_i(t+1)$, $\tilde{M}_i(t+1)$, $i = 1,..,N$ $\tilde{U}(t+1)$ under $\tilde{\pi}$. First we show that the following hold

$$\mathbf{Y}(t+1) \prec \tilde{\mathbf{Y}}(t+1) \tag{5.8}$$

where

$$Y_i(t+1) = X_i(t)(1 - 1\{U(t+1) = 1\}C_i(t+1)M_i(t+1)), \; i = 1,..,N$$

$$\tilde{Y}_i(t+1) = \tilde{X}_i(t)(1 - 1\{\tilde{U}(t+1) = 1\}\tilde{C}_i(t+1)\tilde{M}_i(t+1)), \; i = 1,..,N$$

Let $j(l)$, $\tilde{j}(l)$ be the lth nonempty queue starting from queue 1 in states $\mathbf{X}(t)$, $\tilde{\mathbf{X}}(t)$. If $\tilde{j}(l) > j(l)$ then we have $\sum_{i=1}^{j} X_i(t) = l > \sum_{i=1}^{j} \tilde{X}_i(t)$ which contradicts the induction hypothesis therefore we have

$$\tilde{j}(l) \leq j(l), \; l = 1,..,Q(t) \tag{5.9}$$

and by the assumption about the indexing of the queues

$$p_{\tilde{j}(l)} \leq p_{j(l)} \; l = 1,..,Q(t). \tag{5.10}$$

Because of (5.10) we may construct $C_{j(l)}(t+1)$, $\tilde{C}_{\tilde{j}(l)}(t+1)$ in a common probability space such that

$$\tilde{C}_{\tilde{j}(l)}(t+1) = 1 \Rightarrow C_{j(l)}(t+1) = 1, \quad l = 1,..,Q(t)$$

We distinguish the following cases.

Case 1: No nonempty queue is connected at $t+1$ under $\hat{\pi}$;

In this case no queue is served and we have

$$\sum_{i=1}^{j} Y_i(t+1) = \sum_{i=1}^{j} X_i(t), \; j = 1,..,N \tag{5.11}$$

Since $C_{j(l)}(t+1) = 0, \; l = 1,..,Q(t)$ and because of the coupling of the connectivities we have $\tilde{C}_{\tilde{j}(l)}(t+1) = 0, \; l = 1,..,Q(t)$, therefore no queue with index $j \leq \tilde{j}(Q(t))$ is served and we have

$$\sum_{i=1}^{j} \tilde{Y}_i(t+1) = \sum_{i=1}^{j} \tilde{X}_i(t), \; j \leq \tilde{j}(Q(t)) \tag{5.12}$$

$$\sum_{i=1}^{j} \tilde{Y}_i(t+1) \geq Q(t), \; j > \tilde{j}(Q(t)) \tag{5.13}$$

From (5.11), (5.12) and the induction hypothesis we have $\sum_{i=1}^{j} Y_i(t+1) \leq \sum_{i=1}^{j} \tilde{Y}_i(t+1)$ for $j \leq \tilde{j}(Q(t))$ and from (5.11), (5.13) we have $\sum_{i=1}^{j} Y_i(t+1) = Q(t) \leq \sum_{i=1}^{j} \tilde{Y}_i(t+1)$ for $j > \tilde{j}(Q(t))$. Hence relation (5.8) holds.

Case 2: Some queue is connected at $t+1$ under $\tilde{\pi}$.

If no queue is served under $\tilde{\pi}$ during $t+1$ then $\tilde{Y}_i(t+1) = \tilde{X}_i(t), \; i = 1,..,N$ while $Y_i(t+1) \leq X_i(t), \; i = 1,..,N$ therefore (5.8) follows from the induction hypothesis for $\tau = t+1$. If some queue is served under both policies then give the same service variables to the queues under both policies. If service is not completed then $\mathbf{Y}(t+1) = \mathbf{X}(t)$, $\tilde{\mathbf{Y}}(t+1) = \tilde{\mathbf{X}}(t)$ and (5.8) follows from the induction hypothesis. If service is completed at $t+1$ then let $j_0 = j(l_0)$, $\tilde{j}_0 = \tilde{j}(\tilde{l}_0)$ be the queues served under $\hat{\pi}$ and $\tilde{\pi}$ respectively. Since $\hat{\pi}$ serves the nonempty queue with the smallest probability of being connected we have

$$C_{j(l)}(t+1) = 0, \quad 1 \leq l < l_0$$

From the coupling of the connectivities it is implied that

$$C_{\tilde{j}(l)}(t+1) = 0, \quad 1 \le l < l_0$$

and we have

$$\tilde{j}_0 \ge \tilde{j}(l_0) \tag{5.13a}$$

If $\tilde{j}_0 \ge j_0$ then for $j \ge j_0$ we have

$$\sum_{i=1}^{j} Y_i(t+1) = \sum_{i=1}^{j} X_i(t) - 1 \le \sum_{i=1}^{j} \tilde{X}_i(t) - 1 \le \sum_{i=1}^{j} \tilde{Y}_i(t+1) \tag{5.14}$$

and for $j < j_0$ we have

$$\sum_{i=1}^{j} \tilde{Y}_i(t+1) = \sum_{i=1}^{j} \tilde{X}_i(t) \ge \sum_{i=1}^{j} X_i(t) = \sum_{i=1}^{j} Y_i(t+1) \tag{5.15}$$

From (5.14) and (5.15), (5.8) follows. If $\tilde{j}_0 < j_0$ then for $j < \tilde{j}_0$ we have

$$\sum_{i=1}^{j} \tilde{Y}_i(t+1) = \sum_{i=1}^{j} \tilde{X}_i(t) \ge \sum_{i=1}^{j} X_i(t) = \sum_{i=1}^{j} Y_i(t+1) \tag{5.16}$$

For $j \ge j_0$ we have

$$\sum_{i=1}^{j} Y_i(t+1) = \sum_{i=1}^{j} X_i(t) - 1 \le \sum_{i=1}^{j} \tilde{X}_i(t) - 1 = \sum_{i=1}^{j} \tilde{Y}_i(t+1) \tag{5.17}$$

For $\tilde{j}_0 \le j < j_0$ and because of (5.13a) we have

$$\sum_{i=1}^{j} Y_i(t+1) = \sum_{i=1}^{j} X_i(t) \le l_0 - 1 \le \sum_{i=1}^{j} \tilde{X}_i(t) - 1 = \sum_{i=1}^{j} \tilde{Y}_i(t+1) \tag{5.18}$$

From (5.16), (5.17), (5.18) we get (5.8) for $\tau = t+1$.

Now given that (5.8) holds we show that (5.7) holds at $t+1$. Let $\tilde{m}(l)$, $m(l)$ be the lth empty queue starting from queue 1 for the states $\tilde{\mathbf{Y}}(t+1)$,

$\mathbf{Y}(t+1)$ respectively. Let $Q'(t+1)$, $\tilde{Q}'(t+1)$ be the number of packets in the system when the states are $\mathbf{Y}(t+1)$, $\tilde{\mathbf{Y}}(t+1)$ respectively. We couple the arrivals under the two policies such that

$$A_{m(l)}(t+1) = \tilde{A}_{\tilde{m}(l)}(t+1), \quad l = 1,..,N - \tilde{Q}'(t+1)$$

Consider an arbitrary queue j and let k and $\tilde{k}$ be the number of empty queues with index less than or equal to j under $\hat{\pi}$ and $\tilde{\pi}$ respectively. Because of (5.8) we have $k \geq \tilde{k}$. We get

$$\sum_{i=1}^{j} \tilde{X}_i(t+1) = \sum_{i=1}^{j} \tilde{Y}_i(t+1) + \sum_{l=1}^{\tilde{k}} \tilde{A}_{\tilde{m}(l)}(t+1) \tag{5.19}$$

$$\sum_{i=1}^{j} X_i(t+1) = \sum_{i=1}^{j} Y_i(t+1) + \sum_{l=1}^{k} A_{m(l)}(t+1) \tag{5.20}$$

$$\sum_{i=1}^{j} \tilde{Y}_i(t+1) - \sum_{i+1}^{j} Y_i(t+1) = k - \tilde{k} \tag{5.21}$$

From the coupling of the arrivals we have

$$\sum_{i=1}^{k} A_{m(l)}(t+1) - \sum_{i=1}^{\tilde{k}} \tilde{A}_{\tilde{m}(l)}(t+1) = \sum_{l=\tilde{k}+1}^{k} A_{m(l)}(t+1) \leq k - \tilde{k} \tag{5.22}$$

Subtracting (5.20) from (5.19) and replacing from (5.21), (5.22) we get

$$\sum_{i=1}^{j} \tilde{X}_i(t+1) - \sum_{i=1}^{j} X_i(t+1) \geq 0, \quad j = 1,..,N$$

Hence (5.7) holds for $\tau = t+1$. Notice that

$$\mathbf{X}(t) \prec \tilde{\mathbf{X}}(t) \Rightarrow Q(t) \leq \tilde{Q}(t), \quad t = 1,..$$

therefore (5.7) implies (5.2).

Now we show (5.3). Let $j'(l)$ and $\tilde{j}'(l)$ be the lth nonempty queues, starting from queue 1, for the states $\mathbf{Y}(t+1)$ and $\tilde{\mathbf{Y}}(t+1)$ respectively. Couple the arrivals at $t+1$ as follows

$$A_{m(l)}(t+1) = \tilde{A}_{\tilde{m}(l)}(t+1), \quad l = 1, .., Q'(t+1) \tag{5.23}$$

For the number of blocked packets we have

$$B(t+1) = \sum_{l=1}^{Q'(t+1)} A_{j(l)}(t+1), \quad \tilde{B}(t+1) = \sum_{l=1}^{\tilde{Q}'(t+1)} \tilde{A}_{\tilde{j}(l)}(t+1)$$

and from (5.23)

$$\tilde{B}(t+1) - B(t+1) = \sum_{l=1}^{\tilde{Q}'(t+1)} \tilde{A}_{\tilde{j}(l)}(t+1) - \sum_{l=1}^{Q'(t+1)} A_{j(l)}(t+1)$$

$$= \sum_{l=Q'(t+1)}^{\tilde{Q}'(t+1)} \tilde{A}_{\tilde{j}(l)}(t+1) \geq 0$$

therefore (5.3) holds. ◇

CHAPTER 5

Stabilization of a general queueing network

5.1 Introduction

In this chapter we study a general queueing network where there is routing and flow control at each queue. The control objective is similar to that of chapter 2 that is stabilization of the system for a wide range of arrival and service rates. Necessary and sufficient stabilizability conditions are obtained which are expressed in a direct manner on the arrival and service rates. The stability condition that we obtain for the queueing network allow us to give an independent proof to the well known maxflow-mincut theorem that holds in deterministic flow networks, by constructing an appropriate queueing network for a given flow network and considering the average rates of the customer flows when the network is stable. Finally a model is considered where we assume that the controller obtains delayed information about the queue lengths in the network. Stability results are obtained for this case as well.

Notice that in the model we consider in this section there are no constraints in the servers as there were in the model of chapter 2 and this network does not model directly a radio network. Nevertheless it provides considerable insight on the stabilizability properties of queueing networks as well as its connections with flows in deterministic networks.

This chapter is organized as follows. After the specification of the model in section 5.2 we state and prove our main stability results in section 5.3. In section 5.4 we apply the stability results obtained in section 5.3 in deterministic flow networks. In section 5.5 we consider the network stability problem in the case where the queues get delayed information about the lengths of their neighboring queues.

5.2 The controlled queueing system

Consider a queueing network consisting of M queues and L classes of arriving customers. It is represented by a directed graph G (fig. 7) which contains one node (black) for each class of arriving customers, one node (white) for each queue and one destination node (D). The customers of class l arrive according to a Poisson process with rate a_l. Upon arrival they join one of the queues of the set S_l^c which contains all the terminal queues of the arcs originating from the node that correspond to class l. Each queue i possesses a single exponential server, which may either idle or provide service with rate m_i. A customer of queue i after its service completion is routed to one of the queues of the set S_i that contains all the terminal nodes of the arcs originating at queue i. If node D is contained in the set S_i then the served customers of queue i may be routed out of the system. We assume that the routing of customers upon arrival in the network is controlled and the routing decision may depend on the lengths of all queues of the system,that is the system state, at the arrival time instant t. Similarly the routing of a customer completing service at queue i is controlled and the routing decision is based on the system state at the time instant of service completion. Finally we assume that the service rate of queue i, which can take two values 0 and m_i, is controlled and the decision rule may be a function of the lengths of all queues of the system.

Let us denote by $X_i(t)$ the number of customers in queue i at time t; the customer receiving service at that time is included. The vector of queue lengths of all queues of the system at time t is $\mathbf{X}(t) = (X_i(t) : \ i = 1, ..., M)$ and it takes values in the state space $\mathcal{X} = Z_+^M$. The routing of the arriving customers of class l is specified by a function $R_l^c : \mathcal{X} \rightarrow S_l^c$ in the sense that an arriving customer of class l at time t joins the queue $R_l^c(\mathbf{X}(t-))$ where $\mathbf{X}(t-)$ is the vector of queue lengths just before the time instant t.

The function R_l^c is called the *routing rule* of class l in the following. A served customer of queue i is routed to one of the queues of the set S_i (or out of the system if $D \in S_i$) according to a function $R_i : \mathcal{X} \rightarrow S_i$ in the sense that the customer of queue i completing service at time t joins queue $R_i(\mathbf{X}(t-))$. The function R_i is the *routing rule* of queue i. Finally the service rate of the server of queue i is controlled according to a function $F_i : \mathcal{X} \rightarrow \{0, m_i\}$; the rate of server i at time t is $F_i(\mathbf{X}(t))$. The function F_i is called the *service control rule* of queue i. The control of the service rate available to the queues provide a kind of flow control. An *admissible control policy* for the network consists of a collection of routing rules, one for each customer class and each queue, and service control rules, one for each queue of the system. Let G be the collection of all admissible control policies. Customers of different classes are treated identically by the control policy; also the service times of customers of different classes are identically distributed in each queue. Hence customers of different classes are indistinguishable after they enter the network and they differ only in the set of queues that they can join upon arrival.

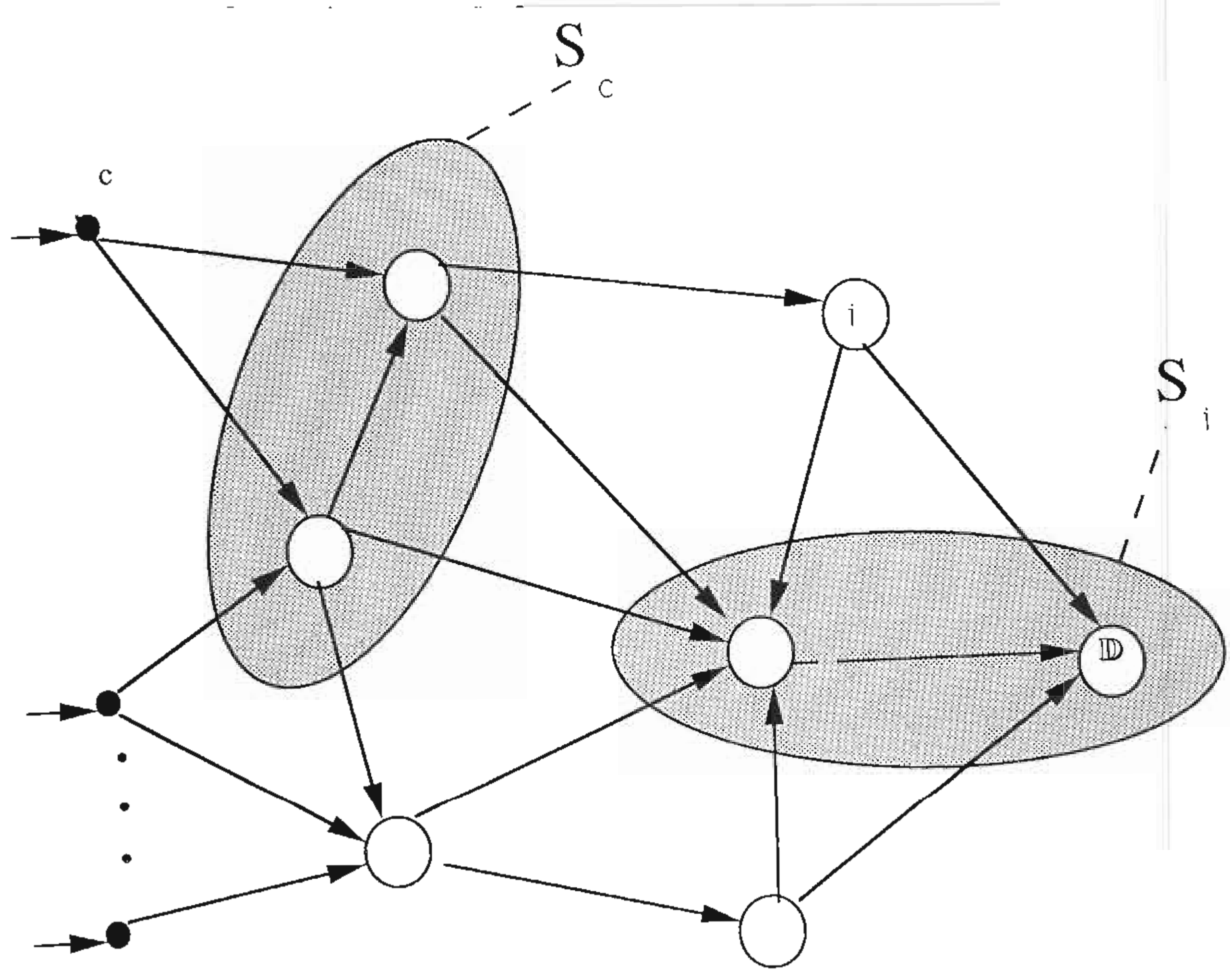

Figure 7. The topology graph of a queueing network.

Remark: If the servers never idle, that is $F_i(\mathbf{x})$ is identically equal to m_i for all i, and the routing, in each queue and each arrival stream, is done by random splitting with some fixed probabilities not dependent on $\mathbf{x}$, then the resulting network is Jacksonian ([4,9]).

When the network is operated by a control policy from G, and since the arrivals are Poissonian and the service times exponentially distributed, the queue length process $\mathbf{X}$ is a Markov chain. The rate $q_{\mathbf{x}\mathbf{x}'}$ of a transition from a state $\mathbf{x}$ to a state $\mathbf{x}'$ is as follows

$$q_{\mathbf{x}\mathbf{x}'} = \begin{cases} m_i, & \text{if } F_i(\mathbf{x}) = m_i,\ x'_i = x_i - 1,\ x'_{R_i(\mathbf{x})} = x_{R_i(\mathbf{x})} + 1 \text{ when } R_i(\mathbf{x}) \neq D, \\ & \text{and } x'_j = x_j \text{ for } j \neq i,\ R_i(\mathbf{x}); \\ a_c, & \text{if } x'_{R^c_l(\mathbf{x})} = x_{R^c_l(\mathbf{x})} + 1, \text{ and } x'_j = x_j \text{ for } j \neq R^c_l(\mathbf{x}). \end{cases}$$

We define stability as follows.

Definition 2.1: The system is *stable* if the distribution of the queue length vector $\mathbf{X}(t)$ converges as $t \to \infty$ to a probability distribution on $\mathcal{X}$.

Since $\mathbf{X}$ is a Markov chain for any policy in G the stability of the system is equivalent to ergodicity of X. Let the *arrival* and *service* rate vectors be $\mathbf{a} = (a_l :\ l = 1, ..., L)$ and $\mathbf{m} = (m_i :\ i = 1, ..., M)$ respectively. We characterize the performance of a policy π by its stability region which is defined in this case as follows.

Definition: The *stability region* C_π of policy π, is the collection of all pairs of vectors $(\mathbf{a}, \mathbf{m})$ for which the system is stable under policy π.

Notice that the stability region is defined to consist of pairs of arrival and service rate vectors unlike section 2.3. The discussion in section 2.3 about the comparison of policies based on their stability regions is valid here as well. Similarly to section 2.3 the *system stability region* is defined by $C = \cup_{\pi \in G} C_\pi$. If a pair $(\mathbf{a}, \mathbf{m})$ belongs to C then it is called *stabilizable.*

Consider the policy π_0 with control rules as follows

$$R_i(\mathbf{x}) = \begin{cases} D, & \text{if } D \in S_i \\ \arg\min_{j \in S_i}(x_j), & \text{otherwise,} \end{cases}$$

$$R_c(\mathbf{x}) = \arg\min_{j \in S_i}(x_j),$$

$$F_i(\mathbf{x}) = \begin{cases} 0, & \text{if } x_i \leq \min_{j \in S_i}\{x_j\} \\ m_i, & \text{otherwise.} \end{cases}$$

If the minimum above is achieved for more than one queue then the argmin is defined to be equal to the minimum index of these queues. Note that in π_0 the routing and the service rate control rules for a queue i use for decision making queue length information from the queues of the set S_i only. The queues of this set usually correspond to neighboring nodes of the physical system under consideration. Henceforth a distributed implementation of the policy is readily available. Policy π_0 has the optimality property that we mentioned earlier. In the next section we will show the optimality of π_0 after we characterize the stability region C of the system.

5.3 Stabilizability results

Given a set of queues S the set C_S is defined to contain every arrival class l the customers of which can not be routed outside of S; that is $S_l \subseteq S$ for every $l \in C_S$. The set Q_S is defined to contain every queue i, the served customers

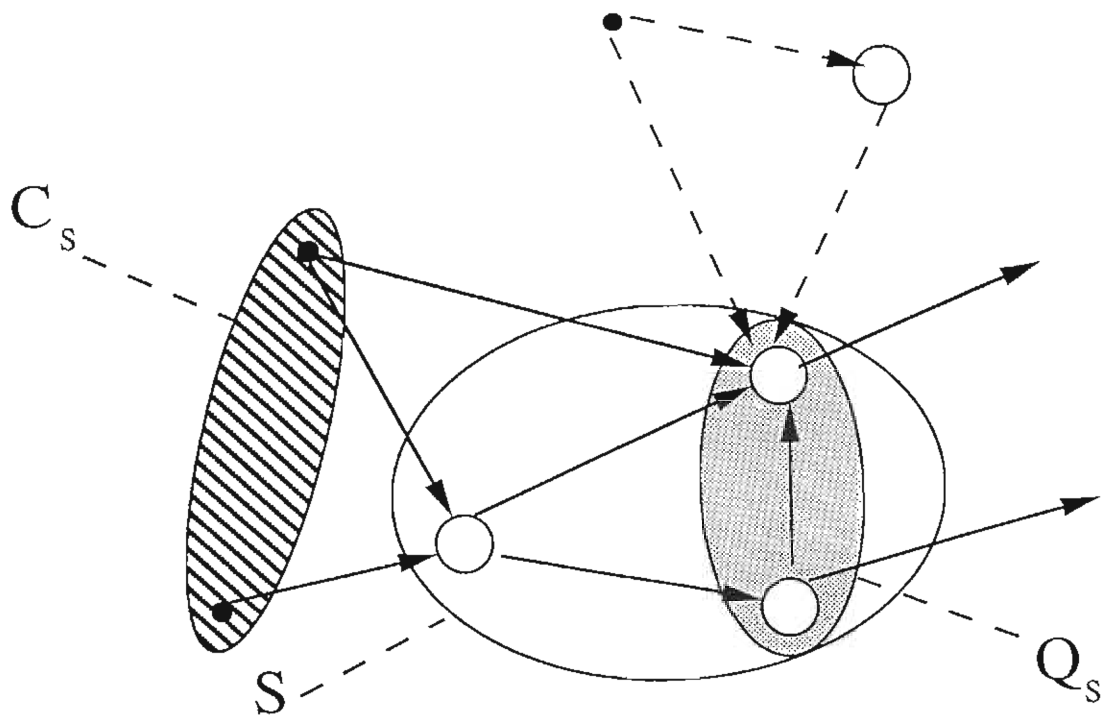

Figure 8. A set of queues S with the corresponding sets of arrival streams and queues that lead customers out of S.

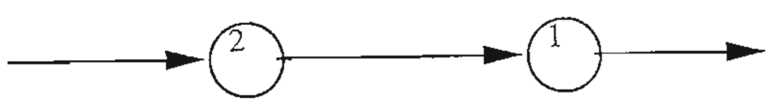

Figure 9. A tandem network.

of which may be routed outside of S; that is $S_i \not\subseteq S$ for every $i \in Q_S$. In fig. 8 the sets C_S and Q_S of arrival streams and queues respectively, are illustrated for a specific set of queues S.

Theorem 3.1:

i) A necessary and sufficient condition for a pair of vectors $(\mathbf{a}, \mathbf{m})$ to belong to C is that

$$\sum_{l \in C_S} a_l < \sum_{i \in Q_S} m_i \ , \qquad \forall S \subset \{1, ..., M\} \tag{3.1}$$

ii) Policy π_0 is optimal in the sense that $C_{\pi_0} = C$.

The proof of the theorem will be given after two lemmas. In the rest of this section, whenever we refer to the process $\mathbf{X}$, we assume that policy π_0 controls the system. The ergodicity of the continous time Markov chain $\mathbf{X}$ is equivalent to the ergodicity of the imbeded discrete time Markov chain ([C082]). This chain, which is denoted by the same symbol as the continous time chain in the following, has the same state space as the continous time and transition probabilities

$$P(\mathbf{X}(t) = \mathbf{y} | \mathbf{X}(t-1) = \mathbf{x}) = -\frac{q_{\mathbf{xy}}}{q_{\mathbf{xx}}}.$$

In the study of the ergodicity in the rest of the section we will refer to the imbeded Markov chain. We need to characterize first some structural properties of the Markov chain X. Recal from section 2.3 the definition of reachability as well as of whether two states communicate or not. The relationship "communicate" is an equivalence relationship. We need to characterize the classes of equivalent states of the chain with respect to this relationship. Note first that $\mathbf{X}$ is not necessarily irreducible, that is $\mathcal{X}$ is not a single equivalence class. Consider for example the simple network in fig. 9. If the initial state $\mathbf{x}_0$ is such that $x_1^0 \leq x_2^0$ then none of the states $\mathbf{x}$ such that $x_1 > x_2$ is reachable. The following lemma provides a classification of the states of the process X. State $\mathbf{0}$ corresponds to the empty network.

Lemma 3.1: If π_0 acts on the queueing system and (3.1) holds, then the subset of the state space

$$\mathcal{R} = \{\mathbf{x} : \ \mathbf{x} \in \mathcal{X} \text{ and } \mathbf{x} \text{ can be reached from state } \mathbf{0}\}$$

is the unique closed class of equivalent states of the Markov chain; furthermore any state of the set $\mathcal{X} - \mathcal{R}$ is transient.

Proof: We show first that the state $\mathbf{0}$ can be reached from any other state $\mathbf{x} \in \mathcal{X}$. For each queue i let w_i be the number of hops (links) of the minimum hop path from the node of queue i to node D in the topology graph of the network. Apparently $w_i > 0$ for $i = 1,..,M$. Consider the linear function $W(\mathbf{x}) = \sum_{i=1}^{M} w_i x_i$ on $\mathcal{X}$. We claim that if for some state $\mathbf{x} \in \mathcal{X}$ we have $W(\mathbf{x}) > 0$ then there exists a transition with positive rate from $\mathbf{x}$ to some state $\mathbf{x}'$ which is such that $W(\mathbf{x}) - W(\mathbf{x}') = 1$. Consider the index $d = \arg\min_{\{j:\ x_j > 0\}}\{w_j\}$ which is well defined since $W(\mathbf{x}) > 0$. We claim that a service completion at queue d will lead the system in the state $\mathbf{x}'$ with the above property. We distinguish the following cases.

Case 1 : $w_d = 1$. Since $w_d = 1$, the queue d may direct customers out of the system; according to policy π_0 all served customers of queue i will be directed out of the system. Hence a service completion at queue d will lead the system in a state $\mathbf{x}'$ such that $x'_d = x_d - 1$ and $x'_i = x_i$ for $i \neq d$. Apparently $W(\mathbf{x}') = W(\mathbf{x}) - 1$.

Case 2 : $w_d > 1$. By the definition of w_i we have that

$$w_i = \begin{cases} 1, & \text{if } D \in S_i, \\ 1 + \min_{j \in S_i}\{w_j\}, & \text{otherwise.} \end{cases}$$

Hence there exists a queue l in S_d such that $x_l = 0$ and $w_l = w_d - 1$. A served customer of queue d will join an empty queue in S_d and we can always define appropriately the indexing of the queues such that the served customer will join the queue l. Apparently the new state $\mathbf{x}'$ will be such that $W(\mathbf{x}') \leq W(\mathbf{x}) - 1$.

We can easily see now that, by selecting the transitions appropriately, from state $\mathbf{x}$ after $W(\mathbf{x})$ transitions, we can eventually reach state $\mathbf{x}'$ such that $W(\mathbf{x}') = 0$. Apparently $\mathbf{x}' = \mathbf{0}$ since for any other state the function W is strictly positive.

By definition of the set R, and from the above result, any of its states communicates with zero; hence any two states of the set communicate as well. Apparently the set R is closed since if a state can be reached by a state in R it can be reached by $\mathbf{0}$ as well and it belongs to R itself. Hence the set R is a closed equivalence class of states. No state outside of R can belong to a closed class of states since any state can reach the state $\mathbf{0} \in R$. Because of that it can not belong to a closed equivalence class of states; the latter condition is necessary for recurrence ([Ja57]). Hence any state $\mathbf{x} \in (\mathcal{X} - R)$ is transient. $\diamond$

Lemma 3.1 implies that the chain $\mathbf{X}$ is not irreducible; hence Foster's criterion is not directly applicable for the study of this chain. We will use a generalization of this criterion which appears in [Tw76] and applies to nonirreducible chains as well. We need first the notion of ϕ-irreducibility which we introduce next.

Definition 3.1. Suppose ϕ is a σ-finite non-trivial measure on the state space $\mathcal{X}$ (countable) of a Markov chain $\mathbf{X}$. Then $\mathbf{X}$ is called ϕ-irreducible if

$$\sum_{t=1}^{\infty} P[\mathbf{x}(t) \in A | \mathbf{x}(0) = x] > 0$$

whenever $\phi(A) > 0$ and for every $\mathbf{x} \in \mathcal{X}$. The following theorem appears in [Tw76].

Theorem 3.2: Let $\mathbf{X}$ be the imbeded chain of a continuous time Markov chain with state space $\mathcal{X}$ and transition rate matrix Q. Let $q_{\mathbf{x}}$ be defined by

$$q_{\mathbf{x}} = \sum_{\mathbf{y} \in X} q_{\mathbf{xy}} \quad , \quad \mathbf{x} \in \mathcal{X} \ .$$

Then a sufficient condition of ergodicity of $\mathbf{X}$ is that we have $\sup_{\mathbf{x} \in \mathcal{X}} q_{\mathbf{x}} < \infty$ and there exists a function $V : \mathcal{X} \to R$, an $\epsilon > 0$ and a finite set $\mathcal{X}_0 \subset \mathcal{X}$ such that

$$-\epsilon > \sum_{\mathbf{y} \in X} \frac{q_{\mathbf{xy}}}{q_{\mathbf{x}}} V(\mathbf{y}) - V(\mathbf{x}) \quad , \quad \mathbf{x} \notin \mathcal{X}_0$$

and

$$\sum_{\mathbf{y}\in X} \frac{q_{\mathbf{xy}}}{q_{\mathbf{x}}} V(\mathbf{y}) < \infty \quad , \quad \mathbf{x} \in \mathcal{X}_0.$$

Lemma 3.2: Under policy π_0 and when condition (3.1) holds then the Markov chain $\mathbf{X}$ is ϕ-irreducible where ϕ is the measure in $\mathcal{X}$ defined as follows

$$\phi(\{\mathbf{x}\}) = \begin{cases} 1, & \text{if } \mathbf{x} \in \mathcal{R}, \\ 0, & \text{if } \mathbf{x} \in \mathcal{X} - \mathcal{R} \end{cases}.$$

Proof: If for a set $A \subset \mathcal{X}$ we have $\phi(A) > 0$ then apparently $A \bigcap \mathcal{R} \neq \emptyset$. Consider an element $\mathbf{y} \in A \bigcap \mathcal{R}$. Since $\mathbf{y} \in \mathcal{R}$ and because of lemma 3.1 we have that from any state $\mathbf{x} \in \mathcal{X}$ there exists a sequence of transitions that lead the system from $\mathbf{x}$ to $\mathbf{y}$. Assume that there exists a sequence of k transitions that lead the system from $\mathbf{x}$ to $\mathbf{y}$. Then from standard Markov chain theory ([KSK76]) and for all $m \geq k$ we have

$$P[\mathbf{X}(t) \in A | \mathbf{X}(0) = \mathbf{x}] \geq P[\mathbf{X}(t) = y | \mathbf{X}(0) = \mathbf{x}] > 0$$

which proves the lemma. ◇

Lemma 3.3: The stability region C_{π_0} contains all pairs of arrival and service rate vectors $(\mathbf{a}, \mathbf{m})$ for which (3.1) holds

Proof: Let $q_{\mathbf{x}}$ be defined by $q_{\mathbf{x}} = \sum_{\mathbf{y}\in\mathcal{X}} q_{\mathbf{xy}}, \quad \mathbf{x} \in \mathcal{X}$. The transition rate $q_{\mathbf{xy}}$ is greater than zero only if the transition from $\mathbf{x}$ to $\mathbf{y}$ corresponds to an arrival or service completion; hence we have

$$q_{\mathbf{x}} \leq \sum_{l=1}^{L} a_l + \sum_{i=1}^{M} m_i \quad \forall \mathbf{x} \in \mathcal{X}. \tag{3.5}$$

Consider the function V defined on the state space $\mathcal{X}$ of the chain by $V(\mathbf{x}) = \sum_{i=1}^{M}(x_i)^2$. As we noted earlier, at each transition of the chain the length of each queue varies at most by one; hence the following

$$\sum_{\mathbf{y}\in\mathcal{X}} \frac{q_{\mathbf{xy}}}{q_{\mathbf{x}}} V(\mathbf{y}) \leq 2V(\mathbf{x}) < \infty, \quad \mathbf{x} \in \mathcal{X}. \tag{3.6}$$

Apparently the set V_b defined by $V_b = \{\mathbf{x} : \mathbf{x} \in \mathcal{X}, \quad V(\mathbf{x}) \leq b\}$ has finite cardinality for all b. In the following we will show that for some fixed $\epsilon > 0$ there exists some b, which may be a function of the arrival and service rates, such that

$$-\epsilon \geq \sum_{\mathbf{y}\in\mathcal{X}} \frac{q_{\mathbf{xy}}}{q_{\mathbf{x}}} V(\mathbf{y}) - V(\mathbf{x}), \quad if\ \mathbf{x} \notin V_b. \tag{3.7}$$

Then, from (3.5-3.7) and using theorem 3.2 we can conclude that $\mathbf{X}$ is ergodic. By simple calculations in (3.7) we get

$$\sum_{\mathbf{y}\in\mathcal{X}} \frac{q_{\mathbf{xy}}}{q_{\mathbf{x}}} V(\mathbf{y}) - V(\mathbf{x}) = \sum_{\mathbf{y}\in\mathcal{X}} \frac{q_{\mathbf{xy}}}{q_{\mathbf{x}}} (\sum_{i=1}^{M} y_i^2 - \sum_{i=1}^{M} x_i^2)$$

$$= \sum_{\mathbf{y}\in\mathcal{X}} \frac{q_{\mathbf{xy}}}{q_{\mathbf{x}}} (\sum_{i=1}^{M} (2x_i(y_i - x_i) + (y_i - x_i)^2)$$

$$= \sum_{y\in\mathcal{X}} \frac{q_{\mathbf{xy}}}{q_{\mathbf{x}}} \sum_{i=1}^{M} 2x_i(y_i - x_i) + \sum_{\mathbf{y}\in\mathcal{X}} \frac{q_{\mathbf{xy}}}{q_{\mathbf{x}}} \sum_{i=1}^{M} (y_i - x_i)^2 \tag{3.8}$$

When $q_{\mathbf{xy}}$ is different than 0, the transition from $\mathbf{x}$ to $\mathbf{y}$ corresponds either to an arrival or to a service completion; hence the the states $\mathbf{x}$ and $\mathbf{y}$ differ in at most two elements and each difference is at most one. Hence we have $\sum_{i=1}^{M}(x_i - y_i)^2 \leq 2$ and for the second term in the right hand side of (3.8) we get

$$\sum_{\mathbf{y}\in\mathcal{X}} \frac{q_{\mathbf{x}\mathbf{y}}}{q_{\mathbf{x}}} \sum_{i=1}^{M} (x_i - y_i)^2 \le 2. \tag{3.9}$$

Now we will bound the first term of the sum in the right hand side of (3.8). Let $a_i(\mathbf{x})$ be the sum of the arrival rates over all customer classes which route the customers upon arrival to queue i when the system is in state $\mathbf{x}$. Define $m_{ij}(\mathbf{x})$ by

$$m_{ij}(\mathbf{x}) = \begin{cases} F_i(\mathbf{x}), & \text{if } R_i(\mathbf{x}) = j \\ 0, & \text{otherwise.} \end{cases}$$

By grouping together, in the first part of (3.8), the terms that correspond to the same queue and since $q_{\mathbf{x}\mathbf{y}} > 0$ whenever the transition from $\mathbf{x}$ to $\mathbf{y}$ corresponds to an arrival or service completion we have

$$\sum_{\mathbf{y}\in\mathcal{X}} \frac{q_{\mathbf{x}\mathbf{y}}}{q_{\mathbf{x}}} \sum_{i=1}^{M} 2x_i(y_i - x_i) = \frac{2}{q_{\mathbf{x}}} \sum_{i=1}^{M} \sum_{\mathbf{y}\in\mathcal{X}} q_{\mathbf{x}\mathbf{y}} x_i (y_i - x_i)$$

$$= \frac{2}{q_{\mathbf{x}}} \sum_{i=1}^{M} x_i \Big(a_i(\mathbf{x}) - F_i(\mathbf{x}) + \sum_{j=1}^{M} m_{ji}(\mathbf{x})\Big) \tag{3.10}$$

When we are in state $\mathbf{x}$ consider a permutation $i_1, i_2, ..., i_M$ of the queues which is such that $x_{i_{m-1}} \le x_{i_m}$, $m = 2, ..., M$ and if $x_{i_{m-1}} = x_{i_m}$ then $i_{m-1} < i_m$. Apparently the permutation is a function of the state Note that if queue i_l routes the served customers to queue i_m then no queue i_k for $k < m$ belongs to S_{i_l}. In view of this observation the right part of (3.10) can be written as

$$\sum_{i=1}^{M} x_i\Big(a_i(\mathbf{x}) - F_i(\mathbf{x}) + \sum_{j=1}^{M} m_{ji}(\mathbf{x})\Big) = \sum_{j=1}^{M} x_{i_j}\Big(a_{i_j}(\mathbf{x}) - F_{i_j}(\mathbf{x}) + \sum_{l=j+1}^{M} m_{i_l i_j}(\mathbf{x})\Big). \tag{3.11}$$

For $j = 1, ..., M-1$ we write

$$a_{i_j}(\mathbf{x}) - F_{i_j}(\mathbf{x}) + \sum_{l=j+1}^{M} m_{i_l i_j}(\mathbf{x}) = \sum_{m=j}^{M} a_{i_m}(\mathbf{x}) - \sum_{m=j+1}^{M} a_{i_m}(\mathbf{x})$$

$$- \sum_{m=j}^{M} F_{i_m}(\mathbf{x}) + \sum_{m=j+1}^{M} F_{i_m}(\mathbf{x}) + \sum_{m=j}^{M} \sum_{l=m+1}^{M} m_{i_l i_m}(\mathbf{x}) - \sum_{m=j+1}^{M} \sum_{l=m+1}^{M} m_{i_l i_m}(\mathbf{x}) \tag{3.11a}$$

By substituting (3.11a) in (3.11) for $j = 1, ..., M-1$ and after some calculations we get

$$\sum_{j=1}^{M} x_{i_j}(a_{i_j}(\mathbf{x}) - F_{i_j}(\mathbf{x}) + \sum_{l=j+1}^{M} m_{i_l i_j}(\mathbf{x})) =$$

$$= \sum_{j=2}^{M} (x_{i_j} - x_{i_{j-1}})(\sum_{l=j}^{M} a_{i_l}(\mathbf{x}) - \sum_{l=j}^{M} F_{i_l}(\mathbf{x}) + \sum_{m=j}^{M} \sum_{l=m+1}^{M} m_{i_l i_m}(\mathbf{x}))$$

$$+ x_{i_1}(\sum_{l=1}^{M} a_{i_l} - \sum_{l=1}^{M} F_{i_l}(\mathbf{x}) + \sum_{m=1}^{M} \sum_{l=m+1}^{M} m_{i_l i_m}(\mathbf{x})). \tag{3.12}$$

Consider the sets of queues $T_j = \{i_l : M \geq l \geq j\}$, $j = 1, ..., L$. If an incomming customer of class l is routed to some queue of T_j it means that $S_l^c \subset T_j$, since otherwise the routed customer would have been routed out of C_{T_j}; hence $l \in C_{T_j}$ and we have

$$\sum_{l=j}^{M} a_{i_l}(\mathbf{x}) = \sum_{l \in C_{T_j}} a_l. \tag{3.13}$$

For any $i_l, i_m \in T_j$ we have $m_{i_l i_m}(\mathbf{x}) > 0$ only if $i_l \notin Q_{T_j}$; thus we get

$$\sum_{m=j}^{M} \sum_{l=m+1}^{M} m_{i_l i_m}(\mathbf{x}) - \sum_{l=j}^{M} F_{i_l}(\mathbf{x}) = - \sum_{l \in Q_{T_j}} F_l(\mathbf{x}). \tag{3.14}$$

Relations (3.10-3.14) imply that

$$\sum_{\mathbf{y} \in \mathcal{X}} \frac{q_{\mathbf{xy}}}{q_{\mathbf{x}}} \sum_{i=1}^{M} 2x_i(y_i - x_i) =$$

$$= \frac{2}{q_{\mathbf{x}}} \sum_{j=2}^{M} (x_{i_j} - x_{i_{j-1}}) (\sum_{l \in C_{T_j}} a_l - \sum_{l \in Q_{T_j}} F_l(\mathbf{x})) + \frac{2}{q_{\mathbf{x}}} (\sum_{l \in C_{T_j}} a_l - \sum_{l \in Q_{T_j}} F_l(\mathbf{x})) \quad (3.15)$$

Whenever $x_{i_j} > x_{i_{j-1}}$, the servers in any queue in Q_{T_j} are active since they can route their customers in some queue out of T_j which has smaller length than they have. Hence we have

$$\sum_{l \in Q_{T_j}} F_l(\mathbf{x}) = \sum_{l \in Q_{T_j}} m_l \quad \text{if} \quad x_{i_j} > x_{i_{j-1}}. \tag{3.15a}$$

From (3.15a), (3.15) can be written as

$$\sum_{\mathbf{y} \in \mathcal{X}} \frac{q_{\mathbf{xy}}}{q_{\mathbf{x}}} \sum_{i=1}^{M} 2x_i(y_i - x_i) =$$

$$= \frac{2}{q_{\mathbf{x}}} \sum_{j=2}^{M} (x_{i_j} - x_{i_{j-1}}) (\sum_{l \in C_{T_j}} a_l - \sum_{l \in Q_{T_j}} m_l) + \frac{2}{q_{\mathbf{x}}} (\sum_{l \in C_{T_j}} a_l - \sum_{l \in Q_{T_j}} m_l). \tag{3.15b}$$

Consider the number c defined by

$$c = \max_{S \subset \{1,\ldots,M\}} \{ \sum_{c \in C_S} a_c - \sum_{i \in Q_S} m_i \}. \tag{3.16}$$

From (3.15b-3.16) we get

$$\sum_{\mathbf{y} \in \mathcal{X}} \frac{q_{\mathbf{xy}}}{q_{\mathbf{x}}} \sum_{i=1}^{M} 2x_i(y_i - x_i) \leq \frac{2}{q_{\mathbf{x}}} c x_{i_M}. \tag{3.17}$$

We can easily see that the relation $V(\mathbf{x}) \geq b$ implies that

$$x_{i_M} \geq \sqrt{\frac{b}{M}} \tag{3.18}$$

We denote by d the right hand side of (3.5) in the following. From (3.17-3.18) we get

$$\sum_{\mathbf{y} \in \mathcal{X}} \frac{q_{\mathbf{xy}}}{q_{\mathbf{x}}} \sum_{i=1}^{M} 2x_i(y_i - x_i) \leq \frac{2}{d} c \sqrt{\frac{b}{M}} \tag{3.20}$$

Equations (3.8), (3.9) and (3.20) give

$$\sum_{\mathbf{y}\in\mathcal{X}} \frac{q_{\mathbf{xy}}}{q_{\mathbf{x}}} V(\mathbf{y}) - V(\mathbf{x}) \leq 2 + \frac{2}{d} c \sqrt{\frac{b}{M}} \tag{3.21}$$

If condition (3.1) holds, then c will be negative as defined. If in (3.21) we replace b by $M(\frac{d}{2c}(2+\epsilon))^2$ then we get the desired relationship (3.7). ◇

Proof of theorem 3.1: We show first the necessity of part i. Assume that for the set of queues S we have

$$\sum_{l\in C_S} a_l \geq \sum_{i\in Q_S} m_i. \tag{3.22}$$

and there exists a policy that stabilizes the network. Then the Markov chain of queue lengths $\mathbf{x}(t)$ under this policy is ergodic and it has a stationary distribution. Assume that we start the network with the stationary distribution; then $\mathbf{X}(t)$ is a stationary ergodic process. Let $X_S(t)$ be the total number of packets at the queues of the set S at time t. Let $D(t)$, $A(t)$ be the processes of departures by any queue of S and arrivals at any queue of the set S respectively, until time t. Then we have

$$X_S(t) = X_S(0) + D(t) - A(t). \tag{3.23}$$

The rate of $D(t)$ varies between 0 and $\sum_{i\in Q_S} m_i$ when all the queues of the set Q_S are either empty, or full and route customers out of the system, respectively. Since $\mathbf{X}$ is stationary and ergodic the ratio $\frac{D(t)}{t}$ converges, almost surely as $t\to\infty$, to a number d; that number is strictly less than $\sum_{i\in Q_S} m_i$ since the network is empty with some positive probability in which case the departure rate from the set S is equal to 0. The rate of the arrival process is at any time greater than or equal to $\sum_{l \in C_S} a_l$. Hence the ratio $\frac{A(t)}{t}$ converges almost surely to a number $\lambda \geq \sum_{l \in C_S} a_l$. Then from (3.22) and (3.23) we can conclude that $X_S(t)$ goes to infinity a.s. as $t \rightarrow \infty$ which contradicts the fact that $\mathbf{X}(t)$ is ergodic.

Lemma (3.3) implies the sufficiency of condition (3.1) in part i. Part ii follows immediately from lemma 3.3 and the above necessity result. $\diamond$

Stabilization of a Jacksonian network

In the network that we considered above assume that a server never idles if its queue is not empty. For each queue i consider the splitting probabilities p_{ij}, $j \in S_i$ such that $0 \leq p_{ij} \leq 1$, $\sum_{j \in S_i} p_{ij} = 1$. At each service completion instant at queue i the served customer is routed within S_i according to these splitting probabilities and independently of everything else in the system. Similarly each arriving customer of class l is routed within S_l^c according to the splitting probabilities p_{lj}^c, $j \in S_l^c$. The above routing policy is called *random splitting* policy in the following. As we mentioned earlier, under any random splitting policy, the queueing network is Jacksonian. We show that condition (3.1) is sufficient for the existence of a random splitting policy that stabilizes the system. The stability condition for a Jacksonian network is that the system of equations

$$a_i = \gamma_i + \sum_{j: i \in S_j} a_j p_{ji} \ , \quad 1 \leq i \leq M \tag{3.24}$$

has a solution $(a_1, ..., a_M)$ such that $a_i < m_i$, $i = 1, ..., M$ ([Wa88]); where p_{ji} are as defined above and γ_i is the total arrival rate at queue i from the outside, that is in our case $\gamma_i = \sum_{l:i\in S_l^c} p_{li} a_l$.

Consider now the network operated under π_0. Under (3.1) the network is ergodic when π_0 acts on it. Consider it in stationary operation and let a_i be the departure rate from queue i. Let furthermore q_{lj}^c be the rate of exogeneous arrivals of class l which are routed to queue j and q_{ij} the rate of the served customers of queue i which are routed to queue j. Since the network is ergodic, at each queue i we have

$$a_i = \sum_{l:j\in S_l^c} q_{lj}^c + \sum_{j:i\in S_j} q_{ji}$$

$$\Rightarrow \quad a_i = \sum_{l:j\in S_l^c} \frac{q_{lj}^c}{a_l} a_l + \sum_{l:j\in S_i^j} \frac{q_{ji}}{a_j} a_j. \tag{3.25}$$

Consider the random splitting policy with splitting probabilities for queue j, $p_{ji} = \frac{q_{ji}}{a_j}$, $i \in S_j$ and for stream l, $p_{lj}^c = \frac{q_{lj}^c}{a_l}$ (apparently the conditions for being splitting probabilities are satisfied). For this random splitting policy, the departure rates of each queue is a solution of the system of equations (3.24) as indicated from (3.25) and the network is ergodic.

5.4 An alternative proof of the maxflow mincut theorem

The maxflow-mincut theorem provides a characterization of the solution of the maxflow problem in deterministic flow networks ([PaS82]). The proof that has been given to this theorem ([FoF53]) is based on duality theory and algorithmic arguments. In this section we show how this theorem is implied by the stability results we obtained in the previous section. The maxflow problem and the maxflow-mincut theorem has been stated in section 2.4 for the specific network studied in that section. For the sake of completeness we briefly state them in this section as well. For more details on this subject the reader is referred to [PS82].

A flow network consists of a connectivity graph $G = (V, E)$, a capacity assignment on the links $C : E \to R^+$, a prespecified origin node v_0 and a prespecified destination node v_d (fig. 10). Without loss of generality we assume that there is no edge terminating at node v_0 or originating at node v_d. A *feasible flow* is a vector $\mathbf{f} = (f_e : e \in E)$ that satisfies the capacity constraints $0 \le f_e \le C(e)$ and the flow conservation equations

$$\sum_{\substack{e \text{ originates} \\ \text{at } v}} f_e = \sum_{\substack{e \text{ terminates} \\ \text{at } v}} f_e \qquad v \in (V - \{v_0, v_d\}). \tag{4.1}$$

Let F be the set of feasible flows. The *flow transfer* f of a feasible flow $\mathbf{f}$ from node v_0 to v_d is

$$f = \sum_{\substack{e \text{ originates} \\ \text{at } v_0}} f_e.$$

The maximum flow problem asks for the maximization of the flow transfer f over the set of feasible flows. That is

$$\max_{\mathbf{f} \in F} f.$$

A flow that achieves the maximum flow transfer is a maxflow.

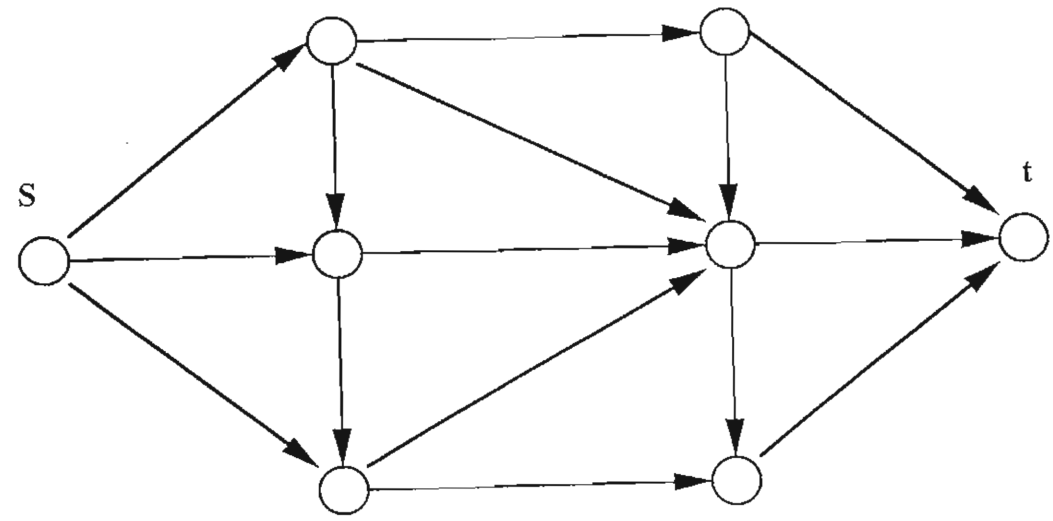

Figure 10. A flow network.

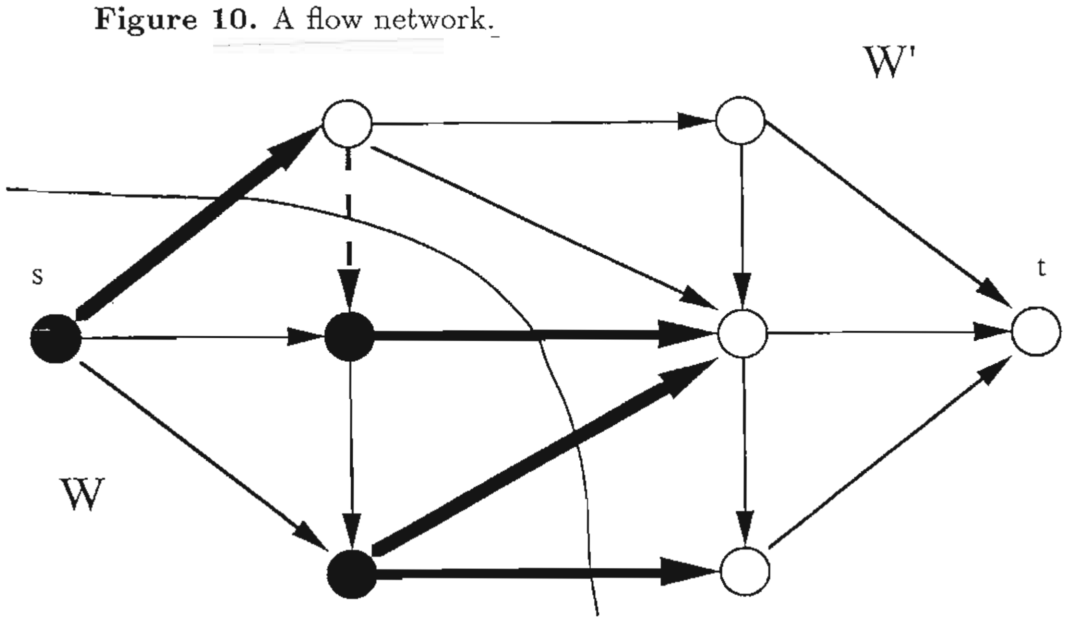

Figure 11. A cut with the forward edges in boldface.

The basic theorem that characterizes the solution of the maximum flow problem is the maxflow-mincut theorem. We need the notion of a *cut* of a flow network in order to state that theorem. This is defined as a partition of the set of nodes V into two sets W, W' such that the set W contains the node v_0 and the set W' contains the node v_d (fig. 11). A *forward* link of the cut is directed from a node of W to a node of W'. The capacity $C(W, W')$ of the cut equals to the sum of the capacities of the forward links. A mincut is a cut with minimum capacity over all the cuts.

Maxflow-mincut Theorem: In a flow network the flow transfer of a maxflow equals to the capacity of a mincut.

The proof which follows is based on the stability properties of the queueing network that we studied in section 5.3.

For a given flow network, let say N, we construct a corresponding queueing network Q_N as follows. We consider one queue $q_{(v,w)}$ for each link (v, w) of the flow network. The served customers of queue $q_{(v,w)}$ can be routed to any queue that corresponds to links originated at node w; if w is the destination node v_d then the served customers of $q_{(v,w)}$ may leave the system. There is only one stream of arriving customers with rate λ; the arriving customers can be routed to any queue $q_{(v_0,w)}$ that corresponds to the link (v_0, w) which originates from node v_0. In the following we are going to use interchangably the links of N and the corresponding queues of Q_N.

Lemma 4.1: If the queueing network Q_N is stabilizable when the arrival rate is λ then there exists a feasible flow $\mathbf{f}$ in the flow network with flow transfer λ.

Proof: If the queueing network is stabilizable then under π_0 the Markov chain $\mathbf{x}(t)$ has a stationary distribution. We start the network with the stationary distribution and the process $\mathbf{X}(t)$ is stationary and ergodic. Consider a vector $\mathbf{f} \in R_+^{|E|}$ such that the element f_e that corresponds to link e equals to the rate of the departure process of the queue that corresponds to link e. Note that since $\mathbf{X}(\mathrm{t})$ is a stationary process, the departure process of each queue is stationary and the rate is well defined. We claim that the flow vector $\mathbf{f}$ is a feasible flow for the network N with flow transfer equal to λ. The rate of the departure process in queue i is less than or equal to its service rate m_i which by definition equals to the capacity of the corresponding link of N. Hence $\mathbf{f}$ satisfies the capacity constraints. Consider all the queues corresponding to links originating at v_0. Any exogeneous arrival is routed to one of these queues. Furthermore these queues receive only exogeneous arrivals. Hence the sum of the arrival rates for the queues originating at v_0 is equal to λ and to the sum of the departure rates from these queues. Consider all links originating from v_0. The sum of their flows is equal to the sum of the departure rates of the corresponding queues, which is equal to the sum of their arrival rates. The later sum equals to the rate λ of the arrival stream and the flow transfer of $\mathbf{f}$ is indeed equal to λ.

It remains to show that $\mathbf{f}$ satisfies the flow conservation equations (4.1). Consider a node $v \in (V - \{v_0, v_d\})$. The sum of the flows of the links originating at v is equal to the sum of the departure rates of the corresponding queues which is equal to the sum of the arrival rates at the same queues. By construction of Q_N, these queues receive traffic only from those queues that correspond to incomming links at node v. Hence the flow conservation equations are satisfied. $\diamond$

Lemma 4.2: The queueing network Q_N is stabilizable if the arrival rate λ is less than the capacity of a mincut of the flow network N.

Proof: We will show that if λ is less than the capacity of a mincut then for every set S of queues the condition (3.1) holds. Then stabilizability follows from theorem 3.2. For every set of queues S consider the set of nodes V_S that contains all nodes for which all the outgoing links correspond to queues that belong to S. If node v_0 does not belong to V_S that means there exists a link originating at v_0 such that the corresponding queue does not belong to S; that is the incomming customers may be routed upon arrival out of S. Hence C_S is empty and condition (3.1) holds. If v_0 belong to V_S then $\sum_{l \in C_S} a_l = \lambda$. Consider the cut $(V_S,\ V - V_S)$ and an arbitrary forward link (v, w). The queue that corresponds to (v, w) belongs to S (otherwise node v would not belong to V_S). From node w there exists an outgoing link such that the corresponding queue does not belong to S (otherwise w would belong to V_S). Hence the queue that corresponds to (v, w) may route customers out of S and belongs to Q_S. Since the queue that corresponds to an arbitrary forward link of $(V_S,\ V - V_S)$ belongs to Q_S we have that

$$\sum_{l \in C_S} a_l = \lambda < C(V_S,\ V - V_S) \leq \sum_{i \in Q_S} m_i.$$

$\diamond$

Proof of the Maxflow-mincut theorem: It is easy to show that for any flow **f** and any cut the total flow is less than the capacity of the cut which readily implies that the solution of the maxflow problem should be less than or equal to the capacity of a mincut. By lemmas 4.1, 4.2 we have that for any λ less than the capacity of a mincut there exists a feasible flow with flow transfer λ. Hence the flow transfer of a maxflow should be greater than or equal to any number smaller than the capacity of a mincut. The theorem then follows. $\diamond$

5.5 Routing with Delayed Information.

Up to this section we have assumed that at each decision time instant at queue i (or at the entry point of class l), the lengths of the queues in S_i at that time are available to queue i. There are several practical systems which are modeled by the above queueing network and in which that assumption does not apply. In those systems the queues of the queueing network correspond to physically different nodes correspond to physically different nodes (locations) at the actual system. The lengths of the queues in S_i are communicated to queue i at certain time instants. The decision at time t is taken according to the lengths of the queues in S_i which have been communicated to queue i most recently and not of the actual lengths at time t. Hence in several cases the decisions are taken based on outdated information about the system state. In this section we study the effect of the outdated information on the system performance. In the following we consider a model for information exchange for which we obtain stability results in the rest of the section.

Assume that the length of queue $j \in S_i$ is communicated to queue i at random time instants that constitute a Poisson process with rate r_{ij}. Let $X_{ij}(t)$ be the most recently communicated value of the length of queue j to queue i. Similarly the length of queue $j \in S_l^c$ is communicated to the entry point of class l where the routing decisions are taken, at random time instants that constitute a Poisson process with rate r_{lj}. The variable $X_{lj}^c(t)$ has a similar interpretation to that of $X_{ij}(t)$. In the rest of this section let $\tilde{\mathbf{X}}(t) = (X_i(t) : i = 1, ..., M;\ X_{ij}(t) : i = 1, ..., M,\ j \in S_i;\ X_{lj}(t) : l = 1, ..., L,\ j \in S_l^c)$ and let $\mathcal{X}$ be the space where this vector lies. The vector of the queue lengths at time t will be denoted by $\mathbf{X}(t)$. The same controls are available to the queues of the network as in the initial model. A control policy is specified by the routing rules R_i, R_l^c, $i = 1, ..., M$, $l = 1, ..., L$ for each queue and class of arriving customers respectively, and the service rate control rules F_i, $i = 1, ..., M$. The interpretation of the control rules is as in section 2. Consider the following control policy π_1, where the decisions at each queue i, at each decision time t

depends on the available information at that queue, that is the values of the variables $X_i(t),\ X_{ij}(t),\ j \in S_i$; its control rules are as follows

$$R_i(\mathbf{x}) = \begin{cases} D, & \text{if } D \in S_i \\ \arg\min_{j \in S_i}(x_{ij}), & \text{otherwise,} \end{cases}$$

$$R_l^c(\mathbf{x}) = \arg\min_{j \in S_l^c}(x_{lj}),$$

$$F_i(\mathbf{x}) = \begin{cases} 0, & \text{if } x_i \leq \min_{j \in S_i}\{x_{ij}\} \\ m_i, & \text{otherwise.} \end{cases}$$

Under π_1 we can easily check that the queue length process $\tilde{X}$ is not a Markov chain. Nevertheless we can still study the ergodicity of $\tilde{X}$ within the Markovian framework. We can do that because the process $\mathbf{X}$ is a Markov chain since in addition to Poisson arrivals and exponential service times, the times of message exchanges form a Poisson process for each pair of neighboring queues. The following proposition characterizes the stability properties of the system.

Theorem 5.1: The necessary and sufficient ergodicity condition for $\mathbf{X}$ under π_1 is that the message exchange rates $r_{ij},\ i = 1, ..., M,\ j \in S_i$ are nonnegative and (3.1) holds for the arrival and service rates.

The proof of the theorem follows after the next lemma.

Lemma 5.1: When all the message exchange rates are positive, condition (3.1) holds and policy π_1 acts on the system the subset of the state space

$$\mathcal{R} = \{x : \ x \in \mathcal{X} \text{ and } x \text{ can be reached from state } \mathbf{0}\ \}$$

is the unique closed class of equivalent states of the Markov chain; any state in the set $\mathcal{X} - \mathcal{R}$ is transient.

Proof: In view of the proof of lemma 3.1, for this proof we just need to show that the state $\mathbf{0}$ is reachable by any other state $\mathbf{x} \in \mathcal{X}$. Consider an arbitrary state $\mathbf{x}$. After a sequence of message transmissions, a state x^1 may be reached which is such that $x^1_{ij}(t) = x^1_j(t)$ for all i, j's. In the model without delayed information there is a sequence of transitions that leads the system from any state $\tilde{x}^1$ (where $\tilde{x}^1$ is the queue length vector that corresponds to state x^1) to $\mathbf{0}$. Consider the same sequence of transitions and after each transition assume that appropriate message exchanges happen such that at each node the actual values of the neighboring queues are available. In this case the control actions of policy π_1 in the model with delayed information will be identical to those in the initial model hence from the proof of 3.1 we have that x^1 will reach eventually state $\mathbf{0}$. $\diamond$

The proof of the theorem follows.

Proof of Theorem 5.1: We use theorem (3.2). Let $q_{\mathbf{x}} = \sum_{\mathbf{y} \in \mathcal{X}} q_{\mathbf{xy}}$. For all $\mathbf{x} \in \mathcal{X}$ we have

$$q_{\mathbf{x}} \leq \sum_{l=1}^{L} a_L + \sum_{i=1}^{M} m_i + \sum_{i=1}^{M} \sum_{j \in S_i} r_{ij} + \sum_{l=1}^{L} \sum_{j \in S_l^c} r_{lj}^c$$

hence

$$\sup_{\mathbf{x} \in \mathcal{X}} (q_{\mathbf{x}}) < \infty. \tag{5.1}$$

On the state space $\mathcal{X}$ consider the functions

$$V_1(\mathbf{x}) = \sum_{i=1}^{M} x_i^2 \ , \ V_2(\mathbf{x})$$

$$= \sum_{i=1}^{M} \sum_{j \in S_i} (x_{ij} - x_j)^2 + \sum_{l=1}^{L} \sum_{j \in S_l^c} (x_{lj} - x_j)^2 \ , \ V(\mathbf{x}) = V_1(\mathbf{x}) + V_2(\mathbf{x}). \quad (5.2)$$

Function V will play the role of a Liapunov function. It can be easily verified, as in (3.6), that

$$\sum_{\mathbf{y} \in \mathcal{X}} \frac{q_{\mathbf{xy}}}{q_{\mathbf{x}}} V(\mathbf{y}) \leq 2V(\mathbf{x}) < \infty \quad (5.3)$$

Consider the set $V_b = \{\mathbf{x} : \mathbf{x} \in \mathcal{X}, \ V(\mathbf{x}) \leq b\}$ which has apparently finite cardinality for each b.We show in the following that for a fixed ϵ there exists a b, which may be a function of the arrival, service and message exchange rates such that

$$-\epsilon \geq \sum_{\mathbf{y} \in \mathcal{X}} \frac{q_{\mathbf{xy}}}{q_{\mathbf{x}}} V(\mathbf{y}) - V(\mathbf{x}), \quad \text{if } \mathbf{x} \notin V_b. \quad (5.4)$$

Then (5.1-2) and (5.4) imply ergodicity of $\mathbf{X}$ from Foster's criterion. Consider the sets

$$A(\mathbf{x}) = \{\mathbf{y} : \text{ an arrival or service completion transfers } \mathbf{x} \textbf{ to } \mathbf{y}\}$$

$$B(\mathbf{x}) = \{\mathbf{y} : \text{ a message transmission transfers } \mathbf{x} \text{ to } \mathbf{y}\}$$

The term $q_{\mathbf{xy}}$ is strictly greater than 0 only if $\mathbf{y} \in A(\mathbf{x})$ or $\mathbf{y} \in B(\mathbf{x})$. Hence the term in the right hand side of (5.4) can be written as

$$\sum_{\mathbf{y} \in \mathcal{X}} \frac{q_{\mathbf{xy}}}{q_{\mathbf{x}}} V(\mathbf{y}) - V(\mathbf{x}) = \sum_{\mathbf{y} \in A(\mathbf{x})} \frac{q_{\mathbf{xy}}}{q_{\mathbf{x}}} (V_1(\mathbf{y}) - V_1(\mathbf{x})) + \sum_{\mathbf{y} \in A(\mathbf{x})} \frac{q_{\mathbf{xy}}}{q_{\mathbf{x}}} (V_2(\mathbf{y}) - V_2(\mathbf{x}))$$

$$+ \sum_{\mathbf{y} \in B(\mathbf{x})} \frac{q_{\mathbf{xy}}}{q_{\mathbf{x}}} (V_1(\mathbf{y}) - V_1(\mathbf{x})) + \sum_{\mathbf{y} \in B(\mathbf{x})} \frac{q_{\mathbf{xy}}}{q_{\mathbf{x}}} (V_2(\mathbf{y}) - V_2(\mathbf{x})). \quad (5.5)$$

For all $\mathbf{y} \in B(\mathbf{x})$ we have $V_1(\mathbf{y}) = V_1(\mathbf{x})$; hence

$$\sum_{\mathbf{y} \in B(\mathbf{x})} \frac{q_{\mathbf{xy}}}{q_{\mathbf{x}}} (V_1(\mathbf{y}) - V_1(\mathbf{x})) = 0. \quad (5.6)$$

By the definition of $B(\mathbf{x})$, for all $\mathbf{y}$ in it there are some queues i, j (or an arrival stream l and a class j) such that the transition from $\mathbf{x}$ to $\mathbf{y}$ has rate r_{ij} (or r^c_{lj}) and $V_2(\mathbf{y}) - V_2(\mathbf{x}) = -(x_{ij} - x_j)^2$. Hence we have

$$\sum_{\mathbf{y}\in B(\mathbf{x})} \frac{q_{\mathbf{xy}}}{q_{\mathbf{x}}}(V_2(\mathbf{y})-V_2(\mathbf{x})) \le h(-\sum_{i=1}^{M}\sum_{j\in S_i}(x_{ij}-x_j)^2-\sum_{l=1}^{L}\sum_{j\in S_l^c}(x_{lj}-x_j)^2) \quad (5.7)$$

where

$$h = \frac{\min\{\min_{\substack{i=1,\dots,M\\ j\in S_i}}\{r_{ij}\}, \min_{\substack{l=1,\dots,L\\ j\in S_l^c}}\{r^c_{lj}\}\}}{\max_{\mathbf{x}\in\bar{X}}\{q_{\mathbf{x}}\}} > 0 \quad (5.8)$$

Relation (5.1) guarantees that h is well defined. The condition $\mathbf{x} \notin V_b(\mathbf{x})$ in (5.4) implies that $V_1(\mathbf{x}) \ge b - V_2(\mathbf{x})$ which from (3.21) implies that

$$\sum_{\mathbf{y}\in A(\mathbf{x})} \frac{q_{\mathbf{xy}}}{q_{\mathbf{x}}}(V_1(\mathbf{y}) - V_1(\mathbf{x})) \le 2 + \frac{2}{d}m\sqrt{\frac{(b - V_2(\mathbf{x}))^+}{M}} \quad (5.9)$$

where $(a)^+$ is the maximum of a and 0. We upper bound now the second term in the sum in the right hand side of (5.5). We have first

$$V_2(\mathbf{y}) - V_2(\mathbf{x}) = \sum_{i=1}^{M}\sum_{j\in S_i}(y_j - x_j)(x_{ij} - x_j + y_{ij} - y_j) +$$

$$\sum_{l=1}^{L}\sum_{j\in S_l^c}(y_j - x_j)(x_{lj} - x_j + y_{lj} - y_j) \quad (5.10)$$

From the definition of $A(\mathbf{x})$ we have that

$$|y_j - x_j| \le 1 \ , \quad |x_{ij} - x_j + y_{ij} - y_j| \le 2|x_{ij} - x_j| + 1 \le 2\sqrt{V_2(\mathbf{x})} + 1 \quad (5.11)$$

and from (5.9),(5.10) we get

$$\sum_{\mathbf{y}\in A(\mathbf{x})} \frac{q_{\mathbf{xy}}}{q_{\mathbf{x}}}(V_2(\mathbf{y}) - V_2(\mathbf{x})) \le C_1\sqrt{V_2(\mathbf{x})} + C_2 \quad (5.12)$$

where C_1, C_2 are positive constants. From (5.5), (5.6), (5.7), (5.9) and (5.12) we get

$$\sum_{\mathbf{y}\in\mathcal{X}} \frac{q_{\mathbf{xy}}}{q_{\mathbf{x}}} V(\mathbf{y}) - V(\mathbf{x}) \leq -hV_2(\mathbf{x}) + C_1\sqrt{V_2(\mathbf{x})} + C_2 + 2 + \frac{2}{d}m\sqrt{\frac{(b - V_2(\mathbf{x}))^+}{M}}. \tag{5.13}$$

The right hand side of (5.12) can indeed become less than $-\epsilon$ for some positive ϵ irrespectively of the value of $V_2(\mathbf{x})$, if b is sufficiently large. Since m is negative we have

$$-hV_2(\mathbf{x}) + C_1\sqrt{V_2(\mathbf{x})} + C_2 + 2 + \frac{2}{d}m\sqrt{\frac{(b - V_2(\mathbf{x}))^+}{M}}$$

$$\leq -hV_2(\mathbf{x}) + C_1\sqrt{V_2(\mathbf{x})} + C_2 + 2;$$

hence we can select a θ such that for all b we have

$$-hV_2(\mathbf{x})+C_1\sqrt{V_2(\mathbf{x})}+C_2+2+\frac{2}{d}m\sqrt{\frac{(b - V_2(\mathbf{x}))^+}{M}} \leq -\epsilon \text{ if } \sqrt{V_2(\mathbf{x})} \geq \theta \tag{5.14}$$

If $\sqrt{V_2(\mathbf{x})} \leq \theta$ then we have

$$-hV_2(\mathbf{x}) + C_1\sqrt{V_2(\mathbf{x})} + C_2 + 2 + \frac{2}{d}m\sqrt{\frac{(b - V_2(\mathbf{x}))^+}{M}}$$

$$\leq C_1\theta + C_2 + 2 + \frac{2}{d}m\sqrt{\frac{(b - \theta^2)^+}{M}}. \tag{5.15}$$

Apparently the right hand side of (5.13) can become less than $-\epsilon$ if b is sufficiently large while inequality (5.13) is not affected by that. This completes the proof. ◇

CHAPTER 6

Conclusion

Several resource allocation problems in multihop radio networks were considered in this dissertation. The problem of joint routing-scheduling for maximum throughput was addressed in a general multihop, multidestination radio network and optimal policies were obtained. The problem of delay optimal scheduling was considered in a tandem radio network; optimal policies and necessary optimality conditions were found. A model of a changing connectivity single hop network were proposed. Necessary and sufficient stabilizability conditions were obtained in this model and stabilizing policies were specified. Scheduling policies that minimize the delay were obtained for both finite and infinite buffer systems. Finally a general queueing network with routing and flow control at each queue were studied and necessary and sufficient stabilizability conditions were obtained. There are several problems, directly related to the ones we have studied, which are left open. We mention few of them next.

In chapter 2 we obtained two routing-scheduling policies that achieve maximum throughput. The second policy π_1, which does not have the implementation difficulties of π_0 involves the selection of an activation set according to some probability distribution on the constraint set S. Policy π_1 depends highly on that probability distribution. It is interesting to study the performance of the system under different distributions. Also notice that π_1, even though it is simpler than π_0, is still centralized; it is important to find distributed implementations of that policy.

We have studied the constrained queueing system under the assumption of slotted operation where the servers are synchronized to start service simultaneously at the beginning of the slot. This assumption appears to be restrictive in certain cases. For example, in the database model, in order for this assumption to hold all of the transactions should have the same length so that they finish their processing simultaneously. Obtaining stabilizing policies in the case where customers have different service times is a problem for further investigation.

In the third chapter, for the system of parallel queues with server activation constraints implied by the tandem topology, we have shown that the class $\tilde{G}$ will contain a delay optimal policy if one exists. Notice that class $\tilde{G}$ contains all policies that maximize parallelization in service. All those policies are myopically optimal in the sense that the activation vector at slot t is such that the number of customers in the system at slot t is the minimum possible given the state of the system at slot $t-1$. It is interesting to study whether similar properties hold for systems of parallel queues with different activation constraints other than those implied by the tandem topology.

Turning to the issue of changing connectivity, an interesting variation of the problem we studied is the case where the connectivity information is not available for decision making and the server allocation is based on information about the queue lengths, the arrivals and the departures. If the connectivity variables in different slots are independent then, as we mentioned in section 4.4.3, the server allocation problem under no connectivity information is equivalent to a server allocation problem in a fixed connectivity system. If the connectivities at neighboring time slots are statistically dependent then the problem of optimal allocation becomes more complicated. The queue lengths are no longer a state of the system and the problem should be as a partially observable Markov Decision Process with the appropriate independence assumptions on the arrival and service processes. The problem of optimal allocation in the latter case of dependent connectivities is open for further investigation.

In our study of changing connectivity we have assumed that each queue is either connected to the server or not, that is the connectivities are binary variables. In certain cases that assumption is inappropriate and the connectivity should be represented by a multivalued variable where the different values correspond to different connectivity qualities. It is of interest to study the resource allocation problem under the assumption of multivalued connectivities.

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